S4A1 - SEMINAR ON ALGEBRAIC GEOMETRY: "JACOBIANS OF CURVES"

Organizers: Peter Scholze, Johannes Anschütz[1]

In this seminar we want to prove the following theorem.

Theorem 1. *Let k be an algebraically closed field and let*

$$f\colon X \to \mathrm{Spec}(k)$$

be a projective, smooth, irreducible curve. Let furthermore $x \in X(k)$ be a k-rational point. Define the Picard functor of X over k by

$$\mathrm{Pic}_{X/k,x}\colon (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Sets}),\ T \mapsto \mathrm{Ker}(\mathrm{Pic}(X_T) \xrightarrow{s_T^*} \mathrm{Pic}(T)),$$

where

$$X_T := X \times_{\mathrm{Spec}(k)} T \ \textit{resp.}\ s_T = x \times \mathrm{Id}_T$$

denote the base change of X resp. x to T. Then the functor $\mathrm{Pic}_{X/k,x}$ is representable by a scheme.

Moreover, we will analyse the representing scheme of $\mathrm{Pic}_{X/k}$ more closely. Some organisatorial remarks are in order. Because of the lack of some extra time each talk must be finished on its assigned day. It is the task of the speaker to arrange the material to fit into this time frame. Moreover, every speaker should contact the second organizer at least one week before his/her talk to clarify its content. We now outline the proof of Theorem 1 and indicate the contents of the talks.

Let X and k be as above. Sending an effective Cartier divisor D on X to its line bundle $\mathcal{O}_X(D)$ (talk 1) defines a map of sets, a special case of the later defined Abel-Jacobi map (talk 8),

$$\mathrm{AJ}'\colon \{\text{effective Cartier divisors on } X \text{ of degree d}\} \to \mathrm{Pic}^d(X),$$

which will be used to prove the representability of the Picard functor. The domain of this map AJ' arises actually as the k-rational points $\mathrm{Hilb}^d_{X/k}(k)$ of a scheme

$$\mathrm{Hilb}^d_{X/k},$$

called the Hilbert scheme of points (talk 3,4). Moreover, this scheme admits a very concrete description. Namely, an effective Cartier divisor of degree d is precisely an unordered tuple $(x_1, \ldots, x_d)$ of k-rational points $x_1, \ldots, x_d$ of X. In other words, we expect that the Hilbert scheme of points $\mathrm{Hilb}^d_{X/k}$ is isomorphic to the symmetric product

$$X^{(d)} := X^d/S_d \cong \mathrm{Hilb}^d_{X/k},$$

where the symmetric group S_d acts on the product X^d by permuting the factors. In fact, this intuition can be made precise (talk 5). We obtain finally a natural transformation

$$\mathrm{AJ}\colon \mathrm{Hilb}^d_{X/k} \to \mathrm{Pic}_{X/k,x}$$

[1]ja@math.uni-bonn.de

of functors, the true Abel-Jacobi map. In order to deduce the representability of the Picard functor $\mathrm{Pic}_{X/k}$ (talk 6,7) we will analyze the geometry of the map AJ more closely. Namely, we will show that if g is the genus of X there exists an open subset, more precisely an open subfunctor, $V \subseteq \mathrm{Hilb}^g_{X/\mathrm{Spec}(k)}$ which maps, via the Abel-Jacobi map AJ, isomorphically onto an open subfunctor of $\mathrm{Pic}^g_{X/k}$. Now, we will use the group structure of $\mathrm{Pic}_{X/k}$. Using this group structure, i.e., the tensor product with a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$, this open subfunctor

$$\mathrm{AJ}(V) \subseteq \mathrm{Pic}_{X/k}$$

can be translated and we finally obtain a covering

$$\mathrm{Pic}_{\mathrm{X/k}} = \bigcup_{\mathcal{L}\in\mathrm{Pic(X)}} \mathcal{L} + \mathrm{AJ}(V)$$

of the Picard functor by representable open subfunctors establishing its representability.

According to the degree of a line bundle the Picard functor of X, and hence any scheme representing it, decomposes into a disjoint union

$$\mathrm{Pic}_{X/k} \cong \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d_{X/k}.$$

The scheme $\mathrm{Pic}^0_{X/k}$, parametrizing line bundles of degree 0 on X, is called the Jacobian of X and it is a basic example of an abelian variety, i.e., of a proper, smooth group scheme over a field.

With this example in mind we discuss abelian varietes a bit further. In particular, we prove that abelian varieties are "abelian", i.e., that there group structure is automatically commutative (talk 10) and determine their torsion (talk 11). Then we move on and discuss dual abelian varieties. However, due to time constraints we will only be able to prove a (restricted) representability in characteristic 0 (talk 13). To construct the dual $\hat{A}$ of an abelian variety A over an algebraically closed field k (of characteristic 0), we first have to construct quotients A/K of A for finite subgroups $K \subseteq A$ (talk 12). In talk 14 we will then prove the self-duality for the Jacobian of a smooth projective curve.

For a compact Riemann surface, e.g., the set $X(\mathbb{C})$ of complex points of X if $k = \mathbb{C}$, the Jacobian can also be constructed (as a complex torus) using the exponential sequence. This will be presented in talk 9. Finally, we need talk 2 to get some background material about smooth morphisms of schemes.

1.Talk: Line bundles on curves (18.04.2017)
Define effective Cartier divisors [Sta17, Tag 01WR] on general schemes and explain their relation with line bundles [Sta17, Tag 01X0]. Then move on to divisors on normal curves over fields and present [GW10, Section (15.9), Section (15.10)] (every occuring curve may be assumed to be normal). Finally state [Liu02, Proposition 7.5.] that a divisor on a projective irreducible curve is ample if and only if its degree is positive (in particular, define ampleness [GW10, Definition 13.44]).

2.Talk: Smooth morphisms (25.04.2017)
Present [GW10, Section (6.8), Section (6.10)], [GW10, Theorem 6.28], its corollary [GW10, Corollary 6.34], [GW10, Example 6.34] and [GW10, Theorem 14.22]. Then

explain as much as possible of [Sta17, Tag 01V8] (or [BLR90, Chapter 2, Proposition 8] which might be more accessible).

3.Talk: The Hilbert scheme of points (02.05.2017)
Follow [Sta17, Chapter 43, Section 2] and present the Tags 0B96-0B99, 0B9A, 0B9B.

4.Talk: The Hilbert scheme of points for relative curves (09.05.2017)
Present [Sta17, Tags 0B9C-0B9J], i.e., [Sta17, Chapter 43, Section 3]. In particular, define relative effective Cartier divisors [Sta17, Tag 062T].

5.Talk: Symmetric powers (16.05.2017)
Present [GW10, Proposition 12.27] resp. [Gro63, Exposé V.1, Proposition 1.1., Proposition 1.8.], deduce the existence of symmetric powers $(X/S)^{(n)}$ for quasi-projective morphisms $f\colon X \to S$ and prove [BLR90, Chapter 9.3., Proposition 3]. Finally, discuss the examples $(\mathbb{A}^1_S/S)^{(n)} \cong \mathbb{A}^n_S$ and $(\mathbb{P}^1_S/S)^{(n)} \cong \mathbb{P}^n_S$.

6.Talk: Representability of the Picard functor I (23.05.2017)
Present [Sta17, Tag 0B9Q] in full detail, in particular recall [Sta17, Tag 01JJ]. Then present [Sta17, Tags 0B9V, 0B9W, 0B9X] (with a more detailed discussion of the cohomological results which are cited there).

7.Talk: Representability of the Picard functor II (30.05.2017)
Present the rest of [Sta17, Chapter 43, Section 6], i.e., Tags 0B9Y, 0B9Z, 0BA0. Finally discuss the cases $g = 0, 1$. Deduce that elliptic curves are canonically equipped with a group structure.

8.Talk: The Abel-Jacobi map in general (13.06.2017)
Recall the notion of relative representability [Sta17, 0023]. Then present [BLR90, Chapter 8.2., Proposition 7] with all necessary preliminaries. This includes [BLR90, Chapter 8.2., Lemma 6] and [BLR90, Chapter 8.1., Theorem 7].

9.Talk: Jacobians of compact Riemann surfaces (20.06.2017)
For this talk some knowledge of complex analysis and deRham cohomology is helpful. Present [Ara, Sections 2-4].

10.Talk: Abelian varieties I (27.06.2017)
Define abelian varieties [Mum08, II.4] and prove that their group structure is abelian [Mum08, Appendix to §4]. State (or prove if possible) the Seesaw Theorem [Mum08, II.5.Corollary 6], the theorem of the cube [Mum08, II.6] and deduce the corollaries 1-4 in [Mum08, II.6].

11.Talk: Abelian varietes II (04.07.2017)
Present the rest of [Mum08, II.6], i.e., start with Application 1. You can omit one of the proofs that $\deg(n_X) = n^{2g}$.

12.Talk: Separable isogenies of abelian varieties (11.07.2017)
Present [Mum08, II.7]. Note that some parts of the main theorem have been covered in talk 5. For the proof of [Mum08, II.7. Proposition 2] you may also cite results on faithfully flat descent if they have been covered in the lecture. If you are very

ambitious you may discuss the case of inseparable isogenies ([Mum08, II.12]) as well.

13.Talk: The dual abelian variety in characteristic 0 (18.07.2017)
Present [Mum08, II.8] till the remarks on page 76 (in the new numbering).

14.Talk: Self-duality of Jacobians (25.07.2017)
Present the rest of [Mum08, II.8]. Then prove [Mil86, Theorem 6.6].

References

[Ara] Donu Arapura. The jacobian of a riemann surface. preprint on webpage at `http://www.math.purdue.edu/~dvb/preprints/jacobian.pdf`.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[Gro63] Alexander Grothendieck. *Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5*, volume 1960/61 of *Séminaire de Géométrie Algébrique*. Institut des Hautes Études Scientifiques, Paris, 1963.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Mil86] J. S. Milne. Jacobian varieties. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 167–212. Springer, New York, 1986.

[Mum08] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

[Sta17] The Stacks Project Authors. *stacks project. http://stacks.math.columbia.edu, 2017.*