S4A3 - GRADUATE SEMINAR ON ADVANCED ALGEBRA: "HILBERT MODULAR SURFACES"

Organizer: Johannes Anschütz[1]
Time and place: WS 17/18, Tuesdays, 16-18h, SR 0.003
Preliminary meeting: Wednesday, 26.07.2017, 16-18h, N0.003.

Let p be a prime congruent to 1 mod 4 and define the real quadratic number field $K := \mathbb{Q}(\sqrt{p})$ with ring of integers $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{p}}{2}]$. In [HVdV74] F. Hirzebruch and A. van de Ven studied the class of the Hilbert modular surfaces $Y(p)$. These surfaces are complex projective surfaces constructed via compactifying and then resolving singularities of quotients $\mathbb{H} \times \mathbb{H}/\Gamma$ of two copies of the upper half plane

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$$

modulo the action of the discrete group $\Gamma := \mathrm{SL}_2(\mathcal{O}_K)$ via Möbius transformations[2]

They investigated where the surfaces $Y(p)$ are placed within the Kodaira classification of algebraic surfaces and obtained the following theorem, which will be the main theorem of this seminar.

Theorem 1 (cf. Theorem III.1 in [HVdV74])**.** *The surfaces $Y(p)$ are*

- *rational surfaces for $p = 5, 13, 17$;*
- *blown up elliptic K3 surfaces for $p = 29, 37, 41$;*
- *honestly elliptic surfaces for $p = 53, 61, 73$;*
- *surfaces of general type for $p \geq 89$.*

The aim of this seminar is to understand this theorem, both in formulation and verification. Thereby we will touch several interesting topics such as Serre's GAGA, intersection theory, blow ups/blow downs, quotient singularities, Kodaira's classification of surfaces, modular curves, modular forms,.... In particular, we will not be able to discuss these topics in all their depth and richness, but we will focus on the aspects needed for the understanding of [HVdV74]. Additional literature is provided by [B01], [HvdG81], [vdG88] or [Hir73].

1.Talk: GAGA (17.10.2017)

Shortly introduce analytic spaces and coherent sheaves on them (cf. [Ser56]). Then present [Gro71, Exposé XII, Chapitre 1-4], mainly focusing on [Gro71, Corollaire 4.3.] and [Gro71, Corollaire 4.5.]. The speaker may choose to follow the classic

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[2] Let $\sigma_i \colon K \to \mathbb{R}, i = 1, 2$, be the two real embeddings of K. Then

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

acts on $(\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}$ via

$$A.(\tau_1, \tau_2) := \left(\frac{\sigma_1(a)\tau_1 + \sigma_1(b)}{\sigma_1(c)\tau_1 + \sigma_1(d)}, \frac{\sigma_2(a)\tau_2 + \sigma_2(b)}{\sigma_2(c)\tau_2 + \sigma_2(d)} \right).$$

[Ser56] instead (or additionally). Finally, sketch the proof that compact Riemann surfaces are algebraic (cf. [GH94, Chapter 2.1]).

2.Talk: Hilbert modular varieties (24.10.2017)
Define Hilbert modular varieties, their cusps and fundamental domains for arbitrary totally real number fields $K/\mathbb{Q}$ ([vdG88, Chapter I.1.,I.3.]). Then prove the structure of elliptic fixed points ([vdG88, Chapter I.5.]), introduce the quotients $\mathbb{H}^n/\Gamma$ (as analytic spaces) and define Hilbert modular forms ([vdG88, Chapter I.6.]). Finally, identify Hilbert modular forms with sections of a suitable line bundle (cf. [vdG88, Chapter 2.7.]). Except in the case $K = \mathbb{Q}$ you may always assume $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$.

3.Talk: Algebraicity of Hilbert modular varieties and the genus of modular curves (31.10.2017)
Provide details on the compactness of the quotients $\overline{\mathbb{H}^n/\Gamma}$ (cf. [vdG88, Chapter 2.7.]). Then prove [vdG88, Theorem 7.1.] (in the reference to [BB66] do not treat the case of general reductive groups $G/\mathbb{Q}$, but only the case relevant for use, namely $G = \mathrm{Res}_{K/\mathbb{Q}}(\mathrm{SL}_2)$, the Weil restriction of scalars of SL_2 for $K/\mathbb{Q}$ a totally real number field). Then compute the genus of modular curves [Shi94, Proposition 1.40].

4.Talk: Intersection theory (7.11.2017)
Follow [B0̌1, Chapter 1] from the beginning and end before [B0̌1, Chapter 1, Theorem 1.22].

5.Talk: Invariants of algebraic surfaces (14.11.2017)
Prove [Har77, Chapter V.1.5,1.6]. Then present [B0̌1, Chapter 5] (a discussion of the Albanese variety can be omitted). Explain the Kodaira classification of algebraic surfaces as presented in [HVdV74, Theorem ROC].

6.Talk: Quotient singularities (21.11.2017)
Present [HVdV74, Chapter II.”The Quotient Singularities”] (perhaps with the aid of [vdG88, Chapter II.6]). In particular, recall the definition of a blow up and compute some explicit resolutions of the quotient singularities (cf. exercise 3.3.-3.6. in [B0̌1]).

7.Talk: Castelnuovo’s theorem (28.11.2017)
Present a proof of Castelnuovo’s theorem [B0̌1, Theorem 3.30].

8.Talk: Criteria for rationality (5.12.2017)
Present the part “Rational Surfaces” in [HVdV74, Chapter I] including, if possible, a proof or parts of a proof for Castelnuovo’s criterion (cf. [B0̌1, Chapter 13]).

9.Talk: Elliptic surfaces (12.12.2017)
Present the part “Elliptic Surfaces” in [HVdV74, Chapter I]. More details about elliptic fibrations can be found in [B0̌1], for example.

10.Talk: Resolution of the cusp singularities (19.12.2017)
Present the part “The Cusps” in [HVdV74, Chapter II] (cf. [vdG88, Chapter II]).

11.Talk: The curves F_N, part I (09.01.2018)
Talk 11 and 12 should cover the rest of [HVdV74, Chapter II]. Therefore the speakers have to decide where to split the material. Additional literature is provided by [HVdV74, Chapter V].

12.Talk: The curves F_N, part II (16.01.2018)
See talk 11.

13.Talk: Proof of the main theorem (23.01.2018)
Prove 1, i.e., [HVdV74, Theorem III.1]. Not every calculation has to be presented in detail, but present at least one argument for every case.

14.Talk: Moduli interpretation of Hilbert modular varieties (30.01.2018)
Present, starting with [vdG88, Chapter X.1], as much as possible of [vdG88, Chapter X].

References

[BB66] W. L. Baily, Jr. and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966.

[B01̆] Lucian Bădescu. *Algebraic surfaces.* Universitext. Springer-Verlag, New York, 2001. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[Gro71] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*, volume 224 of *Lecture notes in mathematics.* Springer-Verlag, 1971.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Hir73] Friedrich E. P. Hirzebruch. Hilbert modular surfaces. *Enseignement Math. (2)*, 19:183–281, 1973.

[HvdG81] Friedrich Hirzebruch and Gerard van der Geer. *Lectures on Hilbert modular surfaces*, volume 77 of *Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics].* Presses de l'Université de Montréal, Montreal, Que., 1981. Based on notes taken by W. Hausmann and F. J. Koll.

[HVdV74] F. Hirzebruch and A. Van de Ven. Hilbert modular surfaces and the classification of algebraic surfaces. *Invent. Math.*, 23:1–29, 1974.

[Ser56] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble*, 6:1–42, 1955–1956.

[Shi94] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan.* Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1.

[vdG88] Gerard van der Geer. *Hilbert modular surfaces*, volume 16 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1988.