Let $p$ be a prime congruent to 1 mod 4 and define the real quadratic number field $K := \mathbb{Q}(\sqrt{p})$ with ring of integers $O_K = \mathbb{Z}[\frac{1+\sqrt{p}}{2}]$. In [HVdV74] F. Hirzebruch and A. van de Ven studied the class of the Hilbert modular surfaces $Y(p)$. These surfaces are complex projective surfaces constructed via compactifying and then resolving singularities of quotients $\mathbb{H} \times \mathbb{H}/\Gamma$ of two copies of the upper half plane $\mathbb{H} := \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \}$ modulo the action of the discrete group $\Gamma := \text{SL}_2(O_K)$ via Möbius transformations.

They investigated where the surfaces $Y(p)$ are placed within the Kodaira classification of algebraic surfaces and obtained the following theorem, which will be the main theorem of this seminar.

**Theorem 1** (cf. Theorem III.1 in [HVdV74]). The surfaces $Y(p)$ are

- rational surfaces for $p = 5, 13, 17$;
- blown up elliptic $K3$ surfaces for $p = 29, 37, 41$;
- honestly elliptic surfaces for $p = 53, 61, 73$;
- surfaces of general type for $p \geq 89$.

The aim of this seminar is to understand this theorem, both in formulation and verification. Thereby we will touch several interesting topics such as Serre’s GAGA, intersection theory, blow ups/blow downs, quotient singularities, Kodaira’s classification of surfaces, modular curves, modular forms,... In particular, we will not be able to discuss these topics in all their depth and richness, but we will focus on the aspects needed for the understanding of [HVdV74]. Additional literature is provided by [Bû01], [HvdG81], [vdG88] or [Hir73].

1. **Talk: GAGA (17.10.2017)**

Shortly introduce analytic spaces and coherent sheaves on them (cf. [Ser56]). Then present [Gro71, Expose XII, Chapitre 1-4], mainly focusing on [Gro71] Corollaire 4.3.] and [Gro71] Corollaire 4.5.]. The speaker may choose to follow the classic
2 Talk: Hilbert modular varieties (24.10.2017)
Define Hilbert modular varieties, their cusps and fundamental domains for arbitrary totally real number fields $K/\mathbb{Q}$ ([vdG88, Chapter I.1.,I.3.]). Then prove the structure of elliptic fixed points ([vdG88, Chapter I.5.]), introduce the quotients $\mathbb{H}^n/\Gamma$ (as analytic spaces) and define Hilbert modular forms ([vdG88, Chapter I.6.]). Finally, identify Hilbert modular forms with sections of a suitable line bundle (cf. [vdG88, Chapter 2.7.]). Except in the case $K = \mathbb{Q}$ you may always assume $\Gamma = \text{SL}_2(\mathcal{O}_K)$.

3 Talk: Algebraicity of Hilbert modular varieties and the genus of modular curves (31.10.2017)
Provide details on the compactness of the quotients $\mathbb{H}^n/\Gamma$ (cf. [vdG88, Chapter 2.7.]). Then prove [vdG88] Theorem 7.1. (in the reference to [BB66] do not treat the case of general reductive groups $G/\mathbb{Q}$, but only the case relevant for use, namely $G = \text{Res}_{K/\mathbb{Q}}(\text{SL}_2)$, the Weil restriction of scalars of $\text{SL}_2$ for $K/\mathbb{Q}$ a totally real number field). Then compute the genus of modular curves [Shi94 Proposition 1.40].

4 Talk: Intersection theory (7.11.2017)
Follow [B˘01, Chapter 1] from the beginning and end before [B˘01, Chapter 1, Theorem 1.22].

5 Talk: Invariants of algebraic surfaces (14.11.2017)
Prove [Har77] Chapter V.1.5,1.6. Then present [B˘01, Chapter 5] (a discussion of the Albanese variety can be omitted). Explain the Kodaira classification of algebraic surfaces as presented in [HVdV74] Theorem ROC].

6 Talk: Quotient singularities (21.11.2017)
Present [HYdV74] Chapter II."The Quotient Singularities" (perhaps with the aid of [vdG88] Chapter II.6)). In particular, recall the definition of a blow up and compute some explicit resolutions of the quotient singularities (cf. exercise 3.3.-3.6. in [B˘01]).

7 Talk: Castelnuovo’s theorem (28.11.2017)
Present a proof of Castelnuovo’s theorem [B˘01 Theorem 3.30].

8 Talk: Criteria for rationality (5.12.2017)
Present the part “Rational Surfaces” in [HVdV74] Chapter I, including, if possible, a proof or parts of a proof for Castelnuovo’s criterion (cf. [B˘01, Chapter 13]).

Present the part “Elliptic Surfaces” in [HVdV74] Chapter I. More details about elliptic fibrations can be found in [B˘01], for example.

10 Talk: Resolution of the cusp singularities (19.12.2017)
11. Talk: The curves $F_N$, part I (09.01.2018)
Talk 11 and 12 should cover the rest of [HvdV74, Chapter II]. Therefore the speakers have to decide where to split the material. Additional literature is provided by [HvdV74, Chapter V].

See talk 11.

13. Talk: Proof of the main theorem (23.01.2018)
Prove 1, i.e., [HvdV74, Theorem III.1]. Not every calculation has to be presented in detail, but present at least one argument for every case.

Present, starting with [vdG88, Chapter X.1], as much as possible of [vdG88, Chapter X].

References


