REDUCTIVE GROUP SCHEMES OVER THE FARGUES-FONTAINE CURVE

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Abstract. For an arbitrary non-archimedean local field we classify reductive group schemes over the corresponding Fargues-Fontaine curve by group schemes over the category of isocrystals. We then classify torsors under such reductive group schemes by a generalization of Kottwitz' set $B(G)$. In particular, we extend a theorem of Fargues on torsors under constant reductive groups to the case of equal characteristic.

1. Introduction

Let E be a non-archimedean local field and let G/E be a reductive group over E. If $E/\mathbb{Q}_p$ is p-adic L. Fargues classified G-torsors on the corresponding Fargues-Fontaine curve X_E (cf. [7]) associated to E (and an algebraically closed perfectoid extension F of the residue field $\mathbb{F}_q$ of E). His result is phrased in terms of R. Kottwitz' set $B(G)$ associated with G (cf. [14]). For a general non-archimedean local field E let

$$L := \widehat{E^{\mathrm{un}}}$$

be the completion of the maximal unramified extension E^{un} of E and let

$$\varphi_L \colon L \to L$$

be the Frobenius of L. In [14] (cf. [13]) R. Kottwitz defined

$$B(G) := G(L)/\varphi_L - \text{conjugacy}$$

as the set of φ_L−conjugacy classes in $G(L)$. Our first main theorem is the classification of G-torsors on the Fargues-Fontaine curve X_E by this set $B(G)$ generalizing L. Fargues' result for $E/\mathbb{Q}_p$ p-adic.

Theorem 1.1 (cf. Theorem 3.11)**.** *There exists a canonical bijection*

$$B(G) \cong H^1_{\acute{e}t}(X_E, G).$$

We mention that this theorem implies the computation of the Brauer group of E (cf. [7, Théorème 2.6.] resp. Theorem 4.2) and that it is a starting point for L. Fargues' program to geometrize the local Langlands correspondence over E (cf. [6]). Similar to [7] we present in Section 4 applications of Theorem 1.1 to the étale and flat cohomology of X_E with locally constant torsion coefficents.

In [7, Remarque 1.2.] L. Fargues asked the question how to generalize Theorem 1.1 about G-torsors on X_E to the case where G is no longer assumed to be constant, but an arbitrary reductive group scheme over X_E. Examples of possibly non-constant reductive group schemes over the Fargues-Fontaine curve X_E can be constructed as follows. Let $\varphi - \mathrm{Mod}_L$ be the category of isocrystals over L and let

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$\mathcal{O}_{\mathbb{G}}$ be a Hopf algebra in $\varphi - \mathrm{Mod}_L$ (such a Hopf algebra is also called an affine group scheme $\mathbb{G}$ over $\varphi - \mathrm{Mod}_L$ (cf. Definition 5.1)). After fixing an embedding $\bar{\mathbb{F}}_q \subseteq F$ there exists a canonical tensor functor (cf. [8, Section 8.2.3.] resp. [12])

$$\mathcal{E}(-)\colon \varphi - \mathrm{Mod}_L \to \mathrm{Bun}_{X_E}$$

from the category $\varphi - \mathrm{Mod}_L$ to the category of vector bundles on X_E. The image $\mathcal{E}(\mathcal{O}_{\mathbb{G}})$ of $\mathcal{O}_{\mathbb{G}}$ is then the Hopf algebra associated with a flat group scheme $\mathcal{G}$ over X_E. We call this group scheme $\mathcal{G}$ the "group scheme associated with the affine group scheme $\mathbb{G}$ over $\varphi - \mathrm{Mod}_L$". Our first theorem about reductive group schemes over the Fargues-Fontaine curve is the following classification result.

Theorem 1.2 (cf. Theorem 5.3)**.** *Every reductive group scheme over the Fargues-Fontaine curve is associated with a, necessarily reductive, group scheme over $\varphi - \mathrm{Mod}_L$.*

The proof of Theorem 1.2 is straightforward, based on the classification of torsors for reductive groups (cf. Theorem 1.1) and the geometric simply-connectedness of X_E (cf. Theorem 4.1).

However, reductive group schemes over X_E are rather close to being constant (cf. Lemma 6.3). For example, they become constant after removing a closed point of X_E (cf. Lemma 6.3).

Based on Theorem 1.2 we can define for a reductive group scheme $\mathcal{G}$ over X_E a natural candidate of a set $B(\mathcal{G})$ generalizing Kottwitz' definition in the constant case (cf. Definition 3.9 and Proposition 5.7). Namely, if $\mathcal{G}$ is associated with the group scheme $\mathbb{G}$ we can set $B(\mathcal{G})$ as the set of φ_L-conjugacy classes in $G(L)$ where G is the reductive group over L associated with the Hopf algebra $\mathcal{O}_{\mathbb{G}}$.[1]

Our second theorem about reductive group schemes over X_E is then the following classification of torsors.

Theorem 1.3 (cf. Theorem 5.9)**.** *For a reductive group scheme $\mathcal{G}$ over X_E there is a canonical bijection*

$$B(\mathcal{G}) \cong H^1_{\acute{e}t}(X_E, \mathcal{G}).$$

The proof of Theorem 1.1 and Theorem 1.3 is based on an excessive use of the Tannakian formalism of $\mathbb{Q}$-graded and $\mathbb{Q}$-filtered fiber functors, which is an easy adaptation of [18] (cf. Section 2, and the classification of vector bundles on X_E (cf. Theorem 3.5), especially the fact that for a semistable vector bundle $\mathcal{E}$ of positive slope the cohomology group

$$H^1(X_E, \mathcal{E}) = 0$$

vanishes. The crucial input in our proof is the fact that sending a vector bundle to its Harder-Narasimhan filtration defines a fully faithful, unfortunately non-exact, tensor functor

$$\mathrm{HN}\colon \mathrm{Bun}_{X_E} \to \mathrm{Fil}^{\mathbb{Q}}\mathrm{Bun}_{X_E}$$

from ordinary vector bundles on X_E to $\mathbb{Q}$-filtered vector bundles on X_E.

Finally, we discuss uniformization results (cf. Section 6) for $\mathcal{G}$-torsors under arbitrary reductive group schemes $\mathcal{G}$ over X_E generalizing the known case that G is constant, quasi-split over E with $E/\mathbb{Q}_p$ p-adic (cf. [7, Théorème 7.2.]).

Theorem 1.4 (cf. Theorem 6.5)**.** *Let $\mathcal{G}$ be a reductive group scheme over X_E. Then every $\mathcal{G}$-torsor becomes trivial after removing a closed point of X_E.*

[1] Later we call this set $B(\mathbb{G})$.

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2. $\mathbb{Q}$-filtered fiber functors

In this section we want to extend the results of [18] about $\mathbb{Z}$-filtered fiber functors to $\mathbb{Q}$-filtered fiber functors. Filtrations on functors indexed by $\mathbb{Q}$ are also discussed in [4], e.g., in Chapter IV.2, or [3].

We follow the definitions and notations of [18, Chapter 2], but for us fiber functors will take values in the category of locally free sheaves of finite ranks and not in the category of arbitrary quasi-coherent modules.[2]

Let k be a field and let $\mathcal{T}$ be a Tannakian category over k. Let Γ be a totally ordered abelian group. Mainly, we will be interested in the case that $\Gamma \subseteq \mathbb{Q}$ is a subgroup. We start by recalling the definition of a Γ-graded fiber functor (cf. [16, Chapitre IV.1]). For this we denote by $\mathrm{Gr}^\Gamma \mathrm{Bun}_S$ the tensor category of Γ-graded vector bundles on a k-scheme S. Equivalently, if $D_{\Gamma,S}$ denotes the constant multiplicative group (over S) with character group Γ, then the category $\mathrm{Gr}^\Gamma \mathrm{Bun}_S$ is equivalent to the category of representations of $D_{\Gamma,S}$ over S on locally free sheaves of finite rank.

Definition 2.1. Let S be a scheme over k. A Γ-graded fiber functor of $\mathcal{T}$ over S is an exact tensor functor

$$\gamma \colon \mathcal{T} \to \mathrm{Gr}^\Gamma \mathrm{Bun}_S.$$

Equivalently, a Γ-graded fiber functor on $\mathcal{T}$ over S consists of a usual fiber functor

$$\omega \colon \mathcal{T} \to \mathrm{Bun}(S),$$

i.e., ω is an exact tensor functor, together with a homomorphism $D_{\Gamma,S} \to \mathrm{Aut}^\otimes(\omega)$ of group schemes over S where $\mathrm{Aut}^\otimes(\omega)$ denotes the group scheme of tensor automorphisms of ω (cf. [16, Chapitre IV.1]). If S is non-empty, then each Γ-graded fiber functor is automatically faithful (cf. [5, Corollarie 2.10.ii)]).

Now we define Γ-filtered fiber functors. We start by defining the category $\mathrm{Fil}^\Gamma \mathrm{Bun}_S$ of Γ-filtered vector bundles on a k-scheme S.

Definition 2.2. Let S be a scheme over k. A Γ-filtered vector bundle $\mathcal{E}$ on S is a vector bundle $\mathcal{E}$ on S together with subbundles, i.e., locally direct summands,

$$F^\lambda \mathcal{E} \subseteq \mathcal{E}$$

for $\lambda \in \Gamma$ such that

$$F^\lambda \mathcal{E} \subseteq F^{\lambda'} \mathcal{E}$$

if $\lambda' \leq \lambda$ and $F^\lambda \mathcal{E} = 0$ for $\lambda \gg 0$ resp. $F^\lambda \mathcal{E} = \mathcal{E}$ for $\lambda \ll 0$. Morphisms of Γ-filtered vector bundles are morphisms

$$f \colon \mathcal{E} \to \mathcal{E}'$$

of vector bundles respecting the subsheaves $F^\lambda \mathcal{E}$, i.e.,

$$f(F^\lambda \mathcal{E}) \subseteq F^\lambda \mathcal{E}'.$$

[2] which is not a restriction as fiber functors take their image in vector bundles

The category of Γ-filtered vector bundles on S is denoted by $\mathrm{Fil}^\Gamma\mathrm{Bun}_S$.

If $(\mathcal{E}, F^\bullet)$ is a Γ-filtered bundle we denote by $F^{>\lambda}\mathcal{E}$ for $\lambda \in \Gamma$ the vector bundle

$$F^{>\lambda}\mathcal{E} := \sum_{\lambda' > \lambda} F^{\lambda'}\mathcal{E}$$

of $\mathcal{E}$. We moreover denote by

$$\mathrm{gr}(\mathcal{E}) := \bigoplus_{\lambda\in\Gamma} \mathrm{gr}^\lambda(\mathcal{E})$$

where

$$\mathrm{gr}^\lambda(\mathcal{E}) := F^\lambda\mathcal{E}/F^{>\lambda}\mathcal{E}$$

the associated Γ-graded vector bundle on S. In this way we obtain a functor

$$\mathrm{gr}\colon \mathrm{Fil}^\Gamma\mathrm{Bun}_S \to \mathrm{Gr}^\Gamma\mathrm{Bun}_S.$$

We note that for a Γ-filtered vector bundle there are, locally on S, only finitely many $\lambda \in \Gamma$ such that

$$\mathrm{gr}^\lambda(\mathcal{E}) \neq 0.$$

If $(\mathcal{E}, F^\bullet)$ and $(\mathcal{F}, F^\bullet)$ are two Γ-filtered vector bundles on S then we define their tensor product

$$(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{F}, F^\bullet)$$

by setting

$$F^\lambda(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{F}) := \sum_{\lambda'+\lambda''=\lambda} F^{\lambda'}\mathcal{E} \otimes_{\mathcal{O}_S} F^{\lambda''}\mathcal{F}.$$

Moreover, we make $\mathrm{Fil}^\Gamma\mathrm{Bun}_S$ into an exact category (in the sense of Quillen) by requiring that a sequence

$$0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$$

is exact if and only if the associated sequence

$$0 \to \mathrm{gr}(\mathcal{E}) \to \mathrm{gr}(\mathcal{E}') \to \mathrm{gr}(\mathcal{E}'') \to 0$$

is exact. In particular, the functor

$$\mathrm{gr}\colon \mathrm{Fil}^\Gamma\mathrm{Bun}_S \to \mathrm{Gr}^\Gamma\mathrm{Bun}_S$$

is then an exact functor. Moreover, gr is even a tensor functor. We can also go into the other direction and associate to a Γ-graded vector bundle a filtered one. Namely, for a Γ-graded vector bundle

$$\mathcal{M} = \bigoplus_{\lambda\in\Gamma} \mathcal{M}^\lambda$$

we can define the Γ-filtered vector bundle

$$(\mathrm{fil}(\mathcal{M}), F^\bullet)$$

by setting

$$F^\lambda\mathrm{fil}(\mathcal{M}) := \bigoplus_{\lambda'\geq\lambda} \mathcal{M}^{\lambda'}.$$

In this way we obtain an exact tensor functor

$$\mathrm{fil}\colon \mathrm{Gr}^\Gamma\mathrm{Bun}_S \to \mathrm{Fil}^\Gamma\mathrm{Bun}_S.$$

We are now able to define what a Γ-filtered fiber functor on $\mathcal{T}$ is (cf. [18, Section 4.1]).

Definition 2.3. Let S be a scheme over k.
i) A Γ-filtered fiber functor of $\mathcal{T}$ over S is an exact tensor functor

$$\omega\colon \mathcal{T} \to \mathrm{Fil}^\Gamma \mathrm{Bun}_S.$$

ii) A splitting of a Γ-filtered fiber functor $\omega\colon \mathcal{T} \to \mathrm{Fil}^\Gamma\mathrm{Bun}_S$ is a Γ-graded fiber functor

$$\gamma\colon \mathcal{T} \to \mathrm{Gr}^\Gamma \mathrm{Bun}_S$$

such that $\omega = \mathrm{fil} \circ \gamma$.

We remark that if a Γ-filtered fiber functor ω admits a splitting then also Γ-filtered fiber functors isomorphic to ω do. Moreover, if ω admits a splitting γ, then $\gamma \cong \mathrm{gr} \circ \omega$.

Clearly, for every morphism $f\colon S' \to S$ of schemes over k the pullback

$$f^*\colon \mathrm{Bun}_S \to \mathrm{Bun}_{S'}$$

of vector bundles induces exact tensor functors

$$f^*\colon \mathrm{Fil}^\Gamma\mathrm{Bun}_S \to \mathrm{Fil}^\Gamma\mathrm{Bun}_{S'}$$

resp.

$$f^*\colon \mathrm{Gr}^\Gamma\mathrm{Bun}_S \to \mathrm{Gr}^\Gamma\mathrm{Bun}_{S'}.$$

Moreover,

$$\mathrm{fil}_{S'} \circ f^* \cong f^* \circ \mathrm{fil}_S\colon \mathrm{Gr}^\Gamma\mathrm{Bun}_S \to \mathrm{Fil}^\Gamma\mathrm{Bun}_{S'}$$

and analogous $\mathrm{gr}_{S'} \circ f^* \cong f^* \circ \mathrm{gr}_S$. Let

$$\underline{\mathrm{Hom}}^\otimes(\mathcal{T}, \mathrm{Fil}^\Gamma\mathrm{Bun})$$

resp.

$$\underline{\mathrm{Hom}}^\otimes(\mathcal{T}, \mathrm{Gr}^\Gamma\mathrm{Bun})$$

be the fibered categories of Γ-filtered resp. Γ-graded fiber functors for $\mathcal{T}$. As in [18, Lemma 4.32, Lemma 3.12.] it follows that both fibered categories are stacks for the fpqc-topology. Moreover, fil resp. gr define morphisms

$$\mathrm{fil}\colon \underline{\mathrm{Hom}}^\otimes(\mathcal{T}, \mathrm{Gr}^\Gamma\mathrm{Bun}) \to \underline{\mathrm{Hom}}^\otimes(\mathcal{T}, \mathrm{Fil}^\Gamma\mathrm{Bun})$$

resp.

$$\mathrm{gr}\colon \underline{\mathrm{Hom}}^\otimes(\mathcal{T}, \mathrm{Fil}^\Gamma\mathrm{Bun}) \to \underline{\mathrm{Hom}}^\otimes(\mathcal{T}, \mathrm{Gr}^\Gamma\mathrm{Bun})$$

of stacks. Finally, for a Γ-filtered fiber functor

$$\omega\colon \mathcal{T} \to \mathrm{Fil}^\Gamma\mathrm{Bun}_S$$

we denote by

$$\mathrm{Spl}(\omega)$$

the functor on S-schemes sending $f\colon S' \to S$ to the set of splittings of the Γ-filtered fiber functor $f^* \circ \omega$. The functor $\mathrm{Spl}(\omega)$ is a sheaf for the fpqc-topology.

The main result of [18, Main theorem 4.14] is the following (in the case we are interested in, i.e., the band of $\mathcal{T}$ is reductive, also cf. [3]).

Theorem 2.4. *Every $\mathbb{Z}$-filtered fiber functor $\omega\colon \mathcal{T} \to \mathrm{Fil}^{\mathbb{Z}}\mathrm{Bun}_S$ is splittable, i.e. admits a splitting, fpqc-locally on S.*

We want to extend Theorem 2.4 to more general groups Γ. As in [18, Definition 4.8.] we make the following definitions.

Definition 2.5. Let
$$\omega\colon \mathcal{T} \to \mathrm{Fil}^{\Gamma}\mathrm{Bun}_S$$
be a Γ-filtered fiber functor on the Tannakian category $\mathcal{T}$. We define the following group sheaves over S.
i) $P(\omega) := \mathrm{Aut}^{\otimes}(\omega)$
ii) $L(\omega) := \mathrm{Aut}^{\otimes}(\mathrm{gr}\circ\omega)$
iii) $U(\omega) := \mathrm{Ker}(P(\omega) \to L(\omega))$

Moreover, we define a canonical filtration
$$U_\lambda(\omega) \subseteq P(\omega),\ 0 \leq \lambda \in \Gamma,$$
as follows. For $\lambda \in \Gamma, \lambda \geq 0$, we set $U_\lambda(\omega)$ as the subgroup of $P(\omega)$ consisting of elements $g \in P(\omega)$ which act trivially on
$$F^{\lambda'}\omega(X)/F^{\lambda'+\lambda}\omega(X)$$
for all $\lambda' \in \Gamma$ and $X \in \mathcal{T}$.

Clearly, for $\lambda \geq \lambda'$ the group
$$U_{\lambda'}(\omega) \subseteq U_\lambda(\omega)$$
is normal. We denote by
$$U_{>\lambda}(\omega)$$
the union in $U_\lambda(\omega)$ of all subgroups $U_{\lambda'}(\omega)$ for $\lambda' > \lambda$ and by
$$\mathrm{gr}^\lambda(U(\omega))$$
the quotient
$$\mathrm{gr}^\lambda(U(\omega)) := U_\lambda(\omega)/U_{>\lambda}(\omega).$$

We will use the following argument to deduce results for Γ-filtered fiber functors where $\Gamma \subseteq \mathbb{Q}$ is a subgroup from results about $\mathbb{Z}$-filtered fiber functors. Let $\Gamma \subseteq \mathbb{Q}$ be a subgroup and let $\omega\colon \mathcal{T} \to \mathrm{Fil}^{\Gamma}\mathrm{Bun}_S$ be a Γ-filtered fiber functor. Define
$$\Gamma_\omega := \{\lambda \in \Gamma \mid \mathrm{gr}^\lambda\omega(X) \neq 0 \text{ for some } X \in \mathcal{T}\}.$$
Then $\Gamma_\omega \subseteq \Gamma$ is a subgroup as gr and ω are tensor functors. If $\mathcal{T}$ has a tensor generator, then (by exactness of gr and ω) the group $\Gamma_\omega \subseteq \Gamma$ is finitely generated by the $\lambda \in \Gamma$ such that $\mathrm{gr}^\lambda(\omega(X)) \neq 0$ for a tensor generator $X \in \mathcal{T}$. Hence, Γ_ω is isomorphic to $\mathbb{Z}$ or $\{0\}$ in this case because $\Gamma_\omega \subseteq \mathbb{Q}$. Moreover, there are fully faithful embeddings
$$\mathrm{Fil}^{\Gamma_\omega}\mathrm{Bun}_S \to \mathrm{Fil}^{\Gamma}\mathrm{Bun}_S$$
resp.
$$\mathrm{Gr}^{\Gamma_\omega}\mathrm{Bun}_S \to \mathrm{Gr}^{\Gamma}\mathrm{Bun}_S$$
and ω factors through a Γ_ω-filtered fiber functor
$$\omega'\colon \mathcal{T} \to \mathrm{Fil}^{\Gamma_\omega}\mathrm{Bun}_S.$$
Moveover,
$$P(\omega) \cong P(\omega'), L(\omega) \cong L(\omega'), U(\omega) \cong U(\omega')$$
etc. Thus all data for ω is defined by a Γ_ω-filtered fiber functor to which we can apply the known results. In particular, we can conclude by Theorem 2.4 that if $\mathcal{T}$ has a tensor generator, then every Γ-filtered fiber functor admits a splitting, fpqc-locally on S. We record the following theorem collecting results about Γ-filtered fiber functors.

To state it let $\pi(\mathcal{T})$ be the fundamental group of $\mathcal{T}$ (cf. [5, Definition 8.13]), an affine group scheme in the Tannakian category $\mathcal{T}$ represented by a Hopf algebra $\mathcal{O}_{\pi(\mathcal{T})}$ in Ind-$\mathcal{T}$. For every fiber functor $\omega\colon \mathcal{T} \to \mathrm{Bun}_S$ it has the property

$$\omega(\pi(\mathcal{T})) \cong \mathrm{Aut}^{\otimes}(\omega)$$

as affine group schemes over S. More precisely, the $\mathcal{O}_S$-Hopf algebra representing $\mathrm{Aut}^{\otimes}(\omega)$ is isomorphic to the Hopf algebra $\omega(\mathcal{O}_{\pi(\mathcal{T})}$. If $\mathcal{T}$ admits a tensor generator we define the Lie algebra

$$\mathrm{Lie}(\pi(\mathcal{T})) := (I/I^2)^{\vee},$$

where $I \subseteq \mathcal{O}_{\pi(\mathcal{T})}$ is the augmentation ideal of $\mathcal{O}_{\pi(\mathcal{T})}$. If $\mathcal{T} = \mathrm{Rep}_k(G)$ for some affine group scheme G over k, then

$$\mathcal{O}_{\pi(\mathcal{T})} = \mathcal{O}_G$$

with G acting on $\mathcal{O}_G$ by conjugation and we see that we recover the usual notion of the Lie algebra of G with its adjoint action.

Theorem 2.6. *Let $\Gamma \subseteq \mathbb{Q}$ be a subgroup. Let $\mathcal{T}$ be a Tannakian category over k and let*

$$\omega\colon \mathcal{T} \to \mathrm{Fil}^{\Gamma}\mathrm{Bun}_S$$

be a Γ-filtered fiber functor. Define

$$G := \mathrm{Aut}^{\otimes}(\mathrm{forg} \circ \omega)$$

as the group scheme over S defined by the usual fiber functor

$$\mathrm{forg} \circ \omega\colon \mathcal{T} \to \mathrm{Fil}^{\Gamma}\mathrm{Bun}_S \to \mathrm{Bun}_S$$

of $\mathcal{T}$ over S. Let $\pi(\mathcal{T})$ be the fundamental group of $\mathcal{T}$. Then
i) For $\lambda \in \Gamma$, $\lambda \geq 0$, the group sheaf $U_\lambda(\omega)$ is representable by a group scheme, affine and faithfully flat over S.
ii) The affine group schemes $U(\omega)$ and $U_\lambda(\omega)$ for $\lambda > 0$ are pro-unipotent and for $\lambda > 0$ the group $\mathrm{gr}^\lambda(U(\omega))$ is abelian and pro-unipotent.
iii) $P(\omega)/U(\omega) \cong L(\omega)$
iv) If $\mathcal{T}$ admits a tensor generator, then for $\lambda \geq 0$

$$\mathrm{Lie}(U_\lambda(\omega)) \cong F^\lambda(\omega(\mathrm{Lie}(\pi(\mathcal{T}))))$$

and

$$\mathrm{Lie}(\mathrm{gr}^\lambda(U(\omega))) \cong \mathrm{gr}^\lambda(\omega(\mathrm{Lie}(\pi(\mathcal{T}))))$$

where $\mathrm{Lie}(\pi(\mathcal{T}))$ is the Lie algebra of the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$.
v) If G is of finite presentation over S (or equivalently, if $\mathcal{T}$ has a tensor generator), then

$$P(\omega), L(\omega), U(\omega), U_\lambda(\omega), \mathrm{gr}^\lambda(U(\omega))$$

for $\lambda \geq 0$ are of finite presentation.
vi) If G is smooth over S, then $P(\omega)$, $U(\omega)$, $U_\lambda(\omega)$ and $\mathrm{gr}^\lambda(U(\omega))$ for $\lambda \geq 0$ are smooth.
vii) If G is reductive over S, then $P(\omega)$ is a parabolic subgroup scheme of G and $U(\omega)$ is its unipotent radical. For $\lambda > 0$ the groups

$$\mathrm{gr}^\lambda(U(\omega))$$

are vector bundles, isomorphic to

$$\mathrm{Lie}(\mathrm{gr}^\lambda(U(\omega))).$$

viii) The sheaf $\mathrm{Spl}(\omega)$ *of splittings of* ω *is an* $U(\omega)$*-torsor over* S *with respect to the fpqc-topology, in particular,* ω *admits a splitting fpqc-locally on* S *and* $\mathrm{Spl}(\omega)$ *is represented by a scheme affine and faithfully flat over* S*.*
ix) If

$$\gamma := \mathrm{gr} \circ \omega \colon \mathcal{T} \to \mathrm{Gr}^{\Gamma}\mathrm{Bun}_S$$

is the associated Γ*-graded fiber functor of* ω*, then for* $\lambda \geq 0$

$$U_\lambda(\omega) \cong U_\lambda(\mathrm{fil} \circ \gamma).$$

Proof. i) If $\mathcal{T}$ admits a tensor generator this follows from [18, Lemma 4.20] and the discussion preceeding this theorem. In the general case, writing $\mathcal{T}$ as a union of sub-Tannakian categories admitting a tensor generator, shows that $U_\lambda(\omega)$ is an inverse limit of schemes, affine and faithfully flat over S and hence itself affine and faithfully flat over S.
ii) The pro-unipotence of $U(\omega)$ and $U_\lambda(\omega)$ for $\lambda > 0$ follows after taking the limit from the case that $\mathcal{T}$ has a tensor generator, say $X \in \mathcal{T}$. Then $U(\omega)$ resp. $U_\lambda(\omega)$ embeds into $\mathrm{GL}(\omega(X))$ as a subgroup of the upper triangular matrices, showing the unipotence. As in [18, Lemma 4.21] it follows that $\mathrm{gr}^\lambda(U(\omega))$ is abelian. As $\mathrm{gr}^\lambda(U(\omega))$ is a quotient of $U_\lambda(\omega)$ it is also pro-unipotent.
iii) This follows as in [18, Lemma 4.23] from viii).
iv) This follows from [16, Proposition IV.2.1.4.1].
v) This follows from [16, Proposition IV.2.1.4.1].
vi) This follows from [18, Lemma 4.20].
vii) This follows from [18, Lemma 4.40] and [18, Proposition 4.25].
viii) By Theorem 2.4, and the discussion preceeding this theorem, the statement is known if $\mathcal{T}$ admits a tensor generator. Moreover, the sheaf $\mathrm{Spl}(\omega)$ is represented by a scheme, affine and faithfully flat over S by [18, Lemma 4.20]. The general case follows by the usual limit argument.
ix) For $\lambda \geq 0$ there exists a canonical map

$$U_\lambda(\omega) \to U_\lambda(\mathrm{fil} \circ \gamma)$$

and thus the statement that this is an isomorphism is fpqc-local on S and we may assume that

$$\omega \cong \mathrm{fil} \circ \gamma$$

is split in which case the statement is trivial. □

3. Admissible fiber functors over the Fargues-Fontaine curve

Fix a local field E with residue field $\mathbb{F}_q$, i.e., either $E \cong \mathbb{F}_q((t))$ or E is a finite extension of $\mathbb{Q}_p$. Moreover, let $F/\bar{\mathbb{F}}_q$ be a perfectoid algebraically closed extension of an algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$. Let $X_{E,F}$ be the schematic Fargues-Fontaine curve associated with E and F (cf. [6, Chapter 1]). In the notation we will omit the field F and we will just write X_E for $X_{E,F}$. Our results will not depend on the field F (assuming that it is algebraically closed).

Let $\mathcal{T}$ be a Tannakian category over E. The basic example will be the category of representations $\mathrm{Rep}_E(G)$ of an affine group scheme G over E. We record the following lemma from the Tannakian formalism (cf. [5]).

Lemma 3.1. *Let k be a field and let G/k be an affine group scheme. Then for every scheme S over k the groupoid of fiber functors*

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_S$$

is equivalent to the groupoid of G-torsors over S for the fpqc-topology. If G is locally of finite presentation, the same holds with "fpqc" replaced by "fppf". If G is locally smooth, the same holds with "fpqc" replaced by "étale".

The following lemma is very important for our proof of the classification of the later defined admissible fiber functors

$$\omega\colon \mathcal{T} \to \mathrm{Bun}_{X_E}$$

of $\mathcal{T}$ over X_E. First we recall the Harder-Narasimhan filtration for a vector bundle $\mathcal{E}$ on X_E (cf. [8, Section 8.2.4.] resp. [12, Corollary 11.7.]). Namely, there exists a canonical $\mathbb{Q}$-filtration $\mathrm{HN}^\bullet(\mathcal{E})$ on $\mathcal{E} \in \mathrm{Bun}_{X_E}$ by subbundles

$$\mathrm{HN}^\lambda(\mathcal{E}), \lambda \in \mathbb{Q},$$

such that for $\lambda \in \mathbb{Q}$ the graded piece

$$\mathrm{gr}^\lambda(\mathcal{E}) = \mathrm{HN}^\lambda(\mathcal{E})/\mathrm{HN}^{>\lambda}(\mathrm{E})$$

is semistable of slope λ.

Lemma 3.2. *Sending a vector bundle $\mathcal{E}$ to the $\mathbb{Q}$-filtered vector bundle*

$$(\mathcal{E}, \mathrm{HN}(\mathcal{E}))$$

defines a fully faithful tensor functor

$$\mathrm{HN}\colon \mathrm{Bun}_{X_E} \to \mathrm{Fil}^{\mathbb{Q}}\mathrm{Bun}_{X_E}.$$

Proof. The Harder-Narasimhan filtration is preserved by every morphism of vector bundles. Hence the functor HN is well-defined and fully faithful. The classification of vector bundles on X_E (cf. [8, Chapter 8], [12] resp. Theorem 3.5) implies that the tensor product of two semistable vector bundles of slope λ resp. μ is again semistable of slope $\lambda + \mu$. This implies that HN is moreover a tensor functor. □

However, note that the functor HN is *not* exact, because semistable vector bundles are successive extensions of line bundles which are possibly of different slopes. Therefore we make the following definition.[3]

Definition 3.3. A fiber functor $\omega\colon \mathcal{T} \to \mathrm{Bun}_{X_E}$ is called admissible if the composition

$$\mathcal{T} \xrightarrow{\omega} \mathrm{Bun}_{X_E} \xrightarrow{\mathrm{HN}} \mathrm{Fil}^{\mathbb{Q}}\mathrm{Bun}_{X_E}$$

is again exact, i.e., a $\mathbb{Q}$-filtered fiber functor.

For example, if $\mathrm{char}(E) = 0$ and $\mathcal{T} \cong \mathrm{Rep}_E(G)$ for a reductive group G over E, then every fiber functor $\omega\colon \mathcal{T} \to \mathrm{Bun}_{X_E}$ is admissible as $\mathcal{T}$ is semisimple in this case.

Admissibility is a convenient notion as it can be checked after base change along a finite field extension E'/E.

[3]In [16] Saavedra-Rivano calls filtered fiber functors admissible if they are fpqc-locally splittable. By [18] this notion is obsolete and thus we think that our terminology is not very confusing. Also our admissible fiber functors are not equipped with a filtration as would be the case in Saavedra-Rivano's notation.

Lemma 3.4. *Let E'/E be a finite field extension and let $\omega\colon \mathcal{T} \to \mathrm{Bun}_{X_E}$ be a fiber functor. Then ω is admissible if and only if the composition*

$$\mathcal{T} \xrightarrow{\omega} \mathrm{Bun}_{X_E} \xrightarrow{f^*} \mathrm{Bun}_{X_{E'}}$$

is admissible where $f\colon X_{E'} \cong X_E \otimes_E E' \to X_E$ is the canonical morphism.

Proof. Because f is faithfully flat it suffices to show that the diagram

$$\begin{array}{ccc} \mathrm{Bun}_{X_E} & \xrightarrow{\mathrm{HN}} & \mathrm{Fil}^{\mathbb{Q}}\mathrm{Bun}_{X_E} \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Bun}_{X_{E'}} & \xrightarrow{\mathrm{HN}} & \mathrm{Fil}^{\mathbb{Q}}\mathrm{Bun}_{X_{E'}} \end{array}$$

commutes, i.e., if $\mathcal{E}$ is a vector bundle on X_E, then the pullback of the Harder-Narasimhan filtration of $\mathcal{E}$ on X_E is the Harder-Narasimhan filtration of $f^*\mathcal{E}$ on $X_{E'}$. We assume first that E' over E is separable. Let $\mathcal{E}$ be a vector bundle on X_E. If μ is the slope of $\mathcal{E}$, then $[E' : E]\mu$ is the slope of $f^*\mathcal{E}$ because f^* preserves ranks, but multiplies degrees by $[E' : E]$. In particular, if $f^*\mathcal{E}$ is semistable, then $\mathcal{E}$ is semistable, because for a destabilizing subbundle $\mathcal{F}$ of $\mathcal{E}$ the pullback $f^*\mathcal{F}$ would again be destabilizing. This implies that it suffices to prove the claim in the case that E'/E is Galois. Then the Harder-Narasimhan filtration of $f^*\mathcal{E}$ must be stable under the Galois action and hence descends to X_E. Moreover, this descended filtration must be the Harder-Narasimhan filtration of $\mathcal{E}$ as its graded pieces are semistable of strictly decreasing slopes because this holds after pullback to $X_{E'}$. In particular, the pullback of the Harder-Narasimhan filtration of $\mathcal{E}$ is the Harder-Narasimhan filtration of $f^*\mathcal{E}$.

Now assume that E'/E is purely inseparable. Again it is clear that a vector bundle $\mathcal{E} \in \mathrm{Bun}_{X_E}$ is semistable if $f^*\mathcal{E}$ is semistable. Conversely, let $\mathcal{E} \in \mathrm{Bun}_{X_E}$ be a semistable vector bundle. We want to show that $f^*\mathcal{E}$ is semistable. For this we may assume that $\mathcal{E}$ is simple. In this case there exists $d \in \mathbb{Z}$ and $d \geq 0$, s.t.

$$\mathcal{E} \cong \pi_*\mathcal{O}_{X_{E_h}}(d)$$

where E_h is an unramified extension of E of degree h and

$$\pi\colon X_{E_h} \to X_E$$

is the canonical morphism. As E'/E is purely inseparable, $E' \otimes_E E_h$ is isomorphic to the unramified extension E'_h of E' of degree h. In particular,

$$f^*\pi_*(\mathcal{O}_{X_{E_h}}(d)) \cong \pi'_*(\mathcal{O}_{X_{E'_h}}(hd)),$$

where $\pi'\colon X_{E'_h} \to X_{E'}$ is the projection, is again semistable. □

Let

$$L := \widehat{E^{\mathrm{un}}}$$

be the completion of the maximal unramified extension of E and let

$$\varphi - \mathrm{Mod}_L$$

be the category of isocrystals over L. As we have fixed an embedding $\bar{\mathbb{F}}_q \subseteq F$ we obtain a canonical exact tensor functor

$$\mathcal{E}(-)\colon \varphi - \mathrm{Mod}_L \to \mathrm{Bun}_{X_E}$$

(cf. [8, Section 8.2.3.] if $E/\mathbb{Q}_p$ resp. [12, Chapter 8] if $E = \mathbb{F}_q((t))$).

We recall properties of this functor.

Theorem 3.5. *The functor*

$$\mathcal{E}(-)\colon \varphi - \mathrm{Mod}_L \to \mathrm{Bun}_{X_E}$$

is an exact faithful tensor functor inducing a bijection on isomorphism classes. Moreover, for $\lambda \in \mathbb{Q}$ it induces an equivalence between semistable isocrystals of slope $-\lambda$ with semistable vector bundles of slope λ.

Proof. Cf. [8, Section 8.2.4.] resp. [12, Proposition 8.6., Theorem 11.1.]. □

One has to be a bit careful when comparing the slope of some isocrystal

$$D \in \varphi - \mathrm{Mod}_L$$

with the slope of the vector bundle $\mathcal{E}(D)$. If $\pi \in E$ is a uniformizer then for $n \in \mathbb{Z}$ the isocrystal

$$D := (L, \pi^n \varphi_L)$$

is of slope n and sent to the line bundle $\mathcal{O}_{X_E}(-n)$ and not to $\mathcal{O}_{X_E}(n)$. This explains the appearance of the sign in Theorem 3.5.

Theorem 3.5 implies that the functor

$$\mathrm{gr} \circ \mathrm{HN}\colon \mathrm{Bun}_{X_E} \to \mathrm{Gr}^{\mathbb{Q}}\mathrm{Bun}_{X_E}$$

preserves duals and symmetric resp. exterior powers because this is true for the functor sending an isocrystal to its decomposition into isoclinic components.

Moreover, the category $\varphi - \mathrm{Mod}_L$ is canonically $\mathbb{Q}$-graded by decomposing an isocrystal into its isoclinic components. We thus obtain a functor

$$\mathcal{E}_{\mathrm{gr}}(-)\colon \varphi - \mathrm{Mod}_L \to \mathrm{Gr}^{\mathbb{Q}}\mathrm{Bun}_{X_E}.$$

Lemma 3.6. *The functor $\mathcal{E}_{\mathrm{gr}}(-)$ identifies the category $\varphi - \mathrm{Mod}_L$ with the full subcategory of $\mathrm{Gr}^{\mathbb{Q}}\mathrm{Bun}_{X_E}$ consisting of $\mathbb{Q}$-graded vector bundles*

$$\mathcal{E} = \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{E}^{\lambda}$$

such that for $\lambda \in \mathbb{Q}$ the vector bundle $\mathcal{E}^{\lambda}$ is semistable of slope λ.

Proof. This follows from the classification of vector bundles on X_E and their homomorphisms (cf. Theorem 3.5). □

We moreover see that the functor

$$\varphi - \mathrm{Mod}_L \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_{X_E} \xrightarrow{\mathrm{HN}} \mathrm{Fil}^{\mathbb{Q}}\mathrm{Bun}_{X_E} \xrightarrow{\mathrm{gr}} \mathrm{Gr}^{\mathbb{Q}}\mathrm{Bun}_{X_E}$$

is fully faithful and exact as it is isomorphic to the functor $\mathcal{E}_{\mathrm{gr}}(-)$. In the following we will sometimes just write $\mathcal{E}(-)$ instead of $\mathcal{E}_{\mathrm{gr}}(-)$.

We now start to classify admissible fiber functors.

Lemma 3.7. *Let $\omega'\colon \mathcal{T} \to \varphi - \mathrm{Mod}_L$ be an exact tensor functor. Then the fiber functor*

$$\mathcal{E}(-) \circ \omega'\colon \mathcal{T} \to \mathrm{Bun}_{X_E}$$

is admissible.

Proof. It suffices to show that the tensor functor

$$\mathrm{gr} \circ \mathrm{HN} \circ \mathcal{E}(-) \circ \omega'$$

is exact. But as was noted above the functor

$$\mathcal{E}_{\mathrm{gr}}(-) \cong \mathrm{gr} \circ \mathrm{HN} \circ \mathcal{E}(-)$$

is exact, which implies the claim. □

Let $\mathcal{T}$ be a Tannakian category and let

$$\omega \colon \mathcal{T} \to \mathrm{Bun}_S$$

be a fiber functor of $\mathcal{T}$ over a non-empty scheme S. If the group scheme $\mathrm{Aut}^{\otimes}(\omega)$ of tensor automorphisms is smooth over S, then this is true for every fiber functor of $\mathcal{T}$ as fiber functors are fpqc-locally isomorphic. Hence, this property of smoothness is in fact intrinsic to $\mathcal{T}$ (and can be rephrased by saying that the band of the gerbe associated to $\mathcal{T}$ is smooth).

We now prove our main theorem about admissible functors.

Theorem 3.8. *Let $\mathcal{T}$ be a Tannakian category admitting a tensor generator and let*

$$\omega \colon \mathcal{T} \to \mathrm{Bun}_{X_E}$$

be an admissible fiber functor such that $\mathrm{Aut}^{\otimes}(\omega)$ *is represented by a reductive group scheme over X_E. Then ω factors as*

$$\omega \cong \mathcal{E}(-) \circ \omega'$$

for an exact tensor functor

$$\omega' \colon \mathcal{T} \to \varphi - \mathrm{Mod}_L.$$

Proof. We consider the tensor functor

$$\psi := \mathrm{HN} \circ \omega \colon \mathcal{T} \to \mathrm{Fil}^{\mathbb{Q}}\mathrm{Bun}_{X_E}.$$

By assumption this tensor functor is a $\mathbb{Q}$-filtered fiber functor, i.e., it is exact. Therefore it gives rise to the $U(\psi)$-torsor

$$\mathrm{Spl}(\psi)$$

of splittings of ψ (cf. Theorem 2.6). Moreover, the group scheme $U(\psi)$ over X_E is smooth (by Theorem 2.6) as $\mathrm{Aut}^{\otimes}(\omega)$ is assumed to be smooth, unipotent and admits a filtration

$$U(\psi) \supseteq U_\lambda(\psi)$$

for $\lambda \geq 0$. Moreover, for $\lambda > 0$ the associated graded pieces $\mathrm{gr}^\lambda(U(\gamma))$ are vector bundles with

$$\mathrm{Lie}(\mathrm{gr}^\lambda(U(\psi))) \cong \mathrm{gr}^\lambda(\mathrm{HN} \circ \omega(\mathrm{Lie}(\pi(\mathcal{T})))).$$

where $\pi(\mathcal{T})$ is the fundamental group of $\mathcal{T}$ (cf. Theorem 2.6). In particular,

$$\mathrm{gr}^\lambda(U(\psi)) \cong \mathrm{Lie}(\mathrm{gr}^\lambda(U(\psi)))$$

is semistable of slope $\lambda > 0$. As only finitely many $\mathrm{gr}^\lambda(U(\psi))$ are non-zero we can conclude

$$H^1_{ét}(X_E, U(\psi)) = 0$$

as for every semistable vector bundle $\mathcal{E}$ of slope $\lambda \geq 0$ on X_E the cohomology group

$$H^1_{ét}(X_E, \mathcal{E}) = 0$$

vanishes (cf. [8, Chapter 8] resp. [12]). In particular, the $U(\psi)$-torsor $\mathrm{Spl}(\psi)$-torsor is trivial and we see that we can choose a splitting

$$\gamma\colon \mathcal{T} \to \mathrm{Gr}^{\mathbb{Q}}\mathrm{Bun}_{X_E}$$

of ψ, i.e.,

$$\psi = \mathrm{fil} \circ \gamma.$$

By construction for every $V \in \mathcal{T}$ and every $\lambda \in \mathbb{Q}$ the locally free sheaf

$$\mathrm{gr}^{\lambda}(\gamma(V))$$

must be semistable of slope λ. In other words (using Lemma 3.6), the functor

$$\gamma\colon \mathcal{T} \to \mathrm{Gr}^{\mathbb{Q}}\mathrm{Bun}_{X_E}$$

factors as

$$\mathcal{E}_{\mathrm{gr}}(-) \circ \omega' = \gamma$$

for an exact tensor functor

$$\omega'\colon \mathcal{T} \to \varphi - \mathrm{Mod}_L.$$

Forgetting the grading shows $\mathcal{E}(-) \circ \omega' \cong \omega$, which proves the claim. □

Definition 3.9. Let $\mathcal{T}$ be a Tannakian category over E. We define $\mathcal{B}(\mathcal{T})$ as the groupoid of exact tensor functors

$$\omega'\colon \mathcal{T} \to \varphi - \mathrm{Mod}_L$$

and $B(\mathcal{T})$ as the set of isomorphism classes of such.

By [4, Lemma 9.1.4.] (cf. Proposition 5.7) this definition agrees with Kottwitz original definition of the set $B(G)$ as φ-conjugacy classes in $G(L)$ if $\mathcal{T} = \mathrm{Rep}_E(G)$ for a connected reductive group G over E.

Recall that we denote by

$$\underline{\mathrm{Hom}}^{\otimes}(\mathcal{T}, \mathrm{Bun}_S)$$

the category of fiber functors of $\mathcal{T}$ over S. For $S = X_E$ we let

$$\underline{\mathrm{Hom}}^{\otimes}_{\mathrm{adm}}(\mathcal{T}, \mathrm{Bun}_{X_E})$$

be the full subcategory of admissible fiber functors of $\mathcal{T}$ over X_E. From Theorem 3.8 we obtain the following classification of admissible fiber functors.

Theorem 3.10. *The composition of the functors*

$$\mathcal{B}(\mathcal{T}) \to \underline{\mathrm{Hom}}^{\otimes}_{\mathrm{adm}}(\mathcal{T}, \mathrm{Bun}_{X_E}),\ \omega' \mapsto \mathcal{E}(-) \circ \omega'$$

and

$$\underline{\mathrm{Hom}}^{\otimes}_{\mathrm{adm}}(\mathcal{T}, \mathrm{Bun}_{X_E}) \to \mathcal{B}(\mathcal{T}),\ \omega \mapsto \mathrm{gr} \circ \mathrm{HN} \circ \omega$$

is naturally isomorphic to the identity. These functors induce a bijection of $B(\mathcal{T})$ with the set of isomorphism classes in $\underline{\mathrm{Hom}}^{\otimes}_{\mathrm{adm}}(\mathcal{T}, \mathrm{Bun}_{X_E})$.

Proof. The first statement is clear by Lemma 3.6. By Theorem 3.8 the canonical map from $B(\mathcal{T})$ to isomorphism classes of admissible fiber functors is surjective, hence bijective by the first statement. □

Let $V \in \mathrm{Rep}_E(G)$ be a representation. We will need to consider the symmetric tensors $\mathrm{TS}_r(V)$ of V ([15, Chapitre 3]), i.e., $\mathrm{TS}_r(V) \subseteq V^{\otimes r}$ is the set of invariants for the permutation action of S_r on $V^{\otimes r}$. By [15, Théorème IV.1., Proposition IV.5.] the vector space $\mathrm{TS}_r(V)$ has the following universal property: For every homogenous polynomial

$$f\colon V \to k$$

of degree r there exists a unique linear form

$$\tilde{f}\colon \mathrm{TS}_r(V) \to k$$

such that $f(v) = \tilde{f}(v \otimes \ldots \otimes v)$ for all $v \in V$. If f is G-invariant, then $\tilde{f}$ will be G-invariant as well.

From a different view point the module $\mathrm{TS}_r(V)$ is isomorphic to the r-the divided power $\Gamma_r(V)$ of V ([15, Proposition IV.5.]). It is clear from the definition that the module of symmetric tensors $\mathrm{TS}_r(V^\vee)$ of the dual $V^\vee$ of V is canonically isomorphic (as a G-representation) to the dual of the r-th symmetric power $\mathrm{Sym}^r(V)$ of V.

Using these symmetric tensors we obtain the following result, generalizing the main theorem of [7] if $E = \mathbb{F}_q((t))$.

Theorem 3.11. *Let G be a reductive group over E. Then there is a canonical bijection*

$$B(G) \cong H^1_{ét}(X_E, G).$$

In other words, every fiber functor

$$\omega\colon \mathrm{Rep}_E(G) \to \mathrm{Bun}_{X_E}$$

is admissible.

Proof. By Theorem 3.10 it suffices to prove the second statement that every fiber functor

$$\omega\colon \mathrm{Rep}_E(G) \to \mathrm{Bun}_{X_E}$$

is automatically admissible because the isomorphism classes of fiber functors of $\mathrm{Rep}_E(G)$ over X_E are in bijection with G-torsors (for the étale topology) over X_E (cf. Lemma 3.1). Let

$$\omega\colon \mathrm{Rep}_E(G) \to \mathrm{Bun}_{X_E}$$

be a fiber functor. We want to show that the composite

$$\mathrm{HN} \circ \omega\colon \mathrm{Rep}_E(G) \to \mathrm{Fil}^{\mathbb{Q}}\mathrm{Bun}_{X_E}$$

is still exact. Equivalently, we can check this after taking the associated graded, i.e., we can check that the functor

$$\mathrm{gr} \circ \mathrm{HN} \circ \omega\colon \mathrm{Rep}_E(G) \to \mathrm{Gr}^{\mathbb{Q}}\mathrm{Bun}_{X_E}$$

is exact. As was remarked after Theorem 3.5 all of the functors $\mathrm{gr} \circ \mathrm{HN}$ and ω are compatible with duals and symmetric resp. exterior powers. Moreover, they preserve modules of symmetric tensors, i.e., S_r-invariants in the r-th tensor powers. Using this a slightly modified proof as in [4, Theorem 5.3.1.] works. We fill out the details. Let

$$0 \to V \to V' \to V'' \to 0$$

be an exact sequence in $\mathrm{Rep}_E(G)$. Set

$$\omega' := \mathrm{gr} \circ \mathrm{HN} \circ \omega.$$

It suffices to prove that the morphism

$$\omega'(\alpha)\colon \omega'(V) \to \omega'(V')$$

is injective. Indeed, dualizing the sequence $0 \to V \to V' \to V'' \to 0$ shows that then the morphism

$$\omega'(V') \to \omega'(V'')$$

has a torsion cokernel. But this morphism is a direct sum of morphisms between semistable vector bundles of the same slope, which implies that this morphism is already surjective. Hence the morphism

$$\omega'(V') \to \omega'(V'')$$

is surjective, which then implies that the sequence

$$0 \to \omega'(V) \to \omega'(V') \to \omega'(V'') \to 0$$

is exact, because

$$\mathrm{rk}(\omega'(V')) = \mathrm{rk}(\omega'(V')) + \mathrm{rk}(\omega'(V'')).$$

Hence, we are left with proving that the morphism

$$\omega'(\alpha)\colon \omega'(V) \to \omega'(V')$$

is injective. But $\omega'(\alpha)$ is injective if and only this is true after taking the exterior power $\Lambda^{\dim V}$. As the functors $\mathrm{gr} \circ \mathrm{HN}$ and ω preserve exterior powers this reduces the claim to the case that V is of dimension one. Tensoring with the dual of V reduces to the case that V is a trivial G-representation. It suffices to proof that $\omega'(\alpha)$ is non-zero. By Haboush's theorem (cf. [11]) there exists some $r \geq 0$ and a G-equivariant homogenous polynomial

$$f\colon V' \to k$$

such that $f_{|V} \neq 0$. Using the universal property of $\mathrm{TS}_r(V')$ we obtain a G-equivariant linear form

$$\tilde{f}\colon \mathrm{TS}_r(V') \to k$$

such that $\tilde{f}$ is non-zero when restricted to $V \cong \mathrm{TS}_r(V)$. In particular, the morphism

$$\mathrm{TS}_r(V) \to \mathrm{TS}_r(V')$$

is split. This implies, using the compatibility of ω' with symmetric tensors, that the morphism

$$\mathrm{TS}_r(\omega'(V)) \to \mathrm{TS}_r(\omega'(V'))$$

is split as well. In particular,

$$\omega'(\alpha)\colon \omega'(V) \to \omega'(V')$$

is non-zero. This finishes the proof. □

We want to explain why our proof differs from the one in [4, Theorem 5.3.1]. In fact the proof there is wrong and we give a counter example to their argument. However, using the above argument with symmetric tensors instead of symmetric powers, the proof in [4, Theorem 5.3.1.] can be fixed. The basic problem is that in positive characteristic the dual of a symmetric power $\mathrm{Sym}^r(V)$ is not canonically isomorphic to the r-th symmetric power $\mathrm{Sym}^r(V^\vee)$ of the dual $V^\vee$ but to the r-th module $\mathrm{TS}_r(V^\vee)$ of symmetric tensors of $V^\vee$. As was remarked before Theorem 3.11 the module $\mathrm{TS}_r(V)$ of symmetric tensors is canonically isomorphic to the r-th divided power $\Gamma_r(V)$ of V. We will rather speak about divided powers in the following.

To give the counter example we assume that E is some field of characteristic 2. Let $G := \mathrm{SL}_2$ and take $V := E^2$ as the standard representation of G with standard basis $x, y \in V$. Note that V is a selfdual representation using the paring

$$V \times V \to E,\ ((x_1, y_1), (x_2, y_2)) \mapsto x_1y_2 + y_1x_2.$$

Let $W := \Gamma_2(V)$ be the second divided power of V. Then there is a short (non-split) exact sequence

$$0 \to V_1 \to W \to \mathrm{Fr}^*(V) \to 0,$$

where V_1 is spanned by the element $x^{[1]}y^{[1]} \in W$ and where $\mathrm{Fr}^*(V)$ denotes the Frobenius twist of V. Note that G acts trivially on V_1. We claim that there does not exist some $r \geq 1$ such that the morphism $V_1 \cong \mathrm{Sym}^r(V_1) \to \mathrm{Sym}^r(W)$ is split. In fact, we prove that for every $r \geq 1$ the restriction morphism

$$(\mathrm{Sym}^r(W)^\vee)^G \to \mathrm{Sym}^r(V_1)^\vee$$

from G-invariants in $\mathrm{Sym}^r(W)^\vee$ to $\mathrm{Sym}^r(V_1)^\vee$ is zero. Using selfduality of V and the fact that symmetric and divided powers are dual to each other it suffices to show that the morphism

$$(\mathrm{Sym}^r(W)^\vee)^G \cong (\Gamma_r(\mathrm{Sym}^2(V)))^G \to \Gamma_r(V_2)$$

is zero where V_2 denotes the quotient $\mathrm{Sym}^2(V)/\mathrm{Fr}^*(V)$ (concretely

$$V_2 = \langle x^2, y^2, xy\rangle / \langle x^2, y^2\rangle).$$

Set $A := x^2, B := y^2, C := xy \in \mathrm{Sym}^2(V)$. A general matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = \mathrm{SL}_2$$

maps these elements to

$$\begin{array}{rcc} A & \mapsto & a^2A + c^2B \\ B & \mapsto & b^2A + d^2B \\ C & \mapsto & C + abA + cdB. \end{array}$$

It suffices to show that there does not exist a G-invariant element $f \in \Gamma_r(\mathrm{Sym}^2(V))$ of the form

$$f = C^{[r]} + g$$

with $g \in \Gamma_r(\mathrm{Sym}^2(V))$ only involving divided power monomials $A^{[k]}B^{[l]}C^{[h]}$ with $h < r$. Namely, elements like g span the kernel of the restriction map to V_2. Let $T \subseteq G$ be the standard torus. The element

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in T$$

maps $A \mapsto \lambda^2 A$, $B \mapsto \lambda^{-2}B$ and $C \mapsto C$. Therefore we can conclude that every G-invariant element $f \in \Gamma_r(\mathrm{Sym}^r(V))$ must be of the form

$$f = \sum_{k+2l=r} a_{l,k} A^{[l]}B^{[l]}C^{[k]},$$

i.e. in each divided power monomial occuring in f the divided powers of A match the divided powers of B. Now apply the element

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

to f. This yields

$$f = \gamma f = \sum_{k+2l=r} a_{l,k} A^{[l]}(B+A)^{[l]}(C+A)^{[k]}$$

by G-invariance of f. Now we collect the coefficient of $A^{[r]}$ using the rules for calculating with divided powers. Namely, this coefficient is given by

$$\kappa := \sum_{k+2l=r} a_{l,k}((l,l,k))$$

where $((l,l,k)) := \frac{(l+l+k)!}{l!l!k!}$. By T-equivariance of f we know that $\kappa = 0$. We claim moreover that

$$\kappa = a_{0,r}$$

which implies $a_{0,r} = 0$, i.e., our claim about f. It suffices to see that if $k + 2l = r$ the number

$$((l,l,k))$$

is divisible by 2 if $k \neq r$ as we are over a field of characteristic 2. But

$$((l,l,k)) = \binom{r}{2l}\binom{2l}{l}$$

and $\binom{2l}{l}$ is divisible by 2 if $l \neq 0$ by Pascal's identity

$$\binom{2l}{l} = \binom{2l-1}{l} + \binom{2l-1}{l-1} = 2\binom{2l-1}{l}.$$

4. Applications

Theorem 3.11 has applications to local class field theory and to the étale and flat cohomology of X_E with finite coefficients (cf. [7, Section 3]). In this section we present these applications to handle, similarly to [7] for $E/\mathbb{Q}_p$, the case $E = \mathbb{F}_q((t))$. However, the methods for $E = \mathbb{F}_q((t))$ are very similar to the one for $E/\mathbb{Q}_p$ and we present the statements uniformly for both cases.

We keep the notations from the last section. Moreover, we denote by $\bar{E}$ a separable closure of E. First we recall the calculation of the étale fundamental group $\pi_1^{\acute{e}t}(X_E, \bar{x})$ of X_E.

Theorem 4.1. *For every geometric point $\bar{x}$ of X_E the canonical morphism*

$$\pi_1^{\acute{e}t}(X_E, \bar{x}) \to \mathrm{Gal}(\bar{E}/E)$$

is an isomorphism.

Proof. This is proven in [8, Théorème 8.6.1.] for $E/\mathbb{Q}_p$, but the same proof applies to $E = \mathbb{F}_q((t))$ using Theorem 3.5. □

Let $\mathrm{Br}(X_E)$ be the Brauer group of X_E. Because X_E is noetherian of Krull dimension 1 [10, Corollaire 2.2.] implies

$$\mathrm{Br}(X_E) \cong H^2_{\acute{e}t}(X_E, \mathbb{G}_m).$$

Theorem 4.2. *We have*

$$\mathrm{Br}(X_E) = H^2_{\acute{e}t}(X_E, \mathbb{G}_m) = 0.$$

Proof. By definition, every element in $\mathrm{Br}(X_E)$ is in the image of

$$H^1_{\acute{e}t}(X_E, \mathrm{PGL}_n) \to H^2_{\acute{e}t}(X_E, \mathbb{G}_m)$$

for some $n \geq 0$. By Theorem 3.11 there is a commutative diagram

$$\begin{array}{ccc} B(\mathrm{GL}_n) & \longrightarrow & B(\mathrm{PGL}_n) \\ \downarrow \cong & & \downarrow \cong \\ H^1_{\acute{e}t}(X_E, \mathrm{Gl}_n) & \longrightarrow & H^1_{\acute{e}t}(X_E, \mathrm{PGL}_n). \end{array}$$

Moreover, using Proposition 5.7 the top horizontal arrow is trivially surjective. In particular, every element in $H^1_{\acute{e}t}(X_E, \mathrm{PGL}_n)$ maps to 0 in $\mathrm{Br}(X_E)$. Thus,

$$\mathrm{Br}(X_E) = 0.$$

□

Let

$$f \colon X_E \to \mathrm{Spec}(E)$$

be the canonical morphism. As $E' = H^0(X_{E'}, \mathcal{O}_{E'})$ for every finite extension E' of E we have

$$f_*(\mathbb{G}_m) = \mathbb{G}_m.$$

Using the Leray spectral sequence for f and Theorem 4.2 one obtains an exact sequence

$$0 \to H^1_{\acute{e}t}(X_E, \mathbb{G}_m) \to H^0(\mathrm{Gal}(\bar{E}/E), H^1_{\acute{e}t}(X_{\bar{E}}, \mathbb{G}_m)) \to H^2_{\acute{e}t}(\mathrm{Spec}(E), \mathbb{G}_m) \to 0.$$

But

$$H^1_{\acute{e}t}(X_E, \mathbb{G}_m) \cong \mathbb{Z}$$

and

$$H^1_{\acute{e}t}(X_{\bar{E}}, \mathbb{G}_m) \cong \mathbb{Q}$$

with trivial $\mathrm{Gal}(\bar{E}/E)$-action. In particular, we obtain the computation of the Brauer group of E

$$\mathrm{Br}(E) \cong \mathbb{Q}/\mathbb{Z}$$

in a rather complicated way. One should note that in order to deduce this theorem we implicitly used Steinbergs theorem that

$$H^1_{\acute{e}t}(\mathrm{Gal}(\bar{E}/E^{\mathrm{un}}), G) = 1$$

for a (connected) reductive group G over the maximal unramified extension E^{un} of E to deduce the concrete description of $B(G)$ (cf. Proposition 5.7) and therefore the surjectivity of

$$B(\mathrm{GL}_n) \to B(\mathrm{PGL}_n).$$

Theorem 4.3. *Let A be a discrete torsion module for* $\mathrm{Gal}(\bar{E}/E)$ *with $nA = 0$ for some n prime to the characteristic of E. Then for $i \leq 2$ the canonical morphism*

$$H^i(\mathrm{Gal}(\bar{E}/E), A) \to H^i_{\acute{e}t}(X_E, A)$$

is an isomorphism.

Proof. Let $f\colon X_E \to \mathrm{Spec}(E)$ be the canonical morphism. It suffices to prove

$$R^i f_*(\mu_l) = 0$$

for $i \leq 2$ and l prime to $\mathrm{char}(E)$. Because l and $\mathrm{char}(E)$ are coprime the Kummer sequence

$$0 \to \mu_l \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

is an exact sequence of étale sheaves. But

$$H^0_{ét}(X_{\bar{E}}, \mathbb{G}_m) \cong \bar{E}^\times$$

$$H^1_{ét}(X_{\bar{E}}, \mathbb{G}_m) \cong \mathbb{Q}$$

and

$$H^2_{ét}(X_{\bar{E}}, \mathbb{G}_m) \cong 0$$

(cf. Theorem 4.2). This implies the claim. □

Now we discuss the case $E = \mathbb{F}_q((t))$ with p-torsion coefficients using flat cohomology.

Theorem 4.4. *Let D be a finite multiplicative group scheme over E. Then the canonical morphism*

$$H^i_{\mathrm{fl}}(\mathit{Spec}(E), D) \to H^i_{\mathrm{fl}}(X_E, D)$$

is an isomorphism for $i \leq 2$. Moreover, if $E = \mathbb{F}_q((t))$ and $D = \mathbb{F}_p$, then

$$H^i_{\mathrm{fl}}(\mathit{Spec}(E), \mathbb{F}_p) \cong H^i_{\mathrm{fl}}(X_E, \mathbb{F}_p)$$

for $i \geq 1$. In particular,

$$H^i_{\mathrm{fl}}(X_E, \mathbb{F}_p) = 0$$

for $i \geq 2$.

Proof. Let again $f\colon X_E \to \mathrm{Spec}(E)$ be the canonical morphism. As in Theorem 4.3 it suffices to prove for every prime $l \in \mathbb{Z}$

$$R^i f_*(\mu_l) = 0$$

resp.

$$R^i f_*(\mathbb{F}_p) = 0$$

for $1 \leq i \leq 2$ resp. all $i \geq 1$. Let E^{alg} be an algebraic closure of E. It suffices to prove

$$H^i_{\mathrm{fl}}(X_{E^{\mathrm{alg}}}, \mu_l) = 0$$

for $1 \leq i \leq 2$ resp.

$$H^i_{\mathrm{fl}}(X_{E^{\mathrm{alg}}}, \mathbb{F}_p) = 0$$

for $i \geq 1$. For coefficients μ_l the statement follows as in Theorem 4.3 using that for $l = p$ the Kummer sequence is exact for the flat topology. The second statement follows using the Artin-Schreier sequence

$$0 \to \mathbb{F}_p \to \mathcal{O}_{X_{E^{\mathrm{alg}}}} \to \mathcal{O}_{X_{E^{\mathrm{alg}}}} \to 0$$

together with the facts that

$$H^i_{\mathrm{fl}}(X_{E^{\mathrm{alg}}}, \mathcal{O}_{X_{E^{\mathrm{alg}}}}) \cong H^i_{ét}(X_{X_{E^{\mathrm{alg}}}}, \mathcal{O}_{X_{E^{\mathrm{alg}}}}) = 0$$

for $i \geq 1$ and $H^0(X_{E^{\mathrm{alg}}}, \mathcal{O}_{X_{E^{\mathrm{alg}}}}) \cong E^{\mathrm{alg}}$. □

In particular, we obtain that

$$H^2_{\mathrm{fl}}(X_E, \mu_{p^r}) \cong H^2_{\mathrm{fl}}(\mathrm{Spec}(E), \mu_{p^r}) \cong \frac{1}{p^r}\mathbb{Z}/Z$$

is canonically isomorphic to the p^r-torsion in the Brauer group $\mathrm{Br}(X_E)$ of E.

Finally, we record the calculation of $H^2_{\acute{e}t}(X_E, T)$ for an arbitrary torus T over E (cf. [7, Théoréme 2.7.]).

Theorem 4.5. *Let T be a torus over E. Then*

$$H^2_{\acute{e}t}(X_E, T) = H^2_{\mathrm{fl}}(X_E, T) = 0.$$

Proof. By [9, Théoréme (11.7)] étale and flat cohomology for tori agree because tori are smooth. Let E' be a finite extension of E splitting T. Then we obtain an exact sequence

$$0 \to T \to \mathrm{Res}_{E'/E} T_{E'} \to T' \to 0$$

of tori, where $\mathrm{Res}_{E'/E} T$ denotes the Weil restriction of $T_{E'}$ to E. By Theorem 3.11 and Theorem 4.2 we obtain a commutative diagram

$$\begin{array}{ccccccc} H^1_{\acute{e}t}(X_{E'}, T_{E'}) & \longrightarrow & H^1_{\acute{e}t}(X_E, T') & \longrightarrow & H^2_{\acute{e}t}(X_E, T) & \longrightarrow & H^2_{\acute{e}t}(X_{E'}, T_{E'}) = 0 \\ \downarrow \cong & & \downarrow \cong & & & & \\ B(\mathrm{Res}_{E'/E} T_{E'}) & \twoheadrightarrow & B(T'), & & & & \end{array}$$

where the lower horizontal arrow is surjective by Steinberg's theorem and Proposition 5.7. This implies that

$$H^2_{\acute{e}t}(X_E, T) = 0$$

as desired. □

Theorem 4.5 is related to Tate-Nakayama duality for tori (cf. [17]). The proétale covering $X_{\bar{E}} \to X_E$ yields a spectral sequence

$$E_2^{ij} = H^i(\mathrm{Gal}(\bar{E}/E), H^j_{\acute{e}t}(X_{\bar{E}}, T_{\bar{E}})) \Rightarrow H^{i+j}_{\acute{e}t}(X_E, T).$$

But

$$H^1_{\acute{e}t}(X_{\bar{E}}, T_{\bar{E}}) \cong X_*(T)_{\mathbb{Q}}$$

and

$$H^2_{\acute{e}t}(X_{\bar{E}}, T_{\bar{E}}) \cong 0,$$

which implies by Theorem 4.5 that there is an exact sequence

$$H^1_{\acute{e}t}(X_E, T) \to H^0(\mathrm{Gal}(\bar{E}/E), X_*(T)_{\mathbb{Q}}) \to H^2(\mathrm{Gal}(\bar{E}/E), T(\bar{E})) \to 0$$

where

$$H^1_{\acute{e}t}(X_E, T) \cong B(T) \cong X_*(T)_\Gamma$$

with $\Gamma = \mathrm{Gal}(\bar{E}/E)$ (cf. [14]).

5. Classification of reductive group schemes

We continue with the notations from the previous two sections. We now want to classify reductive group schemes over the Fargues-Fontaine curve. First we recall the definition of an affine group scheme $\mathbb{G}$ over $\varphi - \mathrm{Mod}_L$ (cf. [4, Definition 9.1.8.]) as they provide the key examples of group schemes over X_E.

Definition 5.1. *An affine group scheme $\mathbb{G}$ over $\varphi - \mathrm{Mod}_L$ is a Hopf algebra object $\mathcal{O}_{\mathbb{G}}$ in the category $\mathrm{Ind} - (\varphi - \mathrm{Mod}_L)$ of Ind-objects of $\varphi - \mathrm{Mod}_L$.*

In other words, an affine group scheme $\mathbb{G}$ over $\varphi - \mathrm{Mod}_L$ consists of an affine group scheme G over L and an isomorphism

$$\varphi_{\mathbb{G}} \colon \varphi_L^* G \cong G$$

such that the Hopf algebra $\mathcal{O}_G$ of G over L is the increasing union of $\varphi_{\mathbb{G}}$-stable subspaces. Therefore we recover [4, Definition 9.1.8.]. An affine group scheme $\mathbb{G}$ over $\varphi - \mathrm{Mod}_L$ is called reductive if the corresponding group scheme G over L is reductive.

If G is an affine group scheme over E, then G gives natural rise to an affine group scheme $\mathbb{G}$ over $\varphi - \mathrm{Mod}_L$ by considering the base change G_L of G to L with its canonical isomorphism $\varphi_L^*(G_L) \cong G_L$. Equivalently, the Hopf algebra underlying $\mathbb{G}$ is given by $\mathcal{O}_G \otimes_E L$ with Frobenius acting on L.

Let $\mathbb{G}$ be an affine group scheme over $\varphi - \mathrm{Mod}_L$. Applying the tensor functor (cf. Theorem 3.5)

$$\mathcal{E}(-) \colon \varphi - \mathrm{Mod}_L \to \mathrm{Bun}_{X_E}$$

to the Hopfalgebra $\mathcal{O}_{\mathbb{G}}$ yields a Hopf algebra

$$\mathcal{O}_{\mathcal{G}} := \mathcal{E}(\mathcal{O}_{\mathbb{G}})$$

over $\mathcal{O}_{X_E}$. Taking the relative Spec of this Hopf algebra

$$\mathcal{G} := \underline{\mathrm{Spec}}(\mathcal{O}_{\mathcal{G}})$$

defines a flat group scheme over X_E. We call $\mathcal{G}$ the group scheme (over X_E) associated with $\mathbb{G}$ and write

$$\mathcal{G} = \mathcal{E}(\mathbb{G})$$

if we want to make this more precise. If $\mathbb{G}$ is reductive, then also $\mathcal{G}$ is reductive (over X_E) as this can be tested fiberwise and then over a large field extension of E.

We record the following general lemma.

Lemma 5.2. *Let k be a field, let G be an affine group scheme over k, let G^{ad} be its adjoint quotient and let S be a k-scheme. Let $\mathcal{Q}$ be a G^{ad}-torsor over S with corresponding inner form $\mathcal{G}_{\mathrm{Q}}$ of G over S. Let*

$$\omega_{\mathcal{Q}} \colon \mathrm{Rep}_k(G^{\mathrm{ad}}) \to \mathrm{Bun}_S$$

be the fiber functor of $\mathrm{Rep}_k(G^{\mathrm{ad}})$ over S associated with $\mathcal{Q}$ (cf. Lemma 3.1). Then

$$\omega_{\mathcal{Q}}(\mathcal{O}_G) \cong \mathcal{O}_{\mathcal{G}_{\mathcal{Q}}}$$

where $\mathcal{O}_G$ is considered as a Hopf algebra in the category of Ind-objects in $\mathrm{Rep}_k(G^{\mathrm{ad}})$ via the adjoint action of G^{ad} and where $\mathcal{O}_{\mathcal{G}_{\mathcal{Q}}}$ denotes the $\mathcal{O}_S$-Hopf algebra of $\mathcal{G}_{\mathcal{Q}}$.

Proof. The fiber functor $\omega_{\mathcal{Q}}$ of $\mathrm{Rep}_k(G^{\mathrm{ad}})$ is given by

$$\omega_{\mathcal{Q}} \colon \mathrm{Rep}_k(G^{\mathrm{ad}}) \to \mathrm{Bun}_S,\ V \mapsto \mathcal{Q} \times^{G^{\mathrm{ad}}} (V \otimes_k \mathcal{O}_S).$$

Moreover, the inner form $\mathcal{G}_{\mathcal{Q}}$ of G is by definition given by the group scheme

$$\mathcal{G}_{\mathcal{Q}} = \mathcal{Q} \times^{G^{\mathrm{ad}}} G$$

over S. Equivalently, this twisting can be done on the Hopf algebra of G. This shows

$$\omega_{\mathcal{Q}}(\mathcal{O}_G) \cong \mathcal{Q} \times^{G^{\mathrm{ad}}} \mathcal{O}_G \cong \mathcal{O}_{\mathcal{G}_{\mathcal{Q}}}$$

and the proof is finished. □

We can now prove the following classification of reductive group schemes over the Fargues-Fontaine curve.

Theorem 5.3. *Let $\mathcal{G}$ be a reductive group scheme over the Fargues-Fontaine curve X_E. Then there exists a reductive group scheme $\mathbb{G}$ over $\varphi - \mathrm{Mod}_L$, unique up to isomorphism, such that $\mathcal{G}$ is isomorphic to the group scheme $\mathcal{E}(\mathbb{G})$ associated with $\mathbb{G}$. There exists moreover a quasi-split group G' over E such that $\mathcal{G}$ is an inner form (over X_E) of G'.*

Proof. Let G_0 be the split reductive group over E such that $\mathcal{G}$ is a form of G_0. Consider the canonical exact sequence

$$1 \to G_0^{\mathrm{ad}} \to \mathrm{Aut}(G_0) \to \mathrm{Out}(G_0) \to 1$$

of group schemes over X_E. It gives rise to an exact sequence of pointed sets

$$H^1_{\acute{e}t}(X_E, G_0^{\mathrm{ad}}) \to H^1_{\acute{e}t}(X_E, \mathrm{Aut}(G_0)) \xrightarrow{\delta} H^1_{\acute{e}t}(X_E, \mathrm{Out}(G_0)).$$

But

$$H^1(\mathrm{Gal}(\bar{E}/E), \mathrm{Out}(G_0)) \cong H^1_{\acute{e}t}(X_E, \mathrm{Out}(G_0))$$

because $\mathrm{Out}(G_0)$ is constant and

$$\pi_1^{\acute{e}t}(X_E) \cong \mathrm{Gal}(\bar{E}/E)$$

by Theorem 4.1. The choice of a pinning of G_0 defines a splitting of δ and the image of

$$H^1(\mathrm{Gal}(\bar{E}/E), \mathrm{Out}(G_0)) \cong H^1_{\acute{e}t}(X_E, \mathrm{Out}(G_0))$$

under this splitting consists precisely of the classes of constant reductive group schemes

$$G' \times_{\mathrm{Spec}(E)} X_E$$

for G' a quasi-split form of G_0 over E. Moreover, the elements in a fiber of δ are all inner forms of each other. In particular, we can see that $\mathcal{G}$ is an inner form (over X_E) of a quasi-split form G' (over E) of G_0.

Let $\mathcal{Q}$ be the $(G')^{\mathrm{ad}}$-torsor over X_E and let

$$\omega_{\mathcal{Q}} \colon \mathrm{Rep}_E((G')^{\mathrm{ad}}) \to \mathrm{Bun}_{X_E}$$

be the corresponding fiber functor. By Lemma 5.2 we obtain

$$\mathcal{O}_{\mathcal{G}} \cong \omega_{\mathcal{Q}}(\mathcal{O}_{G'})$$

where $\mathcal{O}_{G'}$ is considered as a $(G')^{\mathrm{ad}}$-representation via the adjoint action. By Theorem 3.11 we know that the fiber functor $\omega_{\mathcal{Q}}$ is admissible and we can apply theorem Theorem 3.8. This yields an exact tensor functor

$$\omega' \colon \mathrm{Rep}_E((G')^{\mathrm{ad}}) \to \varphi - \mathrm{Mod}_L$$

such that

$$\omega_{\mathcal{Q}} \cong \mathcal{E}(-) \circ \omega'.$$

In particular, we see that $\mathcal{G}$ is isomorphic to the group scheme associated with the group scheme $\mathbb{G}$ given by the Hopf algebra

$$\omega'(\mathcal{O}_{G'})$$

over $\varphi - \mathrm{Mod}_L$. Moreover, as $\mathcal{G}$ is reductive, $\mathbb{G}$ is reductive. As

$$\mathrm{gr} \circ \mathrm{HN} \circ \mathcal{E}(-) \cong \mathrm{Id}_{\varphi - \mathrm{Mod}_L}$$

we see that $\mathbb{G}$ is determined, up to isomorphism, by $\mathcal{G}$. □

In short, the functor from reductive group schemes over $\varphi - \mathrm{Mod}_L$ to reductive group schemes over X_E is faithful and induces a bijection on isomorphism classes. But it is not an equivalence, for $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$ and $\mathcal{G} := \mathrm{GL}(\mathcal{E})$ the global sections $\mathcal{G}(X_E)$ depend on the chosen algebraically closed perfectoid field $F/\mathbb{F}_q$.

We can moreover obtain the following result.

Lemma 5.4. *Let $\tilde{E}/E$ be an algebraic extension of E such that $\tilde{E}$ has cohomological dimension 1 and, if $E = \mathbb{F}_q((t))$, contains the perfection of E. Let $\mathcal{G}$ be a reductive group scheme over X_E. Then there exists a finite subextension $E' \subseteq \tilde{E}$ such that the base change $\mathcal{G}_{X'_E}$ is a pure inner form of a quasi-split group G' over E'.*

Proof. By Theorem 5.3 we already know that $\mathcal{G}$ is an inner form of G' for a quasi-split group G' over E. There exists an exact sequence (in the flat topology)

$$1 \to Z(G') \to G' \to (G')^{\mathrm{ad}} \to 0$$

yielding an exact sequence

$$H^1_{\mathrm{fl}}(X_E, G') \to H^1_{\mathrm{fl}}(X_E, (G')^{\mathrm{ad}}) \to H^2_{\mathrm{fl}}(X_E, Z(G)).$$

Thus it suffices to show that every element in $H^2_{\mathrm{fl}}(X_E, Z(G'))$ maps to zero under a finite extension contained in $\tilde{E}$. Let $T \subseteq G'$ be a maximal torus and consider the exact sequence

$$0 \to Z(G') \to T \to \tilde{T} \to 0.$$

Using the associated long exact sequence and Theorem 4.5 it suffices to prove that

$$\mathrm{Coker}(H^1_{\mathrm{fl}}(X_{\tilde{E}}, T) \to H^1_{\mathrm{fl}}(X_E, \tilde{T})) = 0.$$

As $\tilde{E}$ has cohomological dimension ≤ 1 (and contains the perfection of E if $E = \mathbb{F}_q((t))$) the degrees of finite subextensions E'/E get divisible by arbitrary large integers $n \in \mathbb{Z}$. We can conclude

$$H^1_{\mathrm{fl}}(X_{\tilde{E}}, T) \cong X_*(T)_{\mathbb{Q}}$$

and

$$H^1_{\mathrm{fl}}(X_{\tilde{E}}, \tilde{T}) \cong X_*(\tilde{T})_{\mathbb{Q}}.$$

As $T \to \tilde{T}$ is surjective the morphism

$$X_*(T)_{\mathbb{Q}} \to X_*(\tilde{T})_{\mathbb{Q}}$$

is surjective. This finishes the proof of the lemma. □

For example, looking at the proof shows that one can take $\tilde{E}$ also to be the maximal unramified extension of E. But one can also take $\tilde{E}$ as the composition of a totally ramified extension and the perfection of E, if $E = \mathbb{F}_q((t))$.

We now start to classify torsors under reductive group schemes over X_E.

Definition 5.5. Let $\mathbb{G}$ be a group scheme over $\varphi - \mathrm{Mod}_L$. We define $\mathrm{Rep}_E(\mathbb{G})$ as the category of representations of $\mathbb{G}$ on isocrystals, i.e., as the category of finite dimensional comodules of the Hopf algebra $\mathcal{O}_{\mathbb{G}}$ in the category $\varphi - \mathrm{Mod}_L$.

In other words, an object $V \in \mathrm{Rep}_E(\mathbb{G})$ consists of an isocrystal $V \in \varphi - \mathrm{Mod}_L$ and a coaction $V \to V \otimes_L \mathcal{O}_{\mathbb{G}}$ which is moreover a morphism of Ind-isocrystals. Clearly, the category $\mathrm{Rep}_E(\mathbb{G})$ is Tannakian over E.[4] We denote by

$$\omega'_{\mathrm{can}} \colon \mathrm{Rep}_E(\mathbb{G}) \to \varphi - \mathrm{Mod}_L$$

the canonical exact tensor functor sending a $\mathbb{G}$-representation to its underlying isocrystal. We define

$$\omega_{\mathrm{can}} := \mathcal{E}(-) \circ \omega'_{\mathrm{can}} \colon \mathrm{Rep}_E(\mathbb{G}) \to \mathrm{Bun}_{X_E},$$

and call it the canonical fiber functor of $\mathrm{Rep}_E(\mathbb{G})$ over X_E.

Lemma 5.6. *Let $\mathbb{G}$ be a group scheme over $\varphi - \mathrm{Mod}_L$. Let*

$$\omega \colon \mathrm{Rep}_E(\mathbb{G}) \to \mathrm{Bun}_S$$

be a fiber functor of $\mathrm{Rep}_E(\mathbb{G})$ over S and let

$$\mathcal{G} := \mathrm{Aut}^{\otimes}(\omega)$$

be the corresponding group scheme over S. Let

$$\mathcal{O}_{\mathbb{G}} \in \mathrm{Ind} - \mathrm{Rep}_E(\mathbb{G})$$

be the Hopf algebra underlying $\mathbb{G}$ considered as a representation of $\mathbb{G}$ via the adjoint action. Then there is a natural isomorphism

$$\mathcal{O}_{\mathcal{G}} \cong \omega(\mathcal{O}_{\mathbb{G}})$$

of Hopf algebras. In other words, $\mathcal{O}_{\mathbb{G}}$ with the adjoint action by $\mathbb{G}$ is the fundamental group of $\mathrm{Rep}_E(\mathbb{G})$.

Proof. Let $\mathcal{R}$ be a quasi-coherent $\mathcal{O}_S$-algebra and let

$$f \colon \omega(\mathcal{O}_{\mathbb{G}}) \to \mathcal{R}$$

be a morphism of $\mathcal{O}_S$-algebras. Then for $V \in \mathrm{Rep}_E(\mathbb{G})$ the composition

$$\omega(V) \otimes_{\mathcal{O}_S} \mathcal{R} \to \omega(V) \otimes_{\mathcal{O}_S} \omega(\mathcal{O}_{\mathbb{G}}) \otimes \mathcal{R} \to \omega(V) \otimes_{\mathcal{O}_S} \mathcal{R},$$

where the first morphism is induced by the comultiplication of V, which is a morphism of $\mathbb{G}$-representations if $\mathcal{O}_{\mathbb{G}}$ is equipped with the adjoint action, and the second by f and multiplication in $\mathcal{R}$, is natural in V and a naturally a tensor automorphism, hence defines a $\underline{\mathrm{Spec}}(\mathcal{R})$-valued point of $\mathcal{G}$. Conversely, if

$$\alpha_V \colon \omega(V) \otimes_{\mathcal{O}_S} \mathcal{R} \to \omega(V) \otimes_{\mathcal{O}_S} \mathcal{R}$$

[4] We mention the following possible cause of confusion. If G is a reductive group over E with associated group scheme $\mathbb{G}$ over $\varphi - \mathrm{Mod}_L$, then the category of finite dimensional E-representations $\mathrm{Rep}_E(G)$ of G is the full subcategory of $\mathrm{Rep}_E(\mathbb{G})$ given by representations of $\mathbb{G}$ whose underlying isocrystal is semistable of slope 0.

is a natural tensor automorphism, then evaluating α on the Ind-object $\mathcal{O}_{\mathbb{G}}$ of $\mathrm{Rep}_E(\mathbb{G})$ defines a morphism of $\mathcal{O}_S$-algebras

$$\omega(\mathcal{O}_{\mathbb{G}}) \to \omega(\mathcal{O}_{\mathbb{G}}) \otimes_{\mathcal{O}_S} \mathcal{R} \xrightarrow{\alpha} \omega(\mathcal{O}_{\mathbb{G}}) \otimes_{\mathcal{O}_S} \mathcal{R} \to \mathcal{R},$$

where the first morphism is induced by the unit of $\mathcal{R}$ and the last by the counit of $\mathcal{O}_{\mathbb{G}}$, hence an $\mathcal{R}$-valued point of $\underline{\mathrm{Spec}}(\omega(\mathcal{O}_{\mathbb{G}}))$. One checks that both maps are inverse homomorphisms of groups. Hence, the claim follows. □

Applying this lemma to

$$\omega_{\mathrm{can}} \colon \mathrm{Rep}_E(\mathbb{G}) \to \mathrm{Bun}_{X_E}$$

shows that

$$\mathrm{Aut}^{\otimes}(\omega_{\mathrm{can}}) \cong \mathcal{E}(\mathbb{G})$$

is isomorphic to the group scheme associated with the group scheme $\mathbb{G}$ over $\varphi-\mathrm{Mod}_L$. A similar proof shows moreover, that

$$\mathbb{G} \cong \mathrm{Aut}^{\otimes}(\omega'_{\mathrm{can}})$$

where $\mathrm{Aut}^{\otimes}(\omega'_{\mathrm{can}})$ is the functor sending an algebra object R in $\mathrm{Ind}-(\varphi-\mathrm{Mod}_L)$ to the group of tensor automorphisms of $\omega'_{\mathrm{can}} \otimes R$. In particular, we can conclude that the band of the gerbe of the Tannakian category $\mathrm{Rep}_E(\mathbb{G})$ is reductive if $\mathbb{G}$ is.

We can now give a concrete classification of the category $\mathcal{B}(\mathrm{Rep}_E(\mathbb{G}))$ (cf. Definition 3.9) of a connected group scheme $\mathbb{G}$ over $\varphi-\mathrm{Mod}_L$ (cf. [4, Lemma 9.1.4]). We let $\mathcal{B}(\mathbb{G})$ be the following category: Its objects are elements $b \in \mathbb{G}(L)$ (more precisely, $b \in G(L)$ if G is the group scheme over L underlying $\mathbb{G}$) and the set of morphisms from b' to b are the elements $c \in \mathbb{G}(L)$ such that

$$b = cb'\varphi_{\mathbb{G}}(c)^{-1}$$

where

$$\varphi_{\mathbb{G}} \colon \mathbb{G}(L) \to \mathbb{G}(L)$$

is induced by the given isomorphism $\varphi_{\mathbb{G}}$ of G lying over the Frobenius $\varphi \colon L \to L$ of L.

Given an element $b \in \mathbb{G}(L)$ we can define an exact tensor functor

$$\omega'_b \colon \mathrm{Rep}_E(\mathbb{G}) \to \varphi-\mathrm{Mod}_L$$

by sending a $\mathbb{G}$-representation V to the isocrystal

$$(V, V \xrightarrow{\varphi_V} V \xrightarrow{b} V)$$

where the second morphism denotes the action of $b \in \mathbb{G}(L)$ on V.

Proposition 5.7. *Let $\mathbb{G}$ be a connected affine group scheme over $\varphi-\mathrm{Mod}_L$. Sending $b \in \mathbb{G}(L)$ to ω'_b defines an equivalence of categories*

$$\mathcal{B}(\mathbb{G}) \cong \mathcal{B}(\mathrm{Rep}_E(\mathbb{G})).$$

Proof. Let $\omega'_{\mathrm{can}} \colon \mathrm{Rep}_E(\mathbb{G}) \to \varphi-\mathrm{Mod}_L$ resp.

$$\mathrm{forg} \colon \varphi-\mathrm{Mod}_L \to \mathrm{Vec}_L$$

be the canonical exact tensor functors. If G denotes the underlying connected group scheme of $\mathbb{G}$ over L then by Lemma 5.6

$$\mathrm{Aut}^{\otimes}(\mathrm{forg} \circ \omega'_{\mathrm{can}}) \cong G.$$

Let

$$\omega' \colon \mathrm{Rep}_E(\mathbb{G}) \to \varphi - \mathrm{Mod}_L$$

be an exact tensor functor. By Steinberg's theorem the functors $\mathrm{forg} \circ \omega'$ and $\mathrm{forg} \circ \omega'_{\mathrm{can}}$ are isomorphic. Let

$$\alpha \colon \mathrm{forg} \circ \omega' \to \mathrm{forg} \circ \omega'_{\mathrm{can}}$$

be an L-linear tensor isomorphism. Moreover, both functors come equipped with canonical φ_L-semilinear tensor automorphisms $\sigma', \sigma'_{\mathrm{can}}$. Then

$$\beta := \alpha \circ \sigma' \circ \alpha^{-1} \circ \sigma'_{\mathrm{can}}$$

is an L-linear tensor automorphism of $\mathrm{forg} \circ \omega'_{\mathrm{can}}$. Therefore, by the Tannakian formalism, there exists an element $b \in \mathbb{G}(L)$ such that β is given by the multiplication by b. But then ω' is isomorphic to ω'_b. A similar reasoning shows that the functor $\mathcal{B}(\mathbb{G}) \to \mathcal{B}(\mathrm{Rep}_E(\mathbb{G}))$ is fully faithful. □

Let G/E be a connected affine group scheme and let $\mathbb{G}$ be the connected affine group scheme over $\varphi - \mathrm{Mod}_L$. Using [4, Lemma 9.1.4] (i.e., the same reasoning as in the proof above) we can conclude that there is a canonical chain of equivalences

$$\mathcal{B}(\mathrm{Rep}_E(G)) \cong \mathcal{B}(G) = \mathcal{B}(\mathbb{G}) \cong \mathcal{B}(\mathrm{Rep}_E(\mathbb{G}))$$

where $\mathcal{B}(G)$ is defined exactly as $\mathcal{B}(\mathbb{G})$.

We will need the following lemma. If $\mathcal{T}$ is a Tannakian category over a field k and k'/k a finite extension we denote by $\mathcal{T}_{k'}$ the base change of $\mathcal{T}$ to k' (cf. [18, Construction 2.12]).

Lemma 5.8. *Let E'/E be a composition of a finite totally ramified and a finite purely inseparable extension and let $\mathbb{G}$ be an affine group scheme over $\varphi - \mathrm{Mod}_L$. Then the base change $\mathrm{Rep}_E(\mathbb{G})_{E'}$ of $\mathrm{Rep}_E(\mathbb{G})$ to E' is equivalent to the category $\mathrm{Rep}_{E'}(\mathbb{G}_{E'})$ where $\mathbb{G}_{E'}$ is the base change of $\mathbb{G}$ to E' (i.e., the Hopf algebra of $\mathbb{G}_{E'}$ is given by the Hopf algebra $\mathcal{O}_{\mathbb{G}} \otimes_L L'$ in $\varphi - \mathrm{Mod}_{L'}$ where $L' \cong E' \otimes_E L$ is the completion of the maximal unramified extension of E').*

Proof. An object of $\mathrm{Rep}_E(\mathbb{G})_{E'}$ is, by definition, given by a triple

$$(V \in \varphi - \mathrm{Mod}_L, \alpha \colon E' \mapsto \mathrm{End}_{\mathbb{G}}(V), c \colon V \to V \otimes_L \mathcal{O}_{\mathbb{G}})$$

where V is an isocrystal over L, α an E-algebra homomorphism and c a coaction (in $\varphi - \mathrm{Mod}_L$) of $\mathcal{O}_{\mathbb{G}}$ on V. Moreover, c is E'-linear as α maps into the $\mathbb{G}$-endomorphisms of V. We can map this triple to the $\mathbb{G}_{E'}$-representation $(V, \tilde{c})$ where $V \in \varphi - \mathrm{Mod}_{L'}$ is considered as an isocrystal over $L' \cong E' \otimes_E L$ with E' acting via α on V and $\varphi_V \colon V \to V$ being the given φ_L-semilinear automorphism (which is also $\varphi_{L'}$-linear). Here we have written φ_L resp. $\varphi_{L'} = \mathrm{Id}_{E'} \otimes \varphi_L$ for the Frobenius of L resp. L'. Finally, $\tilde{c}$ is defined as the composition

$$V \xrightarrow{c} V \otimes_L \mathcal{O}_{\mathbb{G}} \cong V \otimes_{L'} (L' \otimes_L \mathcal{O}_{\mathbb{G}})$$

which is a morphism of isocrystals over L'. Conversely, a $\mathbb{G}_{E'}$-representation

$$(W \in \varphi - \mathrm{Mod}_{L'}, \tilde{c} \colon W \to W \otimes_{L'} (L' \otimes_L \mathcal{O}_{\mathbb{G}})$$

can be sent to the triple

$$(W, \alpha \colon E' \to \mathrm{End}_{\mathbb{G}}(W), c \colon W \xrightarrow{\tilde{c}} W \otimes_L \mathcal{O}_{\mathbb{G}})$$

where W is considered as an isocrystal over L, α is the given action of E' on W and c the given coaction (using $W \otimes_{L'} (L' \otimes_L \mathcal{O}_{\mathbb{G}}) \cong W \otimes_L \mathcal{O}_{\mathbb{G}}$). These two functors define inverse equivalences of categories. □

The same proof shows that for the Tannakian category $\mathrm{Rep}_k(G)$ of representations of an affine group scheme G over a field k and a finite extension k' of k the base change $\mathrm{Rep}_k(G)_{k'}$ is equivalent to the Tannakian category $\mathrm{Rep}_{k'}(G_{k'})$.

Theorem 5.9. *Let $\mathcal{G}$ be a reductive group scheme over X_E and let $\mathbb{G}$ be an affine group scheme over $\varphi - \mathrm{Mod}_L$ such that $\mathcal{G}$ is associated to $\mathbb{G}$, i.e.,*

$$\mathcal{G} \cong \mathcal{E}(\mathbb{G}).$$

Then the canonical morphism

$$B(\mathrm{Rep}_E(\mathbb{G})) \to H^1_{ét}(X_E, \mathcal{G})$$

is bijective.

Proof. Let G' be a quasi-split group over E such that $\mathcal{G}$ is an inner form of G' (cf. Theorem 5.3). We first assume that $\mathcal{G}$ is a pure inner form of G'. Then there exists a G'-torsor $\mathcal{Q}$ over X_E such that

$$\mathcal{G} \cong \mathrm{Aut}_{G'}(\mathcal{Q}).$$

By Theorem 3.11, Proposition 5.7 and Lemma 5.6 we can see that there exists an element $b \in G'(L)$ such that

$$\mathcal{O}_{\mathbb{G}} \cong \mathcal{O}_{G'} \otimes_E L$$

with $\varphi_{\mathbb{G}}$ given by the composition

$$\mathcal{O}_{G'} \otimes_E L \xrightarrow{\mathrm{Id} \otimes \varphi_L} \mathcal{O}_{G'} \otimes_E L \xrightarrow{\mathrm{Ad}(b)} \mathcal{O}_{G'} \otimes_E L$$

where

$$\varphi_L \colon L \to L$$

denotes the Frobenius on L and $\mathrm{Ad}(b)$ be adjoint action of b on $\mathcal{O}_{G'} \otimes_E L$. Let $\mathbb{G}'$ be the group scheme over $\varphi - \mathrm{Mod}_L$ associated with G'. One can deduce that

$$\mathrm{Rep}_E(\mathbb{G}) \cong \mathrm{Rep}_E(\mathbb{G}')$$

by mapping a representation V of $\mathbb{G}'$ to the $\mathbb{G}'$-representation bV (cf. [4, Example 9.1.22]). In particular, we can conclude that every fiber functor

$$\omega \colon \mathrm{Rep}_E(\mathbb{G}) \to \mathrm{Bun}_{X_E}$$

for $\mathrm{Rep}_E(\mathbb{G})$ is admissible because this holds true for $\mathrm{Rep}_E(\mathbb{G}')$ by Theorem 3.11 (and the fact these fiber functors identify with fiber functors for $\mathrm{Rep}_E(G')$, cf. Proposition 5.7). By Theorem 3.10 we can conclude in this case.

Now assume that $\mathcal{G}$ is an arbitrary inner form of G'. Let E' be a composition of a finite totally ramified extension of E and a finite purely inseparable extension of E such that $\mathcal{G}_{X_{E'}}$ is a pure inner form of $G'_{E'}$ (the existence of such a field extension is guaranteed by Lemma 5.4 and the discussion following it). We already know (from Theorem 3.10) that the map

$$B(\mathrm{Rep}_E(\mathbb{G})) \to H^1_{ét}(X_E, \mathcal{G})$$

is injective and, again by Theorem 3.10, it suffices to check that every fiber functor

$$\omega \colon \mathrm{Rep}_E(\mathbb{G}) \to \mathrm{Bun}_{X_E}$$

is admissible. Let $\omega\colon \mathrm{Rep}_E(\mathbb{G}) \to \mathrm{Bun}_{X_E}$ be such a fiber functor. By [18, Construction 2.12] the composition

$$\mathrm{Rep}_E(\mathbb{G}) \to \mathrm{Bun}_{X_E} \to \mathrm{Bun}_{X_{E'}}$$

factors over a fiber functor

$$\omega_{E'}\colon \mathrm{Rep}_E(\mathbb{G})_{E'} \to \mathrm{Bun}_{X_{E'}},$$

where $\mathrm{Rep}_E(\mathbb{G})_{E'}$ denotes the base change of the category $\mathrm{Rep}_E(\mathbb{G})$ to E'. By Lemma 5.8 the categories $\mathrm{Rep}_E(\mathbb{G})_{E'}$ and $\mathrm{Rep}_{E'}(\mathbb{G}_{E'})$ are equivalent. As $\mathcal{G}_{X_{E'}}$ is a pure inner form of $G'_{E'}$ (and associated with $\mathbb{G}_{E'}$) the fiber functor $\omega_{E'}$ is admissible by the case already proven. By Lemma 3.4 this implies that ω is admissible. In particular, this finishes the proof of the classification of $\mathcal{G}$-torsors. □

6. Uniformization results

In this section we establish uniformization results for $\mathcal{G}$-torsors over the Fargues-Fontaine curve.

First, we want to prove that reductive group schemes over the Fargues-Fontaine curve become constant after removing a closed point.

We will need the following lemma.

Lemma 6.1. *Let G/E be a quasi-split reductive group over E. Then every element in $B(G)$ admits a reduction to some torus $T \subseteq G$.*

Proof. For $E/\mathbb{Q}_p$ this is [7, Proposition 7.2.]. Using [13, Proposition 13.1] we can extend this argument once we establish that in the case $E = \mathbb{F}_q((t))$ every element in $B(G)$ admits a reduction to some basic element in $B(M)$ for some Levi subgroup $M \subseteq G$. Let $b \in B(G)$ and consider the filtered fiber functor

$$\omega\colon \mathrm{Rep}_E(G) \xrightarrow{b} \varphi - \mathrm{Mod}_L \to \mathrm{Bun}_{X_E} \xrightarrow{HN} \mathrm{Fil}^{\mathbb{Q}}\mathrm{Bun}_{X_E}.$$

It defines a parabolic subgroup scheme

$$P(\omega) \subseteq G \times_E X_E.$$

If $B \subseteq G$ is a Borel subgroup, then there exists a unique standard parabolic

$$B \subseteq P \subseteq G$$

such that for every $x \in X_E$ the groups $P(\omega)_{\bar{x}}$ and P are conjugated. Using [1, Exposé XXVI, Proposition 1.3.] we can conclude that there exists a P-torsor $\mathcal{Q}$ over X_E such that

$$P(\omega) \cong P \times^P \mathcal{Q}$$

with P acting on P via conjugation (in [7, Section 5.1.] this P-torsor would be called the canonical reduction of the torsor associated with P). As in [7, Proposition 5.16.] this P-torsor admits a reduction to a Levi subgroup $M \subseteq P$ because for $\lambda \geq 0$ the vector bundles

$$\mathrm{gr}^\lambda(U(\omega))$$

(cf. Definition 2.5) are semistable of slope $\lambda \geq 0$. By construction this M-torsor is semistable which implies that b reduces to a basic element in $B(M)$ (cf. [7, Proposition 5.12.]). □

As in [7] this implies the following theorem.

Theorem 6.2. *Let G/E be a quasi-split reductive group over E and let $x \in X_E$ be a closed point. Then every G-torsor $\mathcal{Q}$ over X_E is trivial over $X_E \setminus \{x\}$.*

Proof. Using Lemma 6.1 and Theorem 4.5 the same proof as in [7, Théorème 7.1.] works. □

Lemma 6.3. *Let $\mathcal{G}$ be a reductive group scheme over X_E and let $x \in X_E$ be a closed point. Then $\mathcal{G}_{|X_E-\{x\}}$ is isomorphic to a constant quasi-split reductive group.*

Proof. By Theorem 5.3 there exists a quasi-split group G' over E such that $\mathcal{G}$ is an inner form of G'. Let $\mathcal{Q}$ be a $(G')^{\mathrm{ad}}$-torsor such that

$$\mathcal{G} \cong \mathcal{Q} \times^{(G')^{\mathrm{ad}}} G'.$$

Then $(G')^{\mathrm{ad}}$ is again quasi-split as the image of a Borel subgroup $B \subseteq G'$ is a Borel subgroup of $(G')^{\mathrm{ad}}$. By Theorem 6.2 every $(G')^{\mathrm{ad}}$-torseur over X_E is trivial over the punctured curve $X_E - \{x\}$. In particular, $\mathcal{Q}_{X_E-\{x\}}$ is trivial which shows that

$$\mathcal{G}_{|X_E-\{x\}} \cong G' \times_{\mathrm{Spec}(E)} (X_E - \{x\})$$

is isomorphic to a constant quasi-split reductive group. □

In fact, reductive group schemes over X_E are not too far from being constant. Let $\mathcal{G}$ be a reductive group scheme over X_E and write (cf. Theorem 5.3)

$$\mathcal{G} \cong \mathcal{Q} \times^{G^{\mathrm{ad}}} G$$

for a quasi-split reductive group G/E over E and a G^{ad}-torsor $\mathcal{Q}$ over X_E. Let $b \in G^{\mathrm{ad}}(L)$ be an element giving rise to $\mathcal{Q}$ (cf. Proposition 5.7). Assume first that b is basic (cf. [14, 5.1.]). We claim that in this case $\mathcal{G}$ is already constant. Namely, the group G^{ad} is semisimple and it suffices to prove the following lemma showing that $\mathcal{Q}$ is isomorphic to the pullback of a G^{ad}-torsor over E in this case. We denote by $B(H)_{\mathrm{basic}}$ the set of basic elements in $B(H)$ if H is a reductive group over E (cf. [14, 5.1.]).

Lemma 6.4. *Let H/E be a semisimple group. Then the canonical map*

$$H^1(\mathrm{Gal}(\bar{E}/E), H) \to B(H)_{\mathrm{basic}}$$

(cf. [14, 1.8.]*) is bijective.*

Proof. By [14, 4.5.] an element $c \in B(H)$ lies in the image of the canonical injection

$$H^1(\mathrm{Gal}(\bar{E}/E), H) \to B(H)$$

if and only if the associated morphism $\nu_c \colon \mathbb{D} \to H$ is trivial where $\mathbb{D}$ is the constant pro-torus with character group $\mathbb{Q}$ (cf. [14, 4.2.] for the construction of ν_c). Moreover, by definition $c \in B(H)$ is basic if and only if ν_c factors through the center of H ([14, 5.1.]). But if H is semisimple the center of H is finite which implies that every central homomorphism $\nu_c \colon \mathbb{D} \to H$ must be trivial as $\mathbb{D}$ is connected. □

Now assume that $b \in G^{\mathrm{ad}}(L)$ is arbitrary. By [14, Proposition 6.2.] resp. the proof of Lemma 6.1 the element b is φ_L-conjugate to some $b' \in M'(L)$ for some Levi subgroup $M' \subseteq G^{\mathrm{ad}}$ such that b' is basic. In particular,

$$\mathcal{G} \cong \mathcal{Q} \times^{G^{\mathrm{ad}}} G \cong \mathcal{Q}' \times^{M'} G$$

if $\mathcal{Q}'$ denotes the M'-torsor corresponding to b' (cf. Proposition 5.7). Let $M \subseteq G$ be the preimage of $M' \subseteq G^{\mathrm{ad}}$ under the canonical map $G \to G^{\mathrm{ad}}$. As M' is connected the image M'' of M' in $\mathrm{Aut}(M)$ must be contained in the adjoint group M^{ad} of M. Let $\mathcal{Q}''$ be the push forward of the M'-torsor $\mathcal{Q}'$ to M''. The element $c'' \in B(M'')$ corresponding to $\mathcal{Q}''$ will again be basic. Namely, if $c' \in B(M')$ corresponds to the M'-torseur $\mathcal{Q}'$ over X_E, then

$$\nu_{c''} = \alpha \circ \nu_{c'} = 1$$

because the homomorphism $\nu_{c'} \colon \mathbb{D} \to M'$ factors through the center $Z(M')$ of M' and

$$Z(M') \subseteq \mathrm{Ker}(M' \xrightarrow{\alpha} M'').$$

By Lemma 6.4 we can conclude that $\mathcal{G}$ contains the constant reductive group

$$\mathcal{Q}'' \times^{M''} M \cong \mathcal{Q}' \times^{M'} M \subseteq \mathcal{Q}' \times^{M'} G \cong \mathcal{G}.$$

which is of the same rank.

In general reductive group schemes can be non-constant. For example, let $\mathcal{E}$ be a non-semistable vector bundle on X_E. Then the reductive group scheme

$$\mathcal{G} := \mathrm{GL}(\mathcal{E})$$

is non-constant.

Finally we record the following "uniformization result" generalizing [7, Théoème 7.1.] (resp. Theorem 6.2).

Theorem 6.5. *Let $\mathcal{G}$ be a reductive group scheme over X_E, let $x \in X_E$ be a closed point and let $\mathcal{Q}$ be a $\mathcal{G}$-torsor. Then $\mathcal{Q}_{X_E - \{x\}}$ is isomorphic to the trivial $\mathcal{G}_{X_E - \{x\}}$-torsor.*

Proof. By Lemma 6.3 the group scheme

$$\mathcal{G}_{|X_E - \{x\}} \cong G \times_{\mathrm{Spec}(E)} (X_E - \{x\})$$

is isomorphic to a constant quasi-split reductive group scheme. Let $\mathcal{Q}'$ be the $G \times_{\mathrm{Spec}(E)} (X_E - \{x\})$-torseur corresponding to $\mathcal{Q}_{X_E - \{x\}}$ under such an isomorphism. Then $\mathcal{Q}'$ admits an extension to a G-torsor over X_E. Namely, by Beauville-Laszlo glueing (cf. [2] and Lemma 3.1) it suffices to construct a G-torsor over $\mathrm{Spec}(\widehat{\mathcal{O}_{X_E,x}})$ together with an isomorphism to $\mathcal{Q}'$ over $\mathrm{Spec}(\mathrm{Frac}(\widehat{\mathcal{O}_{X_E,x}}))$. But abstractly

$$\widehat{\mathcal{O}_{X_E,x}} \cong k[[t]]$$

is isomorphic to a power series ring by Cohen's structure theorem. In particular,

$$\mathrm{Frac}(\widehat{\mathcal{O}_{X_E,x}}) \cong k((t))$$

is of cohomological dimension 1 as k is algebraically closed in our case. Steinberg's theorem finally implies that every G-torsor over $\mathrm{Spec}(k((t)))$ is trivial and, in particular, admits an extension to $k[[t]]$. Thus, let $\mathcal{Q}''$ be a G-torsor extending $\mathcal{Q}'$. By Theorem 6.2 we can conclude that

$$\mathcal{Q}''_{|X_E - \{x\}} \cong \mathcal{Q}' \cong \mathcal{Q}_{X_E - \{x\}}$$

is isomorphic to the trivial torsor because G' is quasi-split. □

In fact, in Theorem 6.5 we have shown

$$H^1_{\acute{e}t}(X_E - \{x\}, G) = 0$$

for every reductive group G over E.

References

[1] M. Artin, J. E. Bertin, M. Demazure, P. Gabriel, A. Grothendieck, M. Raynaud, and J.-P. Serre. *Schémas en groupes. Fasc. 7: Exposés 23 à 26*, volume 1963/64 of *Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques*. Institut des Hautes Études Scientifiques, Paris, 1965/1966.

[2] Arnaud Beauville and Yves Laszlo. Un lemme de descente. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(3):335–340, 1995.

[3] Christophe Cornut. Filtrations and buildings. preprint on webpage at `https://webusers.imj-prg.fr/~christophe.cornut/papers/FilBuiv3.1.pdf`.

[4] Jean-François Dat, Sascha Orlik, and Michael Rapoport. *Period domains over finite and p-adic fields*, volume 183 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2010.

[5] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 111–195. Birkhäuser Boston, Boston, MA, 1990.

[6] L. Fargues. Geometrization of the local Langlands correspondence: an overview. *ArXiv e-prints*, February 2016.

[7] Laurent Fargues. G-torseurs en théorie de hodge p-adique. preprint on webpage at `https://webusers.imj-prg.fr/~laurent.fargues/Prepublications.html`.

[8] Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de hodge p-adique. preprint on webpage at `https://webusers.imj-prg.fr/~laurent.fargues/Prepublications.html`.

[9] Alexander Grothendieck. Le groupe de Brauer. III. Exemples et compléments. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 88–188. North-Holland, Amsterdam, 1968.

[10] Alexander Grothendieck. Le groupe de Brauer. II. Théorie cohomologique [MR0244270 (39 #5586b)]. In *Séminaire Bourbaki, Vol. 9*, pages Exp. No. 297, 287–307. Soc. Math. France, Paris, 1995.

[11] W. J. Haboush. Reductive groups are geometrically reductive. *Ann. of Math. (2)*, 102(1):67–83, 1975.

[12] Urs Hartl and Richard Pink. Vector bundles with a Frobenius structure on the punctured unit disc. *Compos. Math.*, 140(3):689–716, 2004.

[13] R. Kottwitz. B(G) for all local and global fields. *ArXiv e-prints*, January 2014.

[14] Robert E. Kottwitz. Isocrystals with additional structure. *Compositio Math.*, 56(2):201–220, 1985.

[15] Norbert Roby. Lois polynomes et lois formelles en théorie des modules. *Ann. Sci. École Norm. Sup. (3)*, 80:213–348, 1963.

[16] Neantro Saavedra Rivano. *Catégories Tannakiennes*. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin-New York, 1972.

[17] J. Tate. The cohomology groups of tori in finite Galois extensions of number fields. *Nagoya Math. J.*, 27:709–719, 1966.

[18] Paul Ziegler. Graded and filtered fiber functors on Tannakian categories. *J. Inst. Math. Jussieu*, 14(1):87–130, 2015.

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