In this kleine AG we want to prove the Hodge-Tate decomposition for abelian varieties over $p$-adic fields following the paper [Tate].

The main source will be [Tate], but the notes [Levi] may be very helpful, too. Instead of following the notation in [Tate] we suggest to use the following, more common, ones:

- $K/\mathbb{Q}_p$ a $p$-adic field
- $\mathcal{O}_K$ its ring of integers (instead of $R$ as in [Tate])
- $\mathbb{C}_K$ or $\mathbb{C}_p$ the completion of an algebraic closure $\bar{K}$ of $K$ (instead of $\mathbb{C}$)
- $\mathcal{O}_{\mathbb{C}_K}$ (or $\mathcal{O}_{\mathbb{C}_p}$) the ring of integers of $\mathbb{C}_K$ (or $\mathbb{C}_p$) (instead of $D$)
- $\text{Gal}(\bar{K}/K)$ or $G_K$ the absolute Galois group of $K$ (instead of $G$)
- $1 + m_{\mathbb{C}_K}$ or $1 + m_{\mathbb{C}_p}$ instead of $U$
- $\mu_p$ instead of $\mathbb{G}_m(p)$
- $\mathbb{Z}_p(1)$ or $\mathcal{T}_p \mu_p$ instead of $H$
- $\mathbb{K}_\infty$ instead of $X$
- $T_p G$ instead of $T(G)$, $V_pG := T_pG \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$
- $V_pG/T_pG$ instead of $\Phi(G)$ (shortly explaining why they are isomorphic)
- $\mathbb{C}_K(j) := \mathbb{C}_K \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(j)$
- $H^1_{\text{cont}}(\text{Gal}(\bar{K}/K), -)$ instead of $H^1(G, -)$

Now we briefly outline the topic. The aim is to prove (in talk 5) the Hodge-Tate decomposition, i.e., a $\mathbb{C}_p$-linear, $\text{Gal}(\bar{K}/K)$-equivariant isomorphism

$$ H^1_{\text{et}}(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong H^1(A, \mathcal{O}_A) \otimes_K \mathbb{C}_p \oplus H^0(A, \Omega^1_{A/K}) \otimes_K \mathbb{C}_p(-1), $$

of an abelian variety $A$ over $K$ admitting good reduction, i.e., $A \cong A_K$ for an abelian scheme $A$ over $\mathcal{O}_K$. Instead of dealing with abelian varieties we will consider the Hodge-Tate decomposition

$$ \text{Hom}_{\mathbb{Z}_p}(T_pG, \mathbb{C}_p) \cong t_G'(\mathbb{C}_p) \oplus t_G(\mathbb{C}_p)^\vee(-1), $$

of a $p$-divisible group $G$ over $\mathcal{O}_K$ (recovering the above isomorphism if $G = A[p^\infty]$).

In particular, we want to prove that $\text{Hom}_{\mathbb{Z}_p}(T_pG, \mathbb{C}_p)$ contains a subspace isomorphic to the tangent space $t_G'(\mathbb{C}_p)$ of the dual $p$-divisible group $G'$. To show this we will use the logarithm

$$ \log_{G'} : G'(\mathcal{O}_{\mathbb{C}_p}) \rightarrow t_G'(\mathbb{C}_p) $$

of the $p$-divisible group $G'$. This logarithm will be constructed in talk 2 as a generalization of the usual $p$-adic logarithm

$$ \log_p = \log_{\mu_p}: 1 + m_{\mathbb{C}_p} \rightarrow \mathbb{C}_p. $$
Moreover we need the surprisingly subtle statement \( \mathbb{C}_p^{\text{Gal}(\bar{K}/K)} = K \). This, and the calculation of the continuous Galois cohomology

\[
H^i_{\text{cont}}(\text{Gal}(\bar{K}/K), \mathbb{C}_K(j)),
\]

will be done in talk 3 and talk 4, which therefore contain the technical heart of Tate’s argument. The necessary background on \( p \)-divisible groups, and the crucial statement [Tate] Proposition 3, will be provided by talk 1.

1. **Talk: \( p \)-divisible groups (60 min)**
   Cover sections (2.1)-(2.3) in [Tate]. The proof of [Tate] Proposition 1 can be omitted as well as [Tate] Proposition 2. The main result of this talk is [Tate] Proposition 3. For stating [Tate] Proposition 1 it seems moreover reasonable to define (strict) Ind-schemes\(^2\) and to show that \( p \)-divisible groups (viewed as fppf-sheaves as for example in [Farg, Definition 17]) and formal schemes are both particular cases thereof. As a further example for \( p \)-divisible groups present [Farg, Remark 15].

2. **Talk: The \( p \)-adic Tate module and the logarithm (60 min)**
The material for this talk is [Tate] Section (2.4) (and [Levi, Section 2.1]). The definition of the logarithm should be presented with all details, in particular it should be proven that the logarithm of \( \mu_{p^\infty} \) is the usual \( p \)-adic logarithm. Also spend some time explaining the analytic nature of \( G(O_{\mathbb{C}_p}) \) (as for example in [Levi]).

3. **Talk: Tate-Sen theory I (60 min)**
   Present [Tate] Section (3.1) in detail. It might be especially useful to consider the corresponding parts of [Levi] as well. The main result is [Tate] Proposition 8. Perhaps contact the speaker of talk 4 to distribute some of the material (if you can see that the 60 min do not suffice).

4. **Talk: Tate-Sen theory II (60 min)**
   Present [Tate] Section (3.2),(3.3) (or the corresponding parts of [Levi]). The main result is the calculation of the continuous cohomology \( H^i_{\text{cont}}(\text{Gal}(K/K), \mathbb{C}_K(\chi)) \).

5. **Talk: The Hodge-Tate decomposition (60 min)**
   Now everything is settled to prove the Hodge-Tate decomposition. Present paragraph 4 of [Tate]. If time permits discuss the application [Tate] Theorem 4 (note that the statement about the discriminant, i.e., [Tate] Proposition 2, has not been presented).

References


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\(^2\)i.e. fppf-sheaves which are a filtered direct limit of schemes where the transition maps are closed immersions