①

Let F be a function field over a finite field

$\rightsquigarrow F = k(X)$, $k = \mathbb{F}_q$, X smooth proj geom connected curve over k.

Notat°: Let $\mathbb{A}_F$ ring of adèles of F $\quad \mathbb{A}_F = \prod'_{x\in|X|} F_x$

$\mathcal{O} \subseteq \mathbb{A}_F$ max compact subgp $\quad \mathcal{O} = \prod_{x\in|X|} \mathcal{O}_x$

Let $n \geq 1$, $\ell \neq p$.

Th: (unramified geometric Langlands)

Let $\sigma : Gal(\overline{F}/F) \to Gl_n(\overline{\mathbb{Q}_\ell})$
- everywhere unramified
- geometrically irred
- continuous.

Then, there exists $f_\sigma : GL_n(F)\backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O}) \to \overline{\mathbb{Q}_\ell}$ which is
- cuspidal (for every non-trivial partition $\underline{n} = (n_1, \dots, n_r)$ of n, & every Haar measure on $M_{\underline{n}}(F)\backslash M_{\underline{n}}(\mathbb{A}_F)$

$$\int_{M_{\underline{n}}(F)\backslash M_{\underline{n}}(\mathbb{A}_F)} f(mg)\, dm = 0$$

with $M_{\underline{n}} = GL_{n_1}\times\dots\times GL_{n_r} \subseteq GL_n$)

- an Hecke eigenfunction for σ, i.e.

$\forall x \in |X|$, $\forall i = 1, \dots, n$,

$$T_x^i(f_\sigma) = q_x^{-i(i-1)/2}\, tr\left(\Lambda^i \sigma(\mathrm{Frob}_x)\right) f_\sigma$$

$$\| \quad\quad\quad\quad$$

$$g \mapsto \int_{Gl_n(\mathcal{O}_x)\begin{pmatrix} \underbrace{\pi_x \ \cdots\ \pi_x}_{i\text{-times}} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix} Gl_n(\mathcal{O}_x)} f(gh)\, dh$$

dh ↑ normalized Haar measure on $Gl_n(F_x)$

[More conceptually; $\forall x \in |X|$, Satake isom : $\underbrace{\mathcal{H}_x}_{\text{spherical Hecke alg}} \simeq \mathrm{Rep}(GL_n)_{\overline{\mathbb{Q}_\ell}}$

$\rightsquigarrow$ the character $\chi_{f_\sigma, x}$ of $\mathcal{H}_x$ corresponding to $\overline{\mathbb{Q}_\ell}\cdot f_\sigma$ corresponds to the character $\mathrm{Rep}(Gl_n) \to \overline{\mathbb{Q}_\ell}$, $[V] \mapsto tr(\sigma(\mathrm{Frob}_x), V)$.]

Rk: When $n=1$, can reformulate by saying that the function
$$\mathbb{A}_F^\times / \mathcal{O}^\times \to \overline{\mathbb{Q}}_\ell^\times, \ (a_x)_{x\in|X|} \mapsto \prod_{x\in|X|} \sigma(\mathrm{Frob}_x)^{\mathrm{ord}_x(a_x)}$$
defines a character $f_\sigma : F^\times \backslash \mathbb{A}_F^\times / \mathcal{O}^\times \to \overline{\mathbb{Q}}_\ell^\times$.

How to prove such a theorem? One possible way is to try to geometrize everything

- σ as in Th $\longleftrightarrow$ ℓ-adic rank n local system $E = E_\sigma$ geom irred. on X.

- On the other side, recall (Weil)
$$\underbrace{\mathrm{Bun}_n(\mathbb{F}_q)}_{\substack{\text{set of iso classes}\\ \text{of rank } n \text{ vector bundles on } X}} \simeq GL_n(F) \backslash GL_n(\mathbb{A}_F) / GL_n(\mathcal{O})$$
[recall the argument]

In particular, every $K \in D_c^b(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ gives rise

stack of rk n vb on X

by the sheaf-functions dictionary to
$$f_K : \mathrm{Bun}_n(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$$
$$z \mapsto \mathrm{tr}(\mathrm{Frob}_z, K_z).$$

Idea of proof: Starting from σ, seen as a local system E, produce $\mathrm{Aut}_E \in D_c(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ s.t.
$f_\sigma := f_{\mathrm{Aut}_E}$ satisfies the properties of the theorem.

[cuspidality and Hecke property have geometric analogues]

Rk. $n=1$ Deligne (+ Lang-Rosenlicht), $n=2$ Drinfeld, $n \geq 3$ Laumon, Frenkel-Gaitsgory-Vilonen

First, let's focus on the case $\underline{n=1}$.
Then, we can be more precise:
Recall: Abel-Jacobi map — Picard scheme of X.

$$AJ : X \to \mathrm{Pic}_X = \mathrm{Bun}_1$$
$$x \mapsto \mathcal{O}_X(x).$$

$\underline{\mathrm{Th}}$: Pullback along the Abel-Jacobi map induces an equivalence of cat:

$$\left\{\begin{array}{l} \bar{\mathbb{Q}}_\ell\text{-rank 1-local systems } \mathcal{L} \\ \text{on } \mathrm{Pic}, \text{ s.t.} \\ + \; m^*\mathcal{L} \simeq p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \\ \quad \text{satisfying cocycle condition} \end{array}\right\} \simeq \left\{\begin{array}{l} \bar{\mathbb{Q}}_\ell\text{-rank 1 local} \\ \text{systems on } X \end{array}\right\}$$

Hecke property for GL_1 $\rightsquigarrow$ [+ $m^*\mathcal{L} \simeq p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}$ satisfying cocycle condition] ← "character sheaves on Pic"

$\underline{\mathrm{Rk}}$: Take π_0 of both sides.
- $\pi_0(LHS) = \{\ell\text{-adic characters on } \mathrm{Pic}(k) = F^\times \backslash \mathbb{A}_F^\times / \mathcal{O}^\times\}$
- $\pi_0(RHS) = \{\ell\text{-adic characters of } \pi_1(X) \cong \mathrm{Gal}(\bar{F}/F)^{ab,unr}\}$.

and induced bijection is the one of above Remark.

[For the first, in one direction use the sheaf function dictionary
in the other, use the Lang isogeny on Pic]

$\underline{\mathrm{Pf}}$: Can assume $k = \bar{\mathbb{F}}_q$.
For $d \geq 1$, let $X^{(d)}$ = d-th symmetric product of X
= moduli of effective relative Cartier div of deg d on X

and let $AJ^d : X^{(d)} \to \underbrace{\mathrm{Pic}^d}_{\text{component of deg } d}$

$$D \mapsto \mathcal{O}_X(D)$$

Abel-Jacobi map in deg d (so that $AJ : X \xrightarrow{AJ^1} \mathrm{Pic}^1 \hookrightarrow \mathrm{Pic}$)

Let E rank 1 local system on X. WTS: $\exists$ character sheaf $\mathcal{L}\mathrm{Aut}_E$ s.t. $E = AJ^*\mathcal{L}\mathrm{Aut}_E$.

For $d \geq 1$, consider $E^{\boxtimes d}$ on X^d.

④

Claim 1 (E has rank 1) $E^{\boxtimes d}$ descends to a (rank 1) local system on $X^{(d)}$.

Claim 2 (Riemann-Roch) $\forall d > 2g-1$, AJ^d realizes $X^{(d)}$ as a projective space bundle over Pic^d.

Claim 3 $\forall m \geq 1$, $\mathbb{P}_k^m$ simply connected.

$\Longrightarrow$ $\forall d > 2g-1$, $E^{(d)}$ descends to a local system Aut_E^d on Pic^d.

Moreover, associativity of exterior products $\Rightarrow \forall d, d' > 2g-1$, $m^* \mathrm{Aut}_E^{d+d'} \simeq p_1^* \mathrm{Aut}_E^d \otimes p_2^* \mathrm{Aut}_E^{d'}$.

$\Rightarrow$ Can extend formally Aut_E^d to all $d \in \mathbb{Z}$, and Aut_E on Pic defined in this way is a character sheaf. □.

Rk: This makes sense over $k = \mathbb{C}$ as well (contrary to ones original statement) and is true. Ex: prove it using topology.
(hint: $\mathrm{Pic}^\circ(X) = H_1(X, \mathbb{C})/H_1(X, \mathbb{Z})$ and $\pi_1(X)^{ab} \simeq H_1(X, \mathbb{Z})$)

Summary of proof:

Step ①: Attach to E an object $\mathrm{Aut}'_E = \bigoplus_{d \geq 1} E^{(d)}$
on $\mathrm{Bun}'_1 := \bigsqcup_{d \geq 1} X^{(d)} = \{ \mathcal{O} \hookrightarrow \mathcal{E}, \mathcal{E} \in \mathrm{Pic} \}$

Step ②: Descend Aut'_E to Aut_E along
$r := \bigoplus_{d \geq 1} AJ^d : \mathrm{Bun}'_1 \longrightarrow \mathrm{Bun}_1$
forgetful map.

The strategy of Drinfeld-Laumon-FGV for $n > 1$ is similar.

In the remaining time, let us explain easier step ①.
Idea: use Fourier expansions.

Let $f : GL_n(F)\backslash GL_n(\mathbb{A}_F) \to \overline{\mathbb{Q}}_\ell$ automorphic function.

Fix $\omega \neq 0 \in \Omega^1_{F/k}$, $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$ non-trivial character.

Set: $\Psi : F\backslash \mathbb{A}_F \to \overline{\mathbb{Q}}_\ell^\times$; $(a_x)_{x\in|X|} \mapsto \psi\left(\sum_{x\in|X|} \mathrm{tr}_{k(x)/k}(\mathrm{Res}(a_x\omega))\right)$.

All characters of $F\backslash\mathbb{A}_F$ are of the form:

$$\Psi_\gamma : y \mapsto \Psi(\gamma y),\ \text{some } \gamma\in F. \quad (\text{self-duality of } \mathbb{A}_F).$$

For simplicity, assume from now on $\underline{n=2}$.

$\forall g \in GL_2(\mathbb{A}_F)$, will write the Fourier expansion of

$$N(F)\backslash N(\mathbb{A}_F) \simeq F\backslash\mathbb{A}_F \to \overline{\mathbb{Q}}_\ell \qquad (N = \text{standard unipotent} \subset GL_2).$$
$$n \mapsto f(ng)$$

Get: $\forall g\in GL_2(\mathbb{A}_F)$, $\forall n'\in N(\mathbb{A}_F)$,

$$f(n'g) = \sum_{\gamma\in F}\left(\int_{N(F)\backslash N(\mathbb{A}_F)} f(ng)\Psi^{-1}(\gamma n)\,dn\right)\Psi(\gamma n').$$

Take $n'=1$, f <u>cuspidal</u>:

$$f(g) = \underset{\substack{\uparrow\\ f \text{ cuspidal}}}{\sum_{\gamma\in F^\times}}\left(\int_{N(F)\backslash N(\mathbb{A}_F)} f(ng)\Psi^{-1}(\gamma n)\,dn\right).$$

i.e. $f(g) = \sum_{\gamma\in F^\times} W_{f,\Psi}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right)$, with

$$W_{f,\Psi} : g \mapsto \int_{N(F)\backslash N(\mathbb{A}_F)} f(ng)\Psi^{-1}(n)\,dn$$

The function $W_{f,\Psi}$ lives in the space

$$C^\infty(GL_2(\mathbb{A}_F))^{(N(\mathbb{A}_F),\Psi)} = \left\{\begin{array}{l}\text{smooth functions } W : GL_2(\mathbb{A}_F)\to\overline{\mathbb{Q}}_\ell\\ \text{s.t. } W(ng) = \Psi(n)W(g)\\ \forall n\in N(\mathbb{A}_F),\ g\in GL_2(\mathbb{A}_F)\end{array}\right\}.$$

Let $P\subseteq GL_2$, $P = \left\{\begin{pmatrix} * & * \\ 0 & 1\end{pmatrix}\right\}$

<u>mirabolic subgroup</u>

Prop: There is a $GL_2(\mathbb{A}_F)$-equivariant isomorphism:

$$\Phi : C^\infty(GL_2(\mathbb{A}_F))^{(N(\mathbb{A}_F),\psi)} \xrightarrow{\sim} C^\infty_{cusp}(P(F)\backslash GL_2(\mathbb{A}_F))$$
$$W \longmapsto \sum_{\gamma \in F^\times} W((\gamma,1)\,-)$$

Now comes a result specific to the unramified case.

Th (Casselman-Shalika). For each $x \in |X|$, and each semisimple conjugacy class γ in $GL_n(\overline{\mathbb{Q}}_\ell)$, one can explicitly describe a function $W_{\gamma,x} : GL_2(F_x) \to \overline{\mathbb{Q}}_\ell$, uniquely determined by the properties:

- $W_{\gamma,x}(1) = 1$
- $W_{\gamma,x}$ is right-$GL_2(\mathcal{O}_x)$-invariant.
- $W_{\gamma,x}(ng) = \psi(tr_{k(x)/k}(\mathrm{Res}\; n\omega)) \cdot W_{\gamma,x}(g)$ $\forall g \in GL_2(F_x)$, $n \in N(F_x)$.
- $\forall i = 1, \dots, n$, $T_x^i(W_{\gamma,x}) = q_x^{-i(i-1)/2}\, tr(\Lambda^i(\gamma))\, W_{\gamma,x}$.

For $\sigma : Gal(\overline{F}/F) \to GL_n(\overline{\mathbb{Q}}_\ell)$, everywhere unramified,

let: $W_\sigma : GL_2(\mathbb{A}_F) \to \overline{\mathbb{Q}}_\ell$

$$g \longmapsto \prod_{x \in |X|} W_{\sigma(Frob_x),x}(g_x).$$

and set $\Phi(W_\sigma) =: f'_\sigma$.

By the above, $f'_\sigma \in C^\infty_{cusp}(P(F)\backslash GL_2(\mathbb{A}_F)/GL_2(\mathcal{O}))$

and satisfies: $\forall x \in |X|$, $\forall i = 1, \dots, n$,

$$T_x^i(f'_\sigma) = q_x^{-i(i-1)/2}\, tr(\Lambda^i \sigma(Frob_x)) \cdot f'_\sigma.$$

(unique up to scalar with these properties).

So again, two steps:

<u>step ①</u>: $\sigma \rightsquigarrow f'_\sigma$ as above.

<u>step ②</u>: Need to show that in fact f'_σ is left-$GL_2(F)$-inv, & gives rise to f_σ having all desired properties.

The explicit formulas in the theorem of Casselman-Shalika allow Laumon-FGV to produce $Aut_E \in D(Bun_2, \overline{\mathbb{Q}}_\ell)$ geometrizing f'_σ.