Zhu "Coherent sheaves on the stack of L-parameters"

Let Γ be a group, M affine group scheme over a ring k.

$\rightsquigarrow$ Can define a derived scheme $\mathcal{R}_{\Gamma,M}$ "representations of Γ with values in M"

If A is a k-alg, $\mathcal{R}_{\Gamma,M}(A) = \left\{ \begin{matrix} \text{group morphisms} \\ \Gamma \longrightarrow M(A) \end{matrix} \right\}$

If $\Gamma = F_n$ free group on n letters, $\mathcal{R}_{\Gamma,M} = M^n$. The more relations you have, the more derived $\mathcal{R}_{\Gamma,M}$ can be.

<u>Fact</u>: (2.2.13) Assumptions: • k noetherian, • M smooth affine group scheme$/k$, $\dim = d$.

• Γ finitely generated, of type $FP_\infty(k)$:

$\exists$ resolution $P^\bullet \longrightarrow k$ by finite projective $k[\Gamma]$-mod.

Assume moreover that:

$\forall\ \mathrm{Spec}(k') \to R_{\Gamma,M}$, one has $\begin{cases} H_i(\Gamma, Ad_\rho^*) = 0 & i \geq 2 \\ d - \sum_i (-1)^i \dim H_i(\Gamma, Ad_\rho^*) \geq \dim_\rho {}^{cl}R_{\Gamma,M} \end{cases}$

$\mapsto \rho : \Gamma \to M(k')$

Then $R_{\Gamma,M} = {}^{cl}R_{\Gamma,M}$, is of finite type over k, and l.c.i.

It is moreover smooth at at generic point iff it is flat at this point and $H_2(\Gamma, Ad_\rho^*) = 0$.

Examples: 1) (2.3.2, 2.3.3) Γ finite group of order invertible in k.

Then $R_{\Gamma,M} = {}^{cl}R_{\Gamma,M}$ and smooth of finite type over k.

Even better: Assume that:

- Γ solvable (p-group for example, p invertible in k)
- M is s.t. M° reductive & M/M° finite étale over k.

$\exists$ finite étale k-alg $\mathcal{O}$ s.t.

$$R_{\Gamma,M} \otimes_k \mathcal{O} = \bigsqcup_{\rho} M_{\mathcal{O}} / Z_M(\rho).$$

ρ $\leftarrow$ rep of conjugacy classes of $\Gamma \to M(\overline{\mathrm{Frac}(k)})$.

2) (2.3.7) $q = p^r$, $r > 0$.

M° reductive, M/M° finite étale over k.

$$\Gamma_q = \langle \sigma, \tau, \ \sigma\tau\sigma^{-1} = \tau^q \rangle.$$

$$1 \to \tau^{\mathbb{Z}[\frac{1}{p}]} \to \Gamma_q \to \langle \sigma \rangle \simeq \mathbb{Z} \to 1.$$

Let k be a Dedekind domain over $\mathbb{Z}[\frac{1}{p}]$.

Then $R_{\Gamma_q,M} = {}^{cl}R_{\Gamma_q,M}$, equidim'l of dim d, flat over k and l.c.i.

$\langle \tau \rangle \subset \Gamma_q$ gives $R_{\Gamma_q,M} \to M^{[q]} \subseteq M$

$\chi : M \to M /\!/ M = \mathrm{Spec}\, k[M]^M$.

The map $M \to M$, $x \mapsto x^q$ descends to $M /\!/ M$

Define $\Gamma^{[q]} = \chi^{-1}$(fixed pt locus of the map $M /\!/ M \to M /\!/ M$ induced by $x \mapsto x^q$).

$^{d}\mathcal{R}_{\Gamma_q, M} \xrightarrow{\pi} M^{[q]}$.

$M^{[q]}$ = union of conjugacy classes $C_M(A)$, for elements A in $M^{[q]}$.

The fibers of π are torsors under $Z_\Gamma(A)$.

Note that: $\dim Z_\Gamma(A) + \dim C_\Gamma(A) = \dim \Gamma = d$

Key input: by our assumptions on Γ, the above union is finite.

$\rightsquigarrow$ all generic fibers of $^{d}\mathcal{R}_{\Gamma_q, M}$ have dimension d.

Have a resolution of k as a $k[\Gamma_q]$-module:

$$0 \to k[\Gamma_q] \xrightarrow{\left(\left(1-\sum_{j<q}\tau^j\right)\sigma,\ \tau-1\right)} k[\Gamma_q]^{\oplus 2} \xrightarrow{(1-\tau,\,1-\sigma)} k[\Gamma_q] \longrightarrow k$$

$\rightsquigarrow$ $H_i(\Gamma_q, Ad_\rho^*) = 0$ $i>2$ and $\sum (-1)^i \dim H_i(\Gamma_q, Ad_\rho^*) = 0.$

Criterion gives $R_{\Gamma_q, \mathcal{M}} = {}^{cl}R_{\Gamma_q, \mathcal{M}}$, finite type over k, l.c.i.

all fibres are equidimensional $\Rightarrow$ flat.

3) Same assumptions on M, k Dedekind domain over $\mathbb{Z}[\frac{1}{p}]$.

$$\Gamma = Q \rtimes \Gamma_q \qquad Q \text{ finite } p\text{-group}.$$

Then $R_{\Gamma, M} = {}^{cl}R_{\Gamma, M}$ finite type, flat, equidimensional of dim d, l.c.i.

Pf: $Q \subset \Gamma \to R_{\Gamma, M} \to R_{Q, M}$

By example 1, enough to show $\forall \rho_0 : Q \to M(\mathcal{O})$,

$$^{cl}R^{\rho_0}_{\Gamma, M} := {}^{cl}R_{\Gamma, M} \times_{^{cl}R_{Q,M}} \{\rho_0\}$$

finite type, flat, equidim'l of dim $= \dim Z_\Gamma(\rho_0)$, l.c.i.
$/\mathcal{O}$

Let $N_\Pi(\rho_\circ) =$ normalizer of $\rho_\circ$ in $M_{\mathcal{O}}$.

smooth affine gp scheme and $N_\Pi(\rho_\circ)^\circ = Z_\Pi(\rho_\circ)^\circ$ reductive.

Consider $U \subseteq \mathcal{R}_{\Gamma_q, \pi_0(N_\Pi(\rho_\circ))}$ consisting of $\rho : \Gamma_q \to \pi_0(N_\Pi(\rho_\circ))$

$\pi_0(N_\Pi(\rho_\circ)) = N_\Pi(\rho_\circ)/N_\Pi(\rho_\circ)^\circ$ étale over $\mathcal{O}$

such that

$$\Gamma_q \xrightarrow{\rho} \pi_0(N_\Pi(\rho_\circ)) \longrightarrow \mathrm{Aut}(\rho_\circ(Q))$$

is compatible with the action of Γ_q on Q.

It is open in $\mathcal{R}_{\Gamma_q, \pi_0}(N_\Pi(\rho_\circ))$.

So $\quad {}^{cl}\mathcal{R}^{\rho_\circ}_{\Gamma,\Pi} = {}^{cl}\mathcal{R}_{\Gamma_q, N_\Pi(\rho_\circ)} \times_{{}^{cl}\mathcal{R}_{\Gamma_q, \pi_0(N_\Pi(\rho_\circ))}} U \quad$ open in ${}^{cl}\mathcal{R}_{\Gamma_q, N_\Pi(\rho_\circ)}$

Apply Example 2) to ${}^{cl}\mathcal{R}_{\Gamma_q, N_\Pi(\rho_\circ)}$ to conclude.

Stack of L-parameters:

E local field, G reductive gp $/E$, $\hat{G}/\mathbb{Z}$.
res. field $\mathbb{F}_q$

Let ${}^cG = \hat{G} \rtimes (\mathbb{G}_m \times \Gamma_{\tilde{E}/E})$

Here $\Gamma_{\tilde{E}/E} = \mathrm{Gal}(\tilde{E}/E)$ is such that the action of $\mathrm{Gal}(\bar{E}/E)$ on $\hat{G}$ factors through $\Gamma_{\tilde{E}/E}$

& $\mathbb{G}_m$ acts on $\hat{G}$ as follows: $\mathbb{G}_m \xrightarrow{\rho_{ad}} \hat{G}_{ad} \subset \mathrm{Aut}(\hat{G})$.

Let $d : {}^cG \longrightarrow \mathbb{G}_m \times \Gamma_{\tilde{E}/E}$.

[Recall ${}^LG = \hat{G} \rtimes \Gamma_{\tilde{E}/E}$.]

Recall: $1 \to I_E \to W_E \xrightarrow{\|\cdot\|} \mathbb{Z} \to 1$

and $1 \to P_E \to I_E \to I_E^t \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to 1$

Notation: Let $\chi : W_E \to (\mathbb{Z}[\frac{1}{p}]^\times, P_{\bar{E}/E})$

$(q^{-\|\cdot\|}, pr)$.

Facts: • $\exists$ top splitting $W_E = P_E \rtimes W_E^t$, $W_E^t = W_E / P_E$

$1 \to I_E^t \to W_E^t \to \mathbb{Z} \to 1$.

• $\exists$ embedding $\Gamma_q \xrightarrow{i} W_E^t$

s.t. $i(\tau)$ generator of tame inertia

and $i(\sigma)$ lift of Frobenius.

Let $W_{E,i}$ pull back of W_E along such an embedding.

$W_{E,i} = P_E \rtimes \Gamma_q$.

For any finite extension $L/E^t\tilde{E}$, Galois over E,

get : $1 \to Q_L \to W_{L/E,i} \to P_q \to 1$.

and $\chi \circ i$ well defined on $W_{L/E,i}$ and so we can set:

$$\mathrm{Loc}_{{}^cG,\, L/E,\, i} := \mathcal{R}_{W_{L/E,i},\, {}^cG} \times_{\mathcal{R}_{W_{L/E,i},\, \mathbb{G}_m \times P_{\tilde{E}/E}}} \{\chi \circ i\}$$

over $k = \mathbb{Z}[\frac{1}{p}]$.

i.e. $\forall$ $\mathbb{Z}[\frac{1}{p}]$-algebra A,

$$\mathrm{Loc}_{{}^cG,\, L/E,\, i}(A) = \left\{ \rho : W_{L/E,i} \longrightarrow {}^cG(A),\ \text{s.t. } d \circ \rho = \chi \circ i \right\}.$$

$$\mathrm{Loc}^{\square}_{{}^cG,E,i} = \varinjlim_L \mathrm{Loc}^{\square}_{{}^cG,L/E,i}$$

<u>Claim</u>: This is a disjoint union of classical affine schemes of finite type over $\mathbb{Z}[\frac{1}{p}]$, flat and l.c.i., equidim'l of dim $\dim G = \dim \hat{G}$.

[Follows from $W_{L/E,i} = Q_L \rtimes \mathbb{Z}_q$, Q_L finite p-group].
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Gal}(L/E^t)$

<u>Def</u>: $\mathrm{Loc}_{G,E,i} = \mathrm{Loc}^{\square}_{G,E,i} / \hat{G}$

— Let $\ell \neq p$

<u>Prop</u>: After base change to $\mathbb{Z}_\ell$, $\mathrm{Loc}_{G,E,i}$ is isomorphic to the one introduced yesterday, and so in particular independent of i.

Pf: Need to see:

$\forall$ $\mathbb{Z}_\ell$-alg A, $\rho : W_{L/E,i} \to {}^cG(A)$

extends uniquely to a continuous $\tilde{\rho} : W_E \to {}^cG(A)$.

(i.e. $\forall f \in \mathbb{Z}_\ell[{}^cG]$, $f \circ \tilde{\rho} : W_E \to A$ continuous, A endowed with the topology introduced yesterday).

Key: $\langle \tau \rangle \subseteq P_q$

$\rightsquigarrow \mathrm{Loc}^{\square}_{{}^cG, E, i} \to {}^cG^{[q]}$.

any such element becomes unipotent after taking some power.

Reminder: Picard stacks:

$\mathcal{C}$ site. Picard stack: stack $\mathcal{P}$ on $\mathcal{C}$ with a bifunctor

$$\otimes : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$$

+ associativity and commutativity contraints
s.t. $\forall U \in \mathcal{C}$, $\mathcal{P}(U)$ is a symmetric monoidal groupoid
every object has an inverse.

Deligne: 2-category of Picard stacks is equivalent
to $C^{[-1,0]}$: 2-term complexes of sheaves of abelian groups on $\mathcal{C}$

$$K^{-1} \xrightarrow{d} K^{0}$$

K^{-1} injective.

If $K = (K^{-1} \xrightarrow{d} K^{0})$, $\mathrm{pch}(K)(U)$: . objects $= K^{0}(U)$
. $x, y \in K^{0}(U)$ morphism between x and y is an element $f \in K^{-1}(U)$ s.t. $df = y - x$.

Inverse functor : $\mathcal{P} \to \mathcal{P}^b$.

<u>Ex</u> : A sheaf of ab groups on $\mathcal{C}$.

- $A^b \simeq A[0]$.
- $(BA)^b \simeq A[1]$.

<u>Rk</u> : $\mathcal{P}_1, \mathcal{P}_2$ Picard stacks

$$\left(\underline{Hom}(\mathcal{P}_1, \mathcal{P}_2)\right)^b = \tau_{\leq 0} R\underline{Hom}(\mathcal{P}_1^b, \mathcal{P}_2^b).$$

Assume $\mathcal{C}$ = fppf site of a scheme S. Picard stack $B\mathbb{G}_m$.

<u>Define</u> : if $\mathcal{P}$ Picard stack, $\mathcal{P}^\vee := \underline{Hom}(\mathcal{P}, B\mathbb{G}_m)$.

Ex: 1) A abelian scheme over S.

$$A^\vee = \underline{\mathrm{Hom}}(A, B\mathbb{G}_m) = \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m) = \text{dual abelian scheme.}$$

2) $\Gamma = M_S$ $\quad$ M finitely generated ab. group.

$$\Gamma^\vee = B\, D(\Gamma)$$

Cartier dual of Γ ; $U/S \mapsto \mathrm{Hom}(\Gamma \times_S U, \mathbb{G}_{m,U})$

Conversely, if $G = D(\Gamma)$

then $(BG)^\vee = \Gamma$.

$\mathcal{P}$ Picard stack. Poincaré line bundle on $\mathcal{P} \times_S \mathcal{P}^\vee \begin{array}{c} \xrightarrow{p_1} \mathcal{P} \\ \xrightarrow[p_2]{} \mathcal{P}^\vee \end{array}$
$\hat{\mathcal{L}}_{\mathcal{P}}$

Fourier_Mukai functor : $\Phi_{\mathcal{P}} : D^b(QCoh(\mathcal{P})) \longrightarrow D^b(QCoh(\check{\mathcal{P}}))$

$$\mathcal{F} \mapsto R_{p_{2*}}(Lp_1^* \mathcal{F} \otimes \mathcal{L}_{\mathcal{P}}).$$

Ex : (Laumon) : S scheme, A abelian scheme

$A^{\natural}$ parametrizing iso classes of line bundle on A + flat connection + rigidification

$$0 \to \Omega^1_{A/S} \to A^{\natural} \longrightarrow \hat{A} \longrightarrow 0.$$

$\mathcal{P} = A^{\natural}$ $\quad (\mathcal{P}^\vee)^b = [\hat{A}^0 \xrightarrow{can} A]$

$$\Phi_{\mathcal{P}} : D^b(QCoh(\mathcal{P})) \simeq D^b(QCoh(\mathcal{P}^\vee))$$

$A = \mathrm{Jac}(X)$

Apply to $[\mathbb{G}_m \to A^{4}]$ and $[\hat{A}^\circ \to A \times \mathbb{Z}]$

$$D^b(QCoh(LocSys^{dR}_{\mathbb{G}_m})) \simeq DMod(Bun_{\mathbb{G}_m, X}).$$

Back to the local setting:

Zhu: T torus over E. $Loc_{T, E}$ is a Picard stack $/\mathbb{Z}[\frac{1}{p}]$.

Def: Let $Tor_{T, E}$: Picard groupoid of pairs $(\mathcal{E}, \varphi)$, $\mathcal{E}$ T-torsor over $\breve{E}$

and $\varphi : \sigma^*\mathcal{E} \simeq \mathcal{E}$ as T-torsors.

See it as a constant Picard stack over $\mathbb{Z}[\frac{1}{p}]$.

Conj: $\exists$ natural line bundle on $\mathrm{Tor}_{T,E} \times \mathrm{Loc}_{{}^cT, E}$ s.t.

$$\mathrm{Tor}_{T,E} \simeq \mathrm{Loc}^{\vee}_{{}^cT, E}.$$

$\mathrm{Tor}_{T,E}$: isom classes of objects $B(T)$ (Kottwitz)
aut group of each object $T(E)$.

$$[T(E) \to B(T)].$$

$\mathrm{Loc}_{{}^cT, E}$: points $H^1_{\mathrm{cont}}(W_E, \hat{T})$
automorphisms $\hat{T}^{W_E}$.