Kottwitz set $B(G)$

References:

Demazure – p-div grps (GL_n)

Kottwitz

Rapoport–Richartz

§1 Case of GL_n

$k = \bar{k}$ char $k = p$

$E/\mathbb{Q}_p$ $\breve{E} := W_{\mathcal{O}_E}(k)[\frac{1}{p}]$

Def Isocrystal $\overset{=}{\text{def}}$ (V, Φ) V fin dim $\breve{E}$-Vsp

Φ σ-linear auto.

$\sigma \circlearrowright \breve{E}$

$$\text{Slope}(V, \Phi) := \frac{v_\pi(\det \Phi)}{\dim V} \qquad \pi \in \mathcal{O}_E \text{ unif.}$$

Thm (Dieudonné) (Isocrys) semi-simple

$$\{\text{Simple Obj}\}/\cong \;\longrightarrow\; \mathbb{Q}$$

$$\left(\breve{E}^r, \begin{pmatrix} & & \pi^s \\ 1 & & \\ & \ddots & \\ & 1 & \end{pmatrix}\right) \longleftarrow\!\mathrel{\vert} \lambda = \frac{r}{s} \quad (r, s)$$

$$(V, \Phi) \longmapsto \text{slope}(V, \Phi)$$

$$(V, \Phi) = \bigoplus_{\lambda \in \mathbb{Q}} V^\lambda \quad \text{slope decomposition}$$

Group-theoretic interpretation

$$\left\{ \underset{(V,\Phi)}{\text{Isoc}},\ \dim V = n \right\} \Big/ \cong \quad \cong \quad \underbrace{GL_n(\breve{E}) \Big/ b \sim g b \sigma(g)^{-1}}_{=: B(GL_n)}$$

$$(\breve{E}^n, b\cdot\sigma) \longleftarrow [b]$$

$$(V, \Phi) \longmapsto [(\Phi(v_i)_j)]$$

$v_1, \dots, v_n$ $\breve{E}$-basis of V

Newton map

$$D := \varprojlim_n \mathbb{G}_m = \operatorname{Spec} E[T^{\mathbb{Q}}]$$ torus w/ char grp $\mathbb{Q}$.

$$\operatorname{Hom}_K(D, GL_n) = \left\{ K^n = \bigoplus_{\lambda \in \mathbb{Q}} V_\lambda \right\}$$

Given $b \in GL_n(\breve{E})$, get $\nu_b \in \operatorname{Hom}_{\breve{E}}(D, GL_n)$

Descends to

Def Newton map

$$B(G) \xrightarrow{\nu} {}_{G(\breve{E})} \backslash \operatorname{Hom}_{\breve{E}}(D, GL_n) = @$$

$$\mathcal{N} = \left(X_*(T)_{\mathbb{Q}} / \Omega \right)^{\Gamma} \quad \text{where}$$

$T \subseteq GL_n$ max torus, Ω = Weyl grp,

$\Gamma = \mathrm{Gal}(\bar{E}/E)$

<u>E.g.</u> $T = \mathbb{G}_m^n$ diagonal

$$\underbrace{X_*(T)_{\mathbb{Q}}}/\Omega \xleftarrow{\cong} X_*(T)_{\mathbb{Q}, \mathrm{dom}}$$

$\mathrm{Hom}_{\bar{E}}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{Q}$

$= \mathrm{Hom}_{\bar{E}}(\mathbb{D}, T)$

$$= \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{Q}^n \text{ s.th. } \lambda_1 \geq \cdots \geq \lambda_n \right\}$$

$$\mathrm{Im}(\nu) = \left\{ (\lambda_i) \text{ s.th. multiplicity of } \lambda = \tfrac{r}{s} \text{ lies in } s \cdot \mathbb{Z} \quad (r,s) = 1 \right\}$$

Dieudonné Thm $\Rightarrow$ $\nu([b])$ determines $[b]$

§2 General G (connected reductive)

<u>Def</u> i) G-isocr. $\underset{\mathrm{def}}{=}$ $(\mathrm{Rep}_E G, \otimes) \longrightarrow (\mathrm{Isocr}, \otimes)$

E-linear, exact, faithful, $\otimes$-functors

ii) $B(G) := G(\breve{E}) / b \sim g b \sigma(g)^{-1}$

Prop $B(G) \cong \{G\text{-Isocry.}\}/\cong$

$$[b] \longmapsto \left(\mathcal{I}_b : (V,\rho) \mapsto (\breve{E} \otimes_E V, \rho(b)\sigma)\right)$$

Sketch Recall:

$\{\text{fib. funct. } \mathrm{Rep}_E(G) \to \mathrm{Vect}_K\}$ K/E extn

$\cong \{G\text{-torsors}/K\}$

Given $\mathcal{I}$: G-isocrystal,

$$p \circ \mathcal{I} : \mathrm{Rep}_E G \to \mathrm{Isocr} \xrightarrow{p} \mathrm{Vect}_{\breve{E}}$$

$$(V, \Phi) \longmapsto V$$

corresponds to G-torsor over $\breve{E}$.

Steinberg $\Rightarrow H^1(\breve{E}, G) = \{*\}$

Hence $p \circ \mathcal{I} \underset{\alpha}{\cong} \omega_{std,\breve{E}} : (V,\rho) \mapsto \breve{E} \otimes_E V$

The Φ's yield $\omega_{std,\breve{E}} = \omega_{std,\breve{E}}^{(\sigma)} \xrightarrow[\cong]{\Phi_s} \omega_{std,\breve{E}}$

$\Rightarrow \alpha$ yields $b(\alpha) \in G(\breve{E})$ ... $\square$

Newton map

Given $b \in G(\breve{E})$, $\mathcal{I}_b$ (α is canonical one)

Get $D \xrightarrow{\nu_b} \mathrm{Aut}(\omega_{std,\breve{E}}) = G(\breve{E})$.

Descends to

$$B(G) \xrightarrow{\nu} {}_{G(\breve{E})}\backslash \mathrm{Hom}_{\breve{E}}(D, G) \qquad @$$

Def

$$= \left(X_*(T)_{\mathbb{Q}} / \Omega \right)^{\Gamma}$$

for any maximal $T \subseteq G$.

Is functorial $\nu : B(-) \longrightarrow N(-) :=$ RHS of @

§3 Fibres of Newton map

Def $b \in G(\breve{E})$

$$J_b := \mathrm{Aut}(\mathcal{I}_b) = \left[R/E \mapsto \left\{ g \in G(\breve{E} \otimes_E R) \;\middle|\; g b \sigma(g)^{-1} = b \right\} \right]$$

Prop i) J_b conn. red. grp $/E$

ii) Only depends on $[b]$

(If $g b \sigma(g)^{-1} = b'$, then $g J_b g^{-1} = J_{b'}$.)

__E.g.__ $G = GL_n$

$$(\breve{E}^n, b\cdot\sigma) = \bigoplus_{\lambda\in\mathbb{Q}} V^\lambda \qquad J_b \cong \prod_\lambda GL_{n_\lambda}(D_\lambda)$$

central div alg $/E$
$\mathrm{inv}(D_\lambda) = \pm\lambda$

__In ptic:__ J_b inner form of $\mathrm{Cent}(\nu_b)$, a Levi of GL_n

for $G = GL_n$ defined over E

__Prop__ (Rapoport–Richartz 1.17)

i) J_b is inner form of a __Levi__ of a q-split inner form of G $= \mathrm{Cent}(\nu_b)$ after scalar extn.

ii) $\nu^{-1}(\nu_{[b]}) = H^1(E, J_b)$

$$[b'] \longmapsto J_b\text{-torsor } \mathrm{Isom}^{\otimes}(\mathcal{I}_b, \mathcal{I}_{b'})$$

$$= R/E \longmapsto \{ g \in G(\breve{E} \otimes_E R) \mid g b \sigma(g)^{-1} = b' \}$$

Defn b or $[b]$ basic if

i) ν_b factors through $\mathrm{Cent}(G_{\breve{E}})$ } equivalent

ii) J_b is inner form of G

(In some sense, " $B(G) = \bigcup_{\substack{L \subseteq G \\ \text{Levi}}} B(L)_{\text{basic}}$ " Kottwitz I §6.)

§4 Kottwitz map / Classify $B(G)_{\text{basic}}$

Prop $T \subseteq G$ elliptic max torus $/E$. Then

$$B(G)_{\text{basic}} = \mathrm{Im}\left(B(T) \longrightarrow B(G)\right)$$

E.g. $G = GL_2$ $\quad B(G)_{\text{basic}} \xleftrightarrow{1:1} \frac{1}{2}\mathbb{Z}$

$(V, \Phi) \longmapsto \mathrm{slope}(V, \Phi)$

One direction $\supseteq$ holds:

$$\begin{array}{ccc} X_*(T)^{\Gamma} & \longrightarrow & X_*(\mathrm{Cent}(G)) \\ \| & & \downarrow \\ \mathcal{N}(T) & \longrightarrow & \mathcal{N}(G) \end{array}$$

factors because T elliptic

In partic, ν_b central $\forall\, b \in \mathrm{Im}(B(T) \to B(G))$

T elliptic $\underset{\text{def}}{\iff}$ $T/_{\mathrm{Cen}(G)}$ anisotropic

$$\mathrm{Hom}(\mathbb{G}_m, T/_{\mathrm{Cent}(G)}) = 0$$

Moreover, Kottwitz argues that $B(-)|_{\text{Tori}}$ satisfies

i) $B(\mathrm{Res}_{E'/E}\,\mathbb{G}_m) \cong \mathbb{Z} \quad \forall E'/E$

ii) $\forall\ 1 \to T_1 \to T_2 \to T_3 \to 0$ ex

$$B(T_1) \to B(T_2) \to B(T_3) \to 1$$

exact.

Implies $\exists\ B(-)|_{\text{Tori}} \cong X_*(-)_\Gamma$

+ iso uniquely det by normalization

$$B(\mathbb{G}_m) \cong X_*(\mathbb{G}_m)$$

$$[\pi] \longmapsto 1$$

<u>Def</u> (Borovoi) Algebraic Fundamental grp

$$\pi_1(G) := X_*(T) \Big/ \sum_{\alpha \in \Phi} \mathbb{Z}\cdot\alpha^\vee$$

(Canonically indep of $T \subseteq G$)

<u>Thm</u> (Kottwitz) $\exists\ B(-) \xrightarrow{\kappa} \pi_1(-)_\Gamma$

s.th. $\forall T \quad B(T) \xrightarrow{\cong} X_*(T)_\Gamma = \pi_1(T)_\Gamma$

$$\underbrace{\mathrm{Res}_{E'/E}\, \mathbb{G}_m}_{\text{Torus}/E} := \{ R/E \mapsto \mathbb{G}_m(E' \otimes_E R) \}$$

Weil restriction

$$\dim(-) = [E' : E].$$

1)
$$\begin{tikzcd}
B(G) \arrow[r, "\nu"] \arrow[d, "\kappa"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{N}(G) \arrow[d] \\
\pi_1(G)_\Gamma \arrow[r, "\gamma"] & \pi_1(G)^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}
\end{tikzcd}$$

$$\bar{\mu} \longmapsto |\Gamma \cdot \mu|^{-1} \sum_{\mu' \in \Gamma \cdot \mu} \mu'$$

For GL_n, γ injective

"Newton point determines Kottwitz point"

2) $B(G) \xrightarrow{(\nu, \kappa)} \mathcal{N}(G) \times \pi_1(G)_\Gamma$ <u>injective</u>