

Divisors

$$S = \text{Spa}(C, \mathcal{O}_C)$$

(1)

On $Y_{S,E}$ $\text{Div}_Y^1 = \{ \sum a_Y Y \mid \text{locally finite, } a_Y \in \mathbb{Z}, Y \in Y_{S,E}^d \}$
 $\{ \sum a_Y Y \mid \text{finite sum, } a_Y \in \mathbb{Z}, Y \in Y_{S,E}^d \}$
 $\{ \sum a_Y Y \mid \text{finite, } a_Y \in \mathbb{Z}, Y \in Y_{S,E}^d \}$

~~scribble~~

On $X_{S,E} = Y_{S,E} / \mathbb{G}_m$: $\text{Div}_X^1 = \{ \sum a_x x \mid \text{finite sum, } a_x \in \mathbb{Z}, x \in X^d \}$

$$\text{deg}: \text{Div}_X^1 \rightarrow \mathbb{Z}, \sum a_x x \mapsto \sum a_x$$

(i.e. $\text{deg}(x) = 1$)

~~scribble~~

Want: $\text{Div}_{X_{S,E}}^1 \xrightarrow{\text{deg}} \mathbb{Z}$
 $\downarrow \text{deg}^{-1} \{ \}$
 $\text{Pic}(X_{S,E}) \cong \mathbb{Z}$

(Note: we don't introduce principal divisors)

Lecture: $x \in X^d$ classical $\Rightarrow \mathcal{O}(x) \cong \mathcal{O}(1)$

\Rightarrow can define $\text{deg}(\mathcal{I}) = n$ if $\mathcal{I} \cong \mathcal{O}(n)$

Different viewpoint: $X_{S,E}^{\text{sch}} = \text{Proj}_{\mathbb{Z}} \left(\bigoplus_{d \geq 0} H^0(X_{S,E}, \mathcal{O}(d)) \right) =: P$

1-dim. Noeth. scheme, regular, closed points $\hookrightarrow X^d$

$\Rightarrow \text{Div}_{X^{\text{sch}}}^1$ defined, $\cong \text{Div}_X^1$

~~scribble~~ $\text{Pic} \cong \mathbb{Z}$ ~~scribble~~ $x \in X^d, \mathcal{O}(x) \cong \text{Spec } B_e$

$$B_e := \left(\mathbb{P} \left[\frac{x}{t} \right] \right)_0, \text{ where } V(t) = \{x\}$$

$t \in P_1$

PID
 generated by

Each $f \in B_e \setminus \{0\}$ is a product $\frac{t_1 \cdots t_n}{t^n}$ with $t_1, \dots, t_n, t \in P_1$

$$V(\frac{t}{t}) = \mathbb{Z}$$

$$\Rightarrow \mathcal{O}_X = \mathbb{Z}$$

Now, S arbitrary, $d \geq 0$

The following sets are in bijection

$\neq 0$ \mathbb{H} in each geometric point $s \rightarrow S$

1) $\{ (F, s) \mid \deg F = d, f \in H^0(X_{s,E}, F) \setminus \{0\} \}$

2) $\{ D \in X_{s,E}^{\text{closed}} \mid \text{Cartier divisor, s.t. } \deg D_s \text{ closed Cartier } \forall \text{ geom. pts } s \rightarrow S \text{ of deg. } d \}$

(if S affinoid)

3) $\{ D \in X_{s,E}^{\text{sch}} \mid \text{Cartier divisor, s.t. } D_s \text{ Cartier for all } s \rightarrow S \text{ geom. pts } \deg D_s = d \}$

This defines v -sheaf $\text{Div}^d \rightarrow \text{Spa } \overline{F}_q$, $(F_q = \text{res. fld of } E)$

E.g. $\text{Div}^1 = \text{BC}(\mathcal{O}(1))_{/E^*}^* = \text{Spa}(\overline{F}_q((t^{\frac{1}{p^\infty}}))_{/E^*}$ spatial diamond
 $\text{Spa } E_{\infty} / E^* = \text{Spa } \check{E} / \varphi_{\mathbb{Z}}$ top. space has 1 point

$\text{Div}^d = \text{BC}(\mathcal{O}(d))_{/E^*}^* = (\text{Div}^1)^d_{(S_d)}$
 very strange!

Example

$(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ geom. pt.

perfectoid open unit disk

$\Rightarrow \text{Div}^1_{\mathbb{C}} \cong \text{ID}_{\mathbb{C}}^* / E^*$

$C = \hat{E}$

$\text{Div}^1 \times \text{Div}^1 \cong \text{Spd } \check{E} / \varphi_{\mathbb{Z}} \times \text{Spd } \check{E} / \varphi_{\mathbb{Z}} \cong (\text{Spa}(\mathbb{F}) \times \text{Div}^1)_{/\text{Gal}(\mathbb{F}/\mathbb{E})}$

In part, Div^2 has non trivial geometry

Another example ~~the~~

$$BC(O(\frac{1}{2})) \subset \text{Spd}(\mathbb{F}_q((t^{\frac{1}{2}})))$$

$M_\infty := \{ \begin{matrix} O^2 & \alpha \\ \downarrow & \\ O(\frac{1}{2}) & \end{matrix} \}$ on $\text{Perf}_{\mathbb{F}_q}$
Pisotwise injective maps

$$\Rightarrow M_\infty \rightarrow \text{Spd}_{\mathbb{F}_q} \xrightarrow{\text{Dil}_1} \text{Spd}_{\mathbb{F}_q} \xrightarrow{\text{Dil}_1} \dots \xrightarrow{\text{Dil}_1} \text{Spd}_{\mathbb{F}_q} \xrightarrow{\text{Dil}_1} \dots$$

Nagata \downarrow $V(\det(\alpha))$

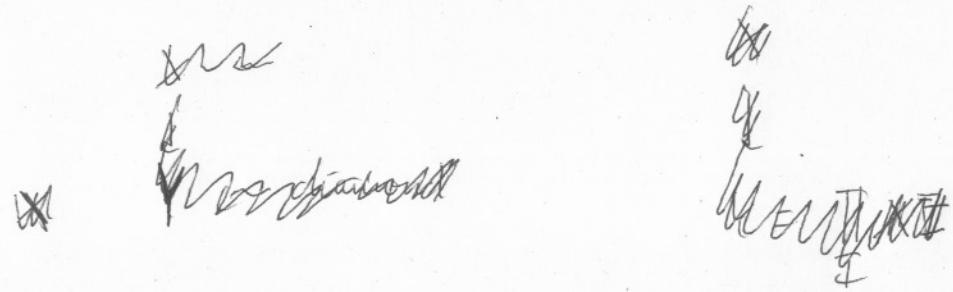
$$BC(O(\frac{1}{2})) \times BC(O(\frac{1}{2}))$$

SW: $M_{\infty, C} \subset \text{geom. pt over Spd}(\check{E}) \quad (C \cong C^\# \text{ unilt}/\check{E})$

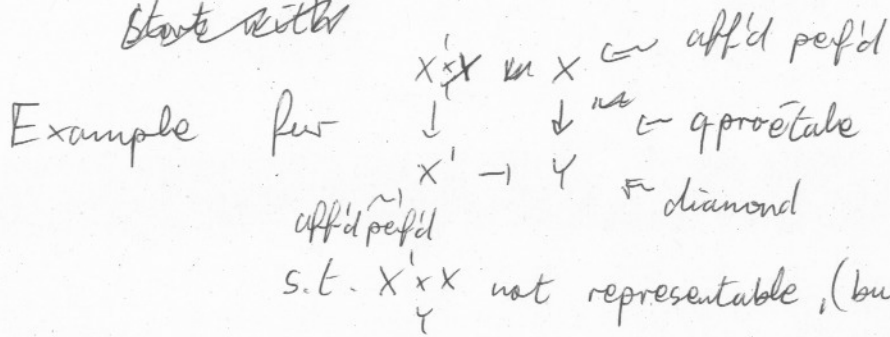
$$\Rightarrow M_{\infty, C} \cong (M_{\infty, C^\#})^{\text{LT}}$$

infinite level Lubin-Tate space

Ex:



Start with



after pullback to
 \hookrightarrow
 stel)

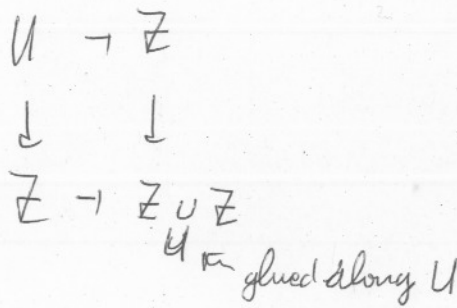
Start with Z aff'd perf'd, $U \subseteq Z$ q.c. not aff'd

Let



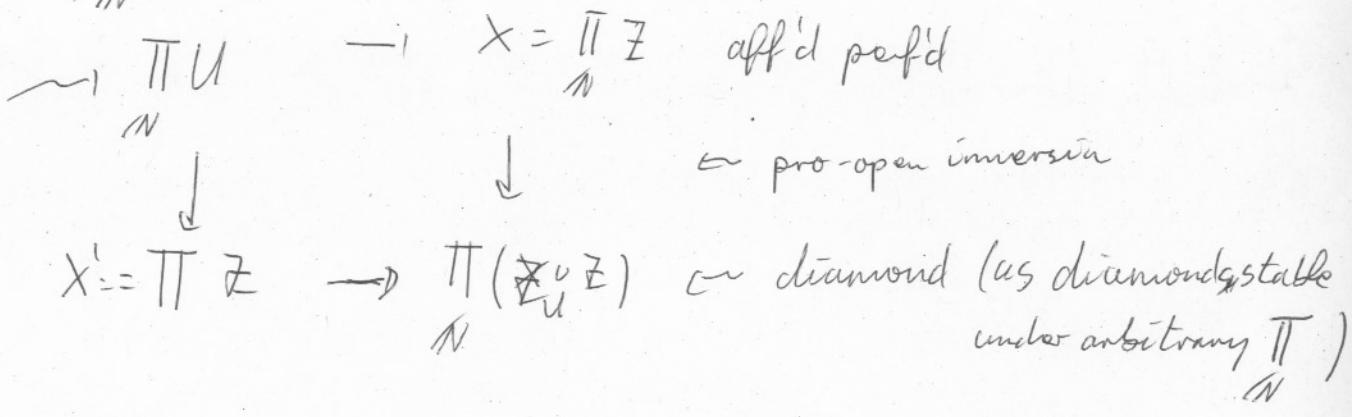
(E.g. $U = \{ |x| \leq 1 \} \cup \{ |y| \leq 1 \}$
 $\subseteq \text{Spa}(K \langle X^{\pm 1}, Y^{\pm 1} \rangle)$
 $Z =$

Consider



Take Π

not repr. by aff'd perf'd



$$\Pi U = \varinjlim_n \Pi U$$

\Rightarrow Each $V \subseteq \Pi U$ q.c. arises via pullback of some q.c. $W \subseteq \Pi U$

Take any ~~std.~~ ^{std.} $S \subseteq T$ (qc pro-open immersion)

$\Rightarrow S$ affinoid perf'd space.

Namely: enough $S \subseteq T$ ~~qc~~ ^{pro-constructible generalizing} (Et. cohom. La 7.6)

$\forall (T \text{ is affinoid, and } S = \bigcap_{f \in \mathcal{O}(T)} \{ |f| \leq 1 \})$

$S \subseteq \{ |f| \leq 1 \}$

$T \rightarrow \pi_0(T)$