Divisors $\qquad S = \mathrm{Spa}(C, \mathcal{O}_C)$

On $Y_{S,E}$: $\quad \mathrm{Div}^1_Y = \{ \sum a_y y$ locally finite, $a_y \in \mathbb{Z}, y \in Y^{cl}_{S,E} \}$

if $U \subseteq Y$ qc, $\{ a_y \mid y \in U, a_y \neq 0 \}$ finite

On $X_{S,E} = Y_{S,E}/\varphi^{\mathbb{Z}}$: $\quad \mathrm{Div}^1_X = \{ \sum a_x x$ finite sum, $a_x \in \mathbb{Z}, x \in X^{cl} \}$

$$\deg : \mathrm{Div}^1_X \to \mathbb{Z}, \quad \sum a_x x \mapsto \sum a_x$$

(i.e. $\deg(x) = 1$)

Want:
$$\begin{tikzcd} \mathrm{Div}^1_{X_{S,E}} \arrow[r, "\deg"] \arrow[d] & \mathbb{Z} \\ \mathrm{Pic}(X_{S,E}) \simeq \mathbb{Z} \arrow[ur, dashed, "\deg, \exists ?"'] & \end{tikzcd}$$

(Note: We don't introduce principal divisors)

Lecture: $x \in X^{cl}$ classical $\Rightarrow \mathcal{O}(x) \simeq \mathcal{O}(1)$

$\Rightarrow$ can ~~define~~ set $\deg(\mathcal{L}) = n$ if $\mathcal{L} \simeq \mathcal{O}(n)$

Different viewpoint: $X^{sch}_{S,E} = \mathrm{Proj}\Big(\overbrace{\bigoplus_{d \geq 0} H^0(X_{S,E}, \mathcal{O}(d))}^{=: P} \Big)$

1-dim. Noeth. schem, regular, closed points $\leftrightarrow X^{cl}$

$\Rightarrow \mathrm{Div}^1_{X^{sch}}$ defined, $\simeq \mathrm{Div}^1_X$

Pic: $x \in X^{cl}$, $\Rightarrow X^{sch} \setminus \{x\} \simeq \mathrm{Spec}\, B_e$,

$B_e := \left(P\left[\frac{1}{t}\right]\right)_0$, where $V(t) = \{x\}$, $t \in P_1$

PID

Each $f \in B_e \setminus \{0\}$ is a product $\frac{t_1 \cdots t_n}{t^n}$ with $t_1, \ldots, t_n, t \in P_1$

$v(\frac{t_1}{t}) = \mathbb{Z}$

$\Rightarrow \mathrm{Cl}_X = \mathbb{Z}$

Now, S arbitrary, $d \geq 0$

The following sets are in bijection

$\neq 0$ in each geometric point $s \to S$

1) $\{(\mathcal{L}, s) \mid \deg \mathcal{L} = d,\ s \in H^0(X_{S,E}, \mathcal{L}) \setminus \{0\}\}$

2) $\{ D \subseteq X_{S,E}$ closed Cartier divisor, s.t. D_s closed Cartier $\forall$ geom. pts $s \to S$ of deg. $d \}$

(if S affinoid)

3) $\{ D \subseteq X^{sch}_{S,E} \mid$ Cartier divisor, s.t. D_s Cartier for all $s \to S$ geom. pts, $\deg D_s = d \}$

This defines v-sheaf $\mathrm{Div}^d \to \mathrm{Spd}\,\overline{\mathbb{F}}_q$, $\mathbb{F}_q$ = res. fld of E

E.g.: * $\mathrm{Div}^1 = BC(\mathcal{O}(1))^* / \underline{E}^* \cong \mathrm{Spa}(\overline{\mathbb{F}}_q((t^{1/p^\infty}))) / \underline{E}^*$ spatial diamond

top. space has 1 point

$\cong \mathrm{Spa}\, E_\infty / \underline{E}^* \cong \mathrm{Spa}\, \breve{E} / \varphi^{\mathbb{Z}}$

* $\mathrm{Div}^d = \cancel{(\mathrm{Div}^1)^d / S_d}\ H^0(X, BC(\mathcal{O}(d)))^* / \underline{E}^* \cong (\mathrm{Div}^1)^d / S_d$

very strange!

Ex.

$(C, \mathcal{O}_C)$ geom. pt.

$\Rightarrow \mathrm{Div}^1_C \cong \mathbb{D}^*_C / \underline{E}^*$ ($\mathbb{D}^*_C$ ~ perfectoid open unit disk)

$C = \widehat{\overline{E}}$

$\mathrm{Div}^1 \times \mathrm{Div}^1 \cong \mathrm{Spd}\, \breve{E}/\varphi^{\mathbb{Z}} \times \cancel{\mathrm{Spd}\,\breve{E}} \cong (\mathrm{Spa}(C) \times \mathrm{Div}^1) / \mathrm{Gal}(C/\breve{E})$

In part, Div^2 has ~~many points~~ non trivial geometry.

Another example

$$BC(\mathcal{O}(\tfrac{1}{q}))^* \simeq \mathrm{Spd}(\mathbb{F}_q[[t^{\frac{1}{q^\infty}}]])$$

$\mathcal{M}_\infty := \{ \mathcal{O}^2 \overset{\alpha}{\hookrightarrow} \mathcal{O}(\tfrac{1}{2}) \}$ on $\mathrm{Perf}_{\mathbb{F}_q}$

fiberwise injective maps

$\Rightarrow$ $\mathcal{M}_\infty \to \mathrm{Div}_1^{\vee} = \mathrm{Spd}\,\breve{E}$, $\alpha \mapsto V(\det(\alpha))$

$\cap$ open

$BC(\mathcal{O}(\tfrac{1}{2}))^* \times BC(\mathcal{O}(\tfrac{1}{2}))^*$

SW: C geom. pt over $\mathrm{Spd}(\breve{E})$ $(\hat{=}\ C^\#$ untilt$/\breve{E})$

$\Rightarrow \mathcal{M}_{\infty, C} \simeq (\mathcal{M}^{LT}_{\infty, C^\#})^\diamond$

infinite level Lubin-Tate space

Ex:

④

~~Start with~~

Example for
$$\begin{tikzcd} X' \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "\text{qpro}"] \\ X' \arrow[r] & Y \end{tikzcd}$$
X ← aff'd perf'd; $X \to Y$ ← qproétale; Y ← diamond; X' aff'd perf'd

s.t. $X' \times_Y X$ not representable, (but repr. ~~if~~ after pullback to $X'' \to X'$ ← std)

Start with Z aff'd perf'd, $U \subseteq Z$ qc. not aff'd

(E.g. $U = \{|x| \leq |\pi|\} \cup \{|y| \leq |\pi|\} \subseteq Z = \mathrm{Spa}(K\langle x^{1/p^\infty}, y^{1/p^\infty}\rangle)$)

~~Set~~

Consider
$$\begin{tikzcd} U \arrow[r] \arrow[d] & Z \arrow[d] \\ Z \arrow[r] & Z \cup_U Z \end{tikzcd}$$
$Z \cup_U Z$ ← glued along U

Take $\prod_{\mathbb{N}}$:

$$\begin{tikzcd} \prod_{\mathbb{N}} U \arrow[r] \arrow[d] & X = \prod_{\mathbb{N}} Z \arrow[d] \\ X' := \prod Z \arrow[r] & \prod_{\mathbb{N}} (Z \cup_U Z) \end{tikzcd}$$

$\prod_{\mathbb{N}} U$ ← not repr. by aff'd perf'd

$X = \prod_{\mathbb{N}} Z$ ← aff'd perf'd

$X \to \prod_{\mathbb{N}} (Z \cup_U Z)$ ← pro-open immersion

$\prod_{\mathbb{N}} (Z \cup_U Z)$ ← diamond (as diamonds stable under arbitrary $\prod_{\mathbb{N}}$)

$$\prod_{\mathbb{N}} U = \varprojlim_n \prod_{i=1}^{n} U$$

$\Rightarrow$ Each $V \subseteq \prod_{\mathbb{N}} U$ q.c. arises via pullback of some qc $W \subseteq \prod_{i=1}^{n} U$

Take any ~~std~~ std. T , $S \subseteq T$ pro-open immersion (qc)

$\Rightarrow$ S affinoid perf'd space.

Namely: enough $S \subseteq T$ ~~qc pro-open immersion~~ pro-constructible generalizing (Et. cohom. La 7.6)

(T ~~= Spa (~~ is affinoid, and $S = \bigcap_{\substack{f \in \mathcal{O}(T) \\ S \subseteq \{|f| \leq 1\}}} \{|f| \leq 1\}$)

$T \to \pi_0(T)$