From Hecke eigenforms to Hecke eigensheaves

①

$X/_{\mathbb{F}_q}$ proj, smooth, geom. conn. curve, $\ell \neq p$

$$K = K(X), \quad \hat{\mathcal{O}} := \prod_{\substack{x \in |X| \\ \mathrm{cld}}} \hat{\mathcal{O}}_{X,x},$$

$$\mathbb{A} := \prod_{\substack{x \in |X| \\ \mathrm{cld}}}{}' K_{X,x}, \quad K_{X,x} = \mathrm{Frac}(\hat{\mathcal{O}}_{X,x})$$

$G/_K$ reductive

Recall: In the LP for K, G one tries to decompose the space ②

$$\mathcal{A}\left(G(K)\backslash G(\mathbb{A})\right) \subseteq C^{\infty}\left(G(K)\backslash G(\mathbb{A}) / \widehat{\mathbb{Q}}_c\right)$$

as a $G(\mathbb{A})$-repr. accord. to arith. data

In part, assume G defined over $\mathbb{F}_q$ & split

$\Rightarrow$ $U := G(\hat{\mathcal{O}}) \leq G(\mathbb{A})$ cpct, open subgrp

Partial Aim: Decompose

$$\mathcal{A}\left(G(K)\backslash G(\mathbb{A})/U\right) \subseteq C^{\infty}\left(G(K)\backslash G(\mathbb{A})/G(\hat{\mathcal{O}})\right)$$

according to the action

of the global unram. Hecke alg ③

$\mathcal{H}(G(\mathbb{A}), U) := \{ f : G(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell \mid$ f biinvariant under U, f cpct support$\}$

$\simeq \{ \bar{f} : U \backslash G(\mathbb{A}) / U \to \overline{\mathbb{Q}}_\ell \mid$ $\bar{f}$ finite supp$\}$

The $\mathcal{H}(G(\mathbb{A}), U)$ action is dfnd via Hecke correspondences

$$\begin{tikzcd}
 & U\backslash G(\mathbb{A})/U & \\
 & G(k)\backslash G(\mathbb{A})\overset{U}{\times} G(\mathbb{A})/U \arrow[u, "\alpha"] \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\
G(k)\backslash G(\mathbb{A})/U & & G(k)\backslash G(\mathbb{A})/U
\end{tikzcd}$$

U acts via $u(g,h) = (gu^{-1}, uh)$

$$\overleftarrow{h}(\overline{g,h}) = \overline{g}$$

$$\overrightarrow{h}(\overline{g,h}) = \overline{gh}$$

$$\alpha(\overline{g,h}) = \overline{h}$$

For $f \in C^{\infty}(G(k)\backslash G(\mathbb{A})/U, \overline{\mathbb{Q}}_\ell)$

$\quad \varphi \in \mathcal{H}(G(\mathbb{A}), U)$

$\Rightarrow \varphi * f := \overrightarrow{h}_!(\overleftarrow{h}^* f \cdot \alpha^* \varphi)$

$= \left(g \mapsto \int_{G(\mathbb{A})} \varphi(h) \cdot f(gh^{-1}) dh \right)$

Recall: 1) $\mathcal{H}(G(\mathbb{A}), U) = \bigotimes'_{\substack{x \in |X| \\ \text{cld}}} \mathcal{H}(G(K_{x,v}), G(\hat{\mathcal{O}}_{x,v}))$

2) $\mathcal{H}(G(K_{x,v}), G(\mathcal{O}_{x,v}))$

"Satake" $\cong K_0(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} \hat{G})$ (e.g. $\hat{G} = GL_n$ if $G = GL_n$)

commutative

$\Rightarrow \forall x \in |X|$ cld, $V \in \mathrm{Rep}\, \hat{G}$

get Hecke operator

$T_{V,x} : C^\infty(G(K) \backslash G(\mathbb{A}) / U, \overline{\mathbb{Q}}_\ell) \circlearrowleft$

Assume $G = GL_n$

$$\rho : Gal(\bar{K}/K) \to \hat{G}(\bar{\mathbb{Q}}_\ell) = GL_n(\bar{\mathbb{Q}}_\ell)$$

irred, everywhere unram.

($\leadsto$ ρ factors $\pi_1^{et}(X, x)$)

$\overset{LP}{\Longrightarrow}$ (Lafforgue Drinfeld)

$$\exists!\ 0 \neq f \in \mathcal{A}(G(K)\backslash G(\mathbb{A})/U),$$

s.t. $\forall x \in |X|$, $V \in Rep\,\hat{G}$

$$T_{V,x} f = tr(\rho(Fr_x)|V) \cdot f$$

Fr_x Frob. at x

$$\rho(Fr_x) \in \hat{G}(\bar{\mathbb{Q}}_\ell) \xrightarrow{r_V} GL(V)$$

⚠ $\mathbb{G}$ $G(K)\backslash G(\mathbb{A})/U = G(\hat{\mathcal{O}}) \simeq \mathrm{Bun}_n(\mathbb{F}_q)$

$\underline{G = Gl_n}$

stack of rk n v.b. on X

$(S \mapsto \{$v.b. on $X \times S\})$

Sketch:

$\xi \in \mathrm{Bun}_n(\mathbb{F}_q)$

$\underset{G=Gl_n}{\Rightarrow}$ ξ_η, $\hat{\xi}_{X,x}$ trivial $\forall x \in X$ cl

Choose $\alpha_\eta : \mathcal{O}_\eta^n \xrightarrow{\sim} \xi_\eta$ $\Leftrightarrow \hat{\xi} = \xi \underset{\mathcal{O}_X}{\otimes} \hat{\mathcal{O}}$ trivial

$\hat{\alpha} : \hat{\xi} \xrightarrow{\sim} \hat{\mathcal{O}}^n$

$\underset{\substack{\text{b.c.}\\ \text{to } \mathbb{A}}}{\Rightarrow}$ $\mathbb{A}^n \xrightarrow[\alpha_\eta]{\sim} \xi \underset{\mathcal{O}}{\otimes} \mathbb{A} \xrightarrow[\hat{\alpha}]{\sim} \mathbb{A}^n$

$\mathbb{A} \supseteq K \underset{\mathcal{O}}{\otimes} \hat{\mathcal{O}}$

$\Rightarrow (\hat{\iota}\circ\alpha_\eta)^{-1} \in G(\mathbb{A})$

& the class of $(\hat{\iota}\circ\alpha_\eta)^{-1}$

in $G(K)\backslash G(\mathbb{A})/G(\hat{\mathcal{O}})$ only

depends on ξ

Grothendieck's sheaf-fct dict.

repl. $C^\infty(G(K)\backslash G(\mathbb{A})/U, \bar{\mathbb{Q}}_\ell)$

by $D(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell) \ni \mathcal{F} \mapsto (y \in \mathrm{Bun}(\mathbb{F}_q) \mapsto \mathrm{tr}(\mathrm{Fr}_q | \mathcal{F}_{\bar{y}}))$

bdd constr.

Q: How to geom. $T_{V,x}$ for $x \in |X|$, $V \in \mathrm{Rep}\,\hat{G}$?

$$
\begin{tikzcd}
 & G(K)\backslash G(\mathbb{A}) \overset{U}{\times} G(\mathbb{A})/U \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "\overrightarrow{h}"] \\
G(K)\backslash G(\mathbb{A})/U & G(K)\backslash G(\mathbb{A})/U
\end{tikzcd}
$$

$$G(K)\backslash G(\mathbb{A})/U \cong \mathrm{Bun}_n(\mathbb{F}_q)$$

not geom.

$$
\begin{tikzcd}
 & \simeq \{ (\xi, \xi', \iota) \mid \xi, \xi' \in \mathrm{Bun}_n(\mathbb{F}_q),\ \iota : \xi_\eta \xrightarrow{\sim} \xi'_\eta \} \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "\overrightarrow{h}"] \\
\xi & \xi'
\end{tikzcd}
$$

Instead, set

$$\mathcal{H}k := \{(\xi, \xi', c, x) \mid \xi, \xi' \in \mathrm{Bun}_n(S),\ x \in X(S),\ c : \xi|_{X_S \setminus \Gamma_x} \simeq \xi'|_{X_S \setminus \Gamma_x}\}$$

$$\begin{tikzcd}
 & \mathcal{H}k \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\
\mathrm{Bun}_n & & \mathrm{Bun}_n \times X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\xi, \xi', c, x) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\xi & & (\xi', x)
\end{tikzcd}$$

Beauville-Laszlo

$\Rightarrow$ fiber over (ξ', x) of $\overrightarrow{h}$

is an affine Grassmannian

More precise,

$$\mathcal{H}k \simeq \mathcal{J} \times^{L^+G_X} \mathrm{Gr}_{G,X}$$

$$\mathcal{J} := \{(\mathcal{E}', x, \gamma : \hat{\mathcal{E}}'_{X,x} \simeq \hat{\mathcal{O}}^n\}$$

$$\Big\downarrow \quad L^+G_X := \underline{\mathrm{Aut}}(\hat{\mathcal{O}}^n) \text{-torsor} \quad (\text{"}G(k(x)[[t_x]])$$

$$\mathrm{Bun}_n \times X$$

$$\mathrm{Gr}_{G,X} = \{(\mathcal{E}, \iota, x) \mid \mathcal{E} \in \mathrm{Bun}_n,\ x \in X,\ \iota : \mathcal{E}|_{X \setminus x} \simeq \mathcal{O}^n_{X \setminus x}\}$$

BD-Grassmannian

More concr., fix $x \in |X|$ cld,

Isom. $\hat{\mathcal{O}}_{X,x} \simeq k((t))$

$$\Rightarrow \quad x \times_X \mathrm{Gr}_{G,X} \simeq \mathrm{Gr}_n = LG/L^+G$$

$$LG(R) := G(R((t))) \supseteq L^+G(R) = G(R[[t]])$$

R k-alg

Alternatively,

$$Gr_n(R) = \{ \Lambda \subseteq R((t))^n,\ \Lambda \text{ fin. proj. } R[[t]]\text{-mod, s.t. } R((t)) \otimes_{R[[t]]} \Lambda \cong R((t))^n \}$$

⚠ Geometric Satake

$\exists$ can. isom.

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} \hat{G} \cong \mathrm{Perv}_{L^+G}(\underset{\substack{\wr \\ \text{ind-scheme} \\ \text{proj.}}}{Gr_n})$$

$$V \longmapsto \mathrm{Sat}(V)$$

" $\hat{=}$ k sheaves on $L^+G \backslash LG / L^+G$ with cpct support "

$\Rightarrow$ Get Hecke operators T_V, $V \in \mathrm{Rep}\,\check{G}$

Consider

$$
\begin{tikzcd}
 & \mathcal{H}k \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times X
\end{tikzcd}
$$

$\mathcal{F} \in D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$

$T_V(\mathcal{F}) = \overrightarrow{h}_!\left(\overleftarrow{h}^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathrm{Sat}(V)}\right)$

$\uparrow$

$D(\mathrm{Bun}_G \times X, \overline{\mathbb{Q}}_\ell)$ relative version of $\mathrm{Sat}(V)$

($\rightsquigarrow$ analogs for $T_{V,x}$ by restr. to a $\mathrm{Bun}_G \times \{x\}$)

For $V = \Lambda^i(V_{st})$ this spec.

V ∈ $\operatorname{Rep} \hat{G}$ ∈ $\operatorname{Rep} Gl_n$

V_{st}: stand. repr.

$\subseteq \xi'(x)$

$$\begin{tikzcd}[column sep=small] & \{(\xi, \xi', L, x) \mid \xi \subseteq \xi' \subseteq \xi(x), \ \xi'/\xi \text{ [redacted] of rk } i\} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \times X \end{tikzcd}$$

rel. Grassm. $Gr($[redacted]$n, i)$

$$\widetilde{Sat}(V) = \overline{\mathbb{Q}}_e[\dim L]$$
$$= \overline{\mathbb{Q}}_e[i(n-1-i)]$$

Q: What is an Hecke eigensheaf?

Want "$T_{V,x} f = tr(\rho(Fr_x)|V) \cdot f$"

$\rho : \pi_1^{et}(X,x) \to \hat{G} = Gl_n(\bar{\mathbb{Q}}_\ell)$

$\hat{G} \xrightarrow{r_V} GL(V)$, $V \in Rep\,\hat{G}$

Under sheaf-fct dict.

"$tr(\rho(Fr_x)|V) \cdot f$" $\hat{=}$ $V \otimes \mathcal{F}_x$

$\circlearrowleft$ Fr_x via ρ

Note:

$\rho \;\hat{=}\; \mathbb{L}_\rho$ $\bar{\mathbb{Q}}_\ell$ - loc. system

Def: A Hecke eigensheaf with eigenvalue $\mathbb{L}_\rho$ is an object $\mathcal{F} \in D(Bun_G, \bar{\mathbb{Q}}_\ell)$

together with isoms.

$$\gamma_V : T_V(\mathcal{F}) \simeq \mathcal{F} \boxtimes \underbrace{r_{V*}\mathbb{L}_\sigma}$$

in $D(\mathrm{Bun}_{n,k}\times X, \overline{\mathbb{Q}}_\ell)$

s.t. γ_V satisfy various comp.

$\overline{\mathbb{Q}}_\ell$-loc. sys. corresp.

$$\pi_1(X,x) \xrightarrow{\sigma} \underset{\hat{G}(\overline{\mathbb{Q}}_\ell)}{\overset{\|}{GL_n(\overline{\mathbb{Q}}_\ell)}} \xrightarrow{r_V} GL(V)$$

Thm (Frenkel–Gaitsgory–Vilonen)

k of char $p>0$, X/k sm. proj. curve, geom. conn.

$\forall$ irred $\overline{\mathbb{Q}}_\ell$-loc. system of rk n

$\exists$ non-zero perverse weak (= less comp. for γ_V) Hecke eigensheaf $\mathcal{F}$ with eigenvalue L.

$\mathcal{F} \in D(\mathrm{Bun}_G)$

actually, ...

$$\begin{tikzcd}
 & Z \arrow[dr] & \\
Y' \arrow[ur] \arrow[rr, "g'"] \arrow[d] & & Y \arrow[d, "qcqs"'] \\
X' \arrow[rr] \arrow[d, equal] & & X \arrow[d, equal] \\
\mathrm{Spa}(C', C'^+) & & \mathrm{Spa}(C, C^+)
\end{tikzcd}$$

$$Y'_v \to Y_v$$

$$\begin{tikzcd}
\underline{I} \times \mathrm{Spa}(C, C^+) \arrow[d] \\
\underline{S} \times \mathrm{Spa}(C, C^+)
\end{tikzcd}$$

$$\mathcal{F} \to Rg'_* g'^* \mathcal{F}$$

$$D^+_{et}(Y_v, \Lambda) \subseteq D^+(Y_v, \Lambda)$$

Y loc. spat. diam.

$$D^+_{*}(Y_{\acute{E}t}, \Lambda) \to D^+(Y_v, \Lambda)$$

$$\begin{array}{c} \cup| \\ D^+_{\acute{e}t}(Y, \Lambda) \end{array}$$

$$D^+_{*}(Y_{\acute{E}t}, \Lambda) \xrightarrow{\sim} D^+_{\acute{e}t}(Y, \Lambda)$$

$$\begin{array}{ccc} & & Y_{pro\acute{e}t} \\ & \nearrow & \downarrow \\ \pi : Y_v & \to & Y_{\acute{E}t} \\ & & \| \\ & & \{ Y' \xrightarrow{\acute{e}tale} Y \} \\ & & \xi \end{array}$$

Y ~~[illegible] loc. system on~~ F

$\downarrow$ ~~covering~~ cover $\to$ $|$

~~Y~~ X

~~X~~ X top. space, $\mathbb{Q}$-loc. sys. on X is a sheaf F of $\mathbb{Q}$-v.s. s.t. $\exists$ covering of X, $\cup U_i$ s.t. $F_{|U_i} \simeq \mathbb{Q}^n$

$\mathbb{Z}_\ell$-loc. system.

$\cong$ comp. system.

of $\mathbb{Z}/\ell^n$ - loc. systems

$\cong$

étale sheaves of

$\mathbb{Z}/\ell^n$-modules, loc. isom

to $\left(\mathbb{Z}/\ell^n\right)^m$

$\mathcal{F}$

$\downarrow$

X

$\simeq$ repr. of $\pi_1^{ét}(X, x)$

on free, finite $\mathbb{Z}/\ell^n$-

modules

Best hope:

$$\mathrm{QCoh}(LS_n) \overset{\sim}{=} D(\mathrm{Bun}_G, \bar{\Phi}_e)$$

comp. with Hecke action