

From Hecke eigenforms to
Hecke eigen sheaves

①

X/\mathbb{F}_q proj, smooth, geom. conn.
curve, $l \neq p$

$$K = K(x), \hat{\Theta} = \prod_{\substack{x \in |x| \\ \text{cl}}} \hat{\mathcal{O}}_{x,x}$$

$$A := \prod_{\substack{x \in |A| \\ \text{cl}}} K_{x,x}, \\ K_{x,x} = \text{Frac}(\hat{\mathcal{O}}_{x,x})$$

G/K reductive

Recall: In the LP for K, G
one tries to

decompose the space

(2)

$$\mathcal{A}(G(k) \backslash G(A)) \cong C^\infty(G(k) \backslash G(A) / \hat{\mathcal{O}}_e)$$

as a $G(A)$ -repr. accord.
to arith. data

In part, assume G ^{sp} defined over
 \mathbb{F}_q & split

$\Rightarrow U := G(\hat{\mathcal{O}}) \subseteq G(A)$ cpet, open
subgroup

Partial

Aim: Decompose

$$\mathcal{A}(G(k) \backslash G(A) / U) \cong C^\infty(G(k) \backslash G(A) / G(k))$$

according to the action

of the global unram. Hecke alg $\textcircled{3}$

$$\mathcal{H}(G(A), U) := \{ f: G(A) \rightarrow \overline{\mathbb{Q}_e} \mid$$

f biinvariant under

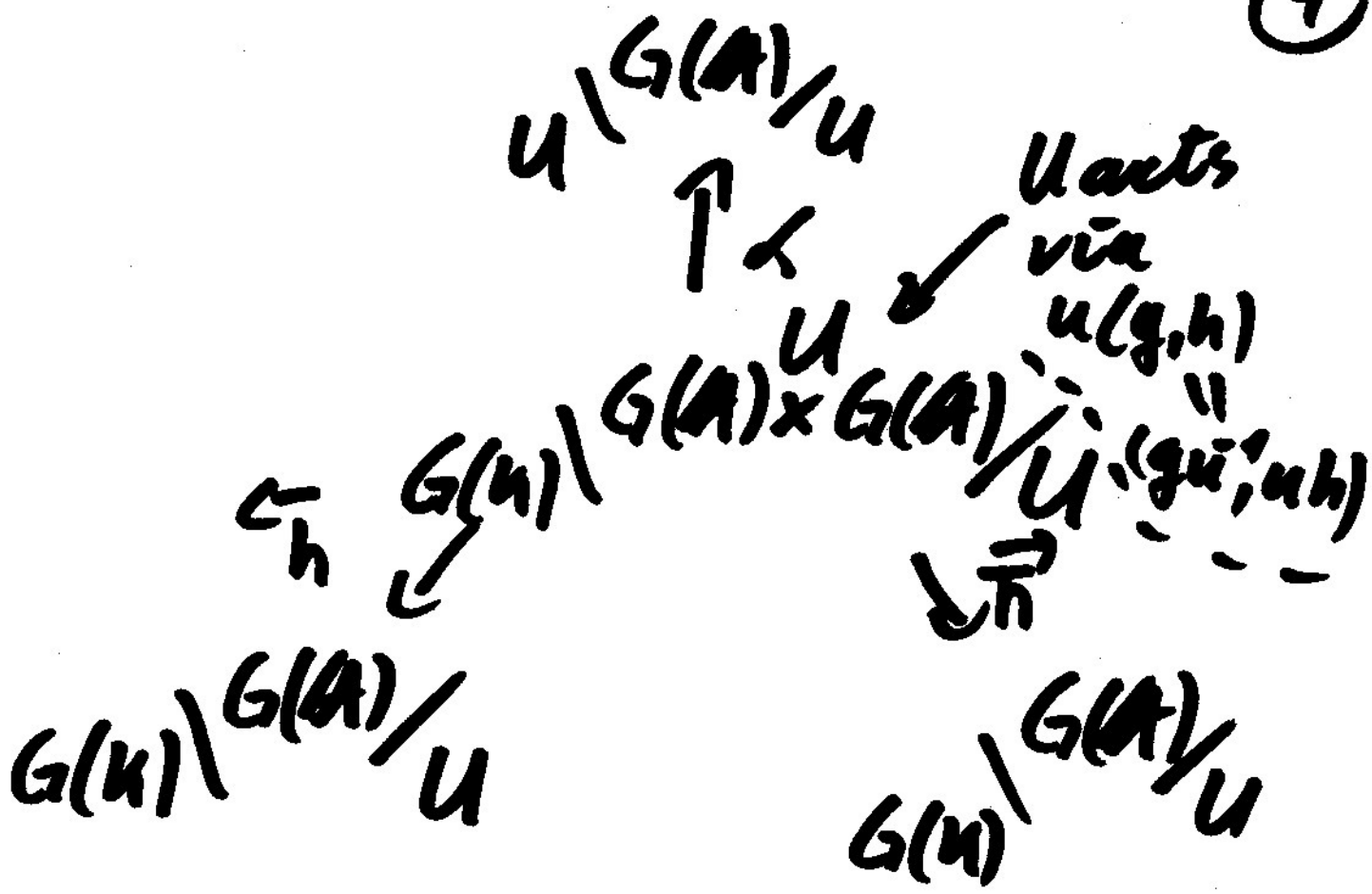
U , f cpt support $\}$

$$\simeq \{ \bar{f}: U \backslash G(A) / U \rightarrow \overline{\mathbb{Q}_e} \mid$$

\bar{f} finite supp $\}$

The $\mathcal{H}(G(A), U)$ action is defnd
via Hecke correspondences

(4)



$$\bar{h}(g, h) = \bar{g}$$

$$\vec{h}(g, h) = \overline{gh}$$

$$\alpha(g, h) = \bar{h}$$

For $f \in C^\infty(G(k) \backslash G(A) / U, \bar{Q}_e)$

$$\varphi \in \mathcal{H}(G(A), U)$$

$$\Rightarrow \varphi * f := \int_{G(A)} (\int_{G(A)} \varphi(h) \cdot f(gh^{-1}) dh) \quad (5)$$

$$= \int_{G(A)} \varphi(h) \cdot f(gh^{-1}) dh$$

Recall: 1) $\mathcal{H}(G(A), U) = \bigoplus'_{x \in |X|_{\text{cl}}} \mathcal{H}(G(U_{x,v}), G(\hat{O}_{x,v}))$

2) $\mathcal{H}(G(U_{x,v}), G(\hat{O}_{x,v}))$

"Satake" $\cong K_0(\text{Rep}_{\hat{O}_e} \hat{G})$ (e.g. $\hat{G} = GL_n$ if $G = GL_n$)

commutative \nearrow

$\Rightarrow \forall x \in |X|_{\text{cl}}, V \in \text{Rep } \hat{G}$

get Hecke operator

$$T_{V,x}: C^\infty(G(U) \backslash G(A) / U, \hat{O}_e) \rightarrow$$

Assume $G = GL_n$

⑥

$$\rho: \text{Gal}(\bar{K}/K) \rightarrow G^{\wedge(\bar{\mathcal{O}}_e)} = GL_n(\bar{\mathcal{O}}_e)$$

irred, everywhere unram.

$$(\sim) \rho \text{ factors } \pi_x^{\text{ét}}(X, x)$$

LP

=>

(Lafforgue
Drinfeld)

$$\exists! 0 \neq f \in \mathcal{A}(G(K) \backslash G(\mathbb{A})/u),$$

$$\text{s.t. } \forall x \in |X|, \forall V \in \text{Rep } \hat{G}$$

$$T_{V, x} f = \text{tr}(\rho(F_x)|V) \cdot f$$

F_x Frob. at x

$$\rho(F_x) \in G(\bar{\mathcal{O}}_e) \xrightarrow{V} GL(V)$$

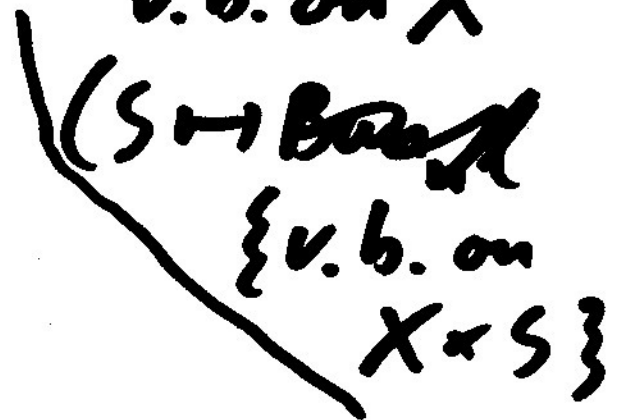


$$G(k) \backslash G(A) / U = G(\hat{O}) \cong \text{Bun}_n(\mathbb{F}_q)$$

$G = GL_n$

stack of rk n
v. b. on X

Sketch:



$$\xi \in \text{Bun}_n(\mathbb{F}_q)$$

\Rightarrow $\xi_\eta, \hat{\xi}_{X,x}$ trivial $\forall x \in X$

Choose $\alpha_\eta : \mathcal{O}_\eta^n \cong \xi_\eta$

$\hat{\alpha} : \hat{\xi} \cong \hat{\mathcal{O}}^n$

$\hat{\xi} = \xi \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}^n$
trivial

\Rightarrow b.c. to A
 $A^n \xrightarrow{\alpha_\eta} \xi \otimes_{\mathcal{O}} A \xrightarrow{\hat{\alpha}} A^n$

$K \otimes_{\mathbb{F}} \hat{\mathcal{O}}$

$$\Rightarrow (\hat{\alpha} \circ \tau \alpha \gamma)^{-1} \in G(A)$$

outside

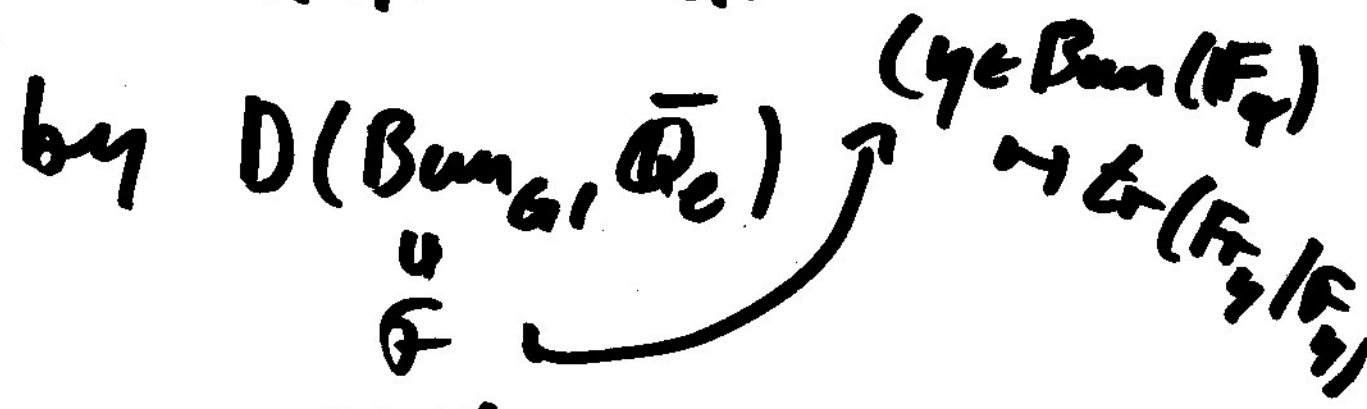
& the class of $(\hat{\alpha} \circ \tau \alpha \gamma)^{-1}$

in $G(U) \setminus G(A) / G(\hat{O})$ only

depends on ξ

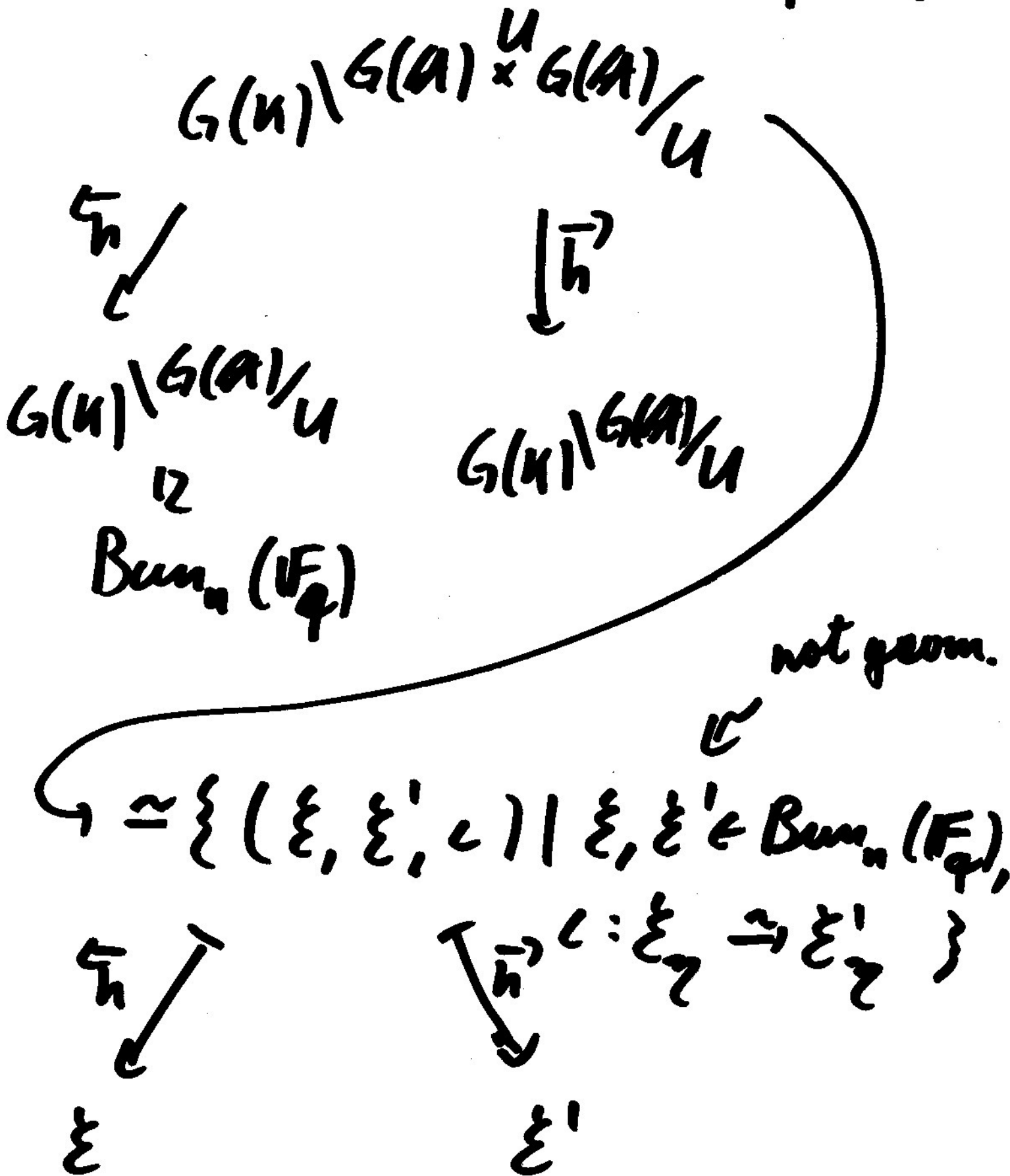
Grothendieck's sheaf-fit dict.

$$\text{repl. } C^\infty(G(U) \setminus G(A) / U, \bar{O}_e)$$



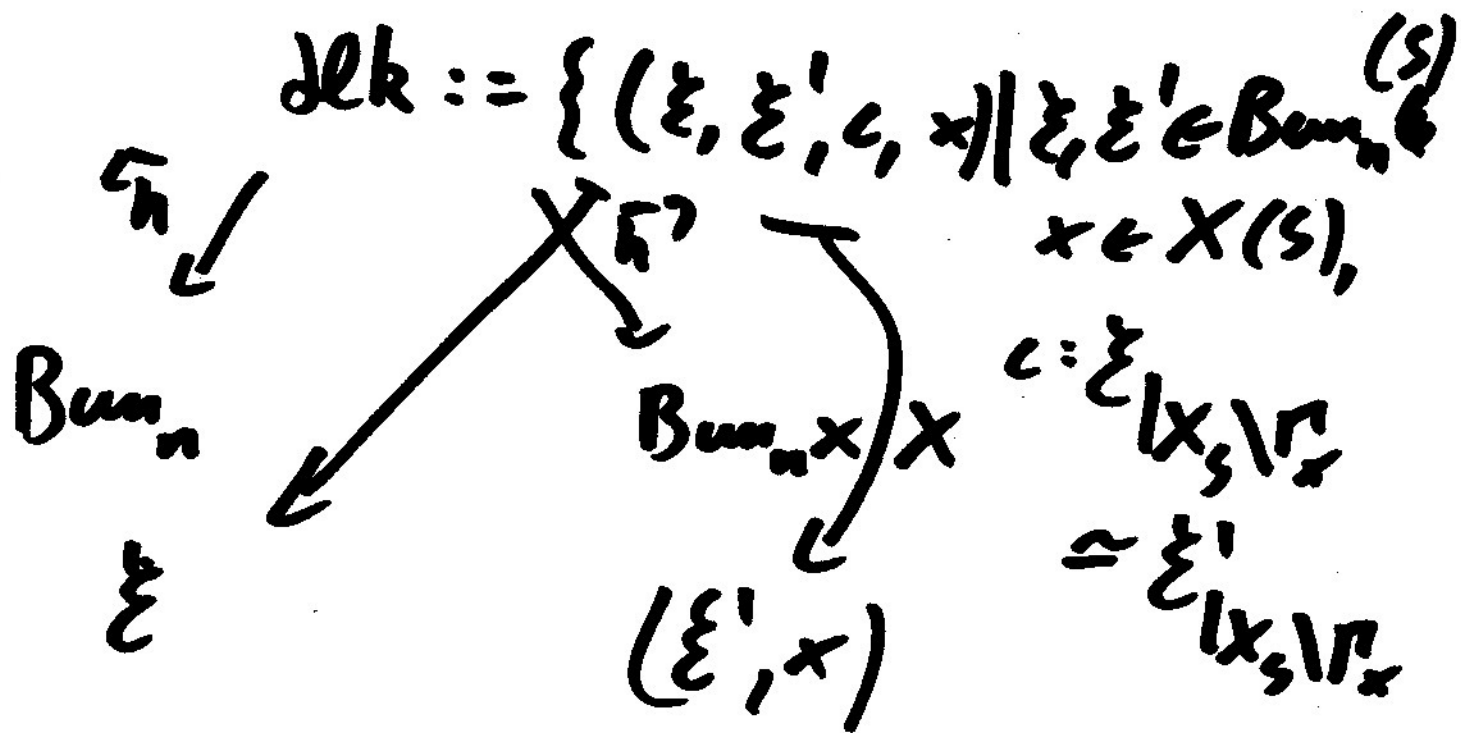
const
bdd const.

Q: How to geom. $T_{V,x}$ for $x \in |X|$,
 $V \in \text{Rep } \mathcal{B}$? ⑨



Instead, set

(10)



Beauville-Laszlo

\Rightarrow fiber over (ξ', x) of h'

is an affine Grassmannian

More precise,

$$\mathcal{L}k \simeq \mathcal{Y}_x^+ \times_x \text{Gr}_{G, X}$$

$$\mathcal{Y} := \{(\xi, \lambda, x, \gamma : \hat{\xi}_{\lambda, x} \cong \hat{\mathcal{O}}^n)\} \quad (1)$$

$$\downarrow \mathcal{L}^+ G_x := \underline{\text{Aut}}(\hat{\mathcal{O}}^n) \quad ("G(k(x)[[t]])$$

Bun_n × X - torsor

$$\text{Gr}_{G, X} = \{(\xi, \lambda, x) \mid \xi \in \text{Bun}_n, x \in X, \mathcal{L} : \xi|_{X \times \text{pt}} \cong \mathcal{O}^n\}$$

BD-Grassmannian

More concr., fix $x \in X$ old,

$$\text{Isom. } \hat{\mathcal{O}}_{\lambda, x} \cong k[[t]]$$

$$\Rightarrow \begin{matrix} x \times X & \text{Gr}_{G, X} & \cong & \text{Gr}_n & \cong & \text{LG} / \text{L}^+ G \end{matrix}$$

(Note: The diagram above is heavily crossed out with scribbles. The scribbles contain the text "L^+ G" and "L G".)

$$\text{LG}(R) := G(R[[t]]) \cong \text{L}^+ G(R)$$

R k-alg \cong $G(R[[t]])$

Alternatively,

(5)

$$\text{Gr}_n(R) = \left\{ \mathcal{L} \subseteq R((t))^n, \mathcal{L} \text{ fin. proj. } R[[t]]\text{-mod,} \right. \\ \left. \text{s.t. } R((t)) \otimes_{R[[t]]} \mathcal{L} \cong R((t))^n \right\}$$

⚠ Geometric Satate

∃ can. isom.

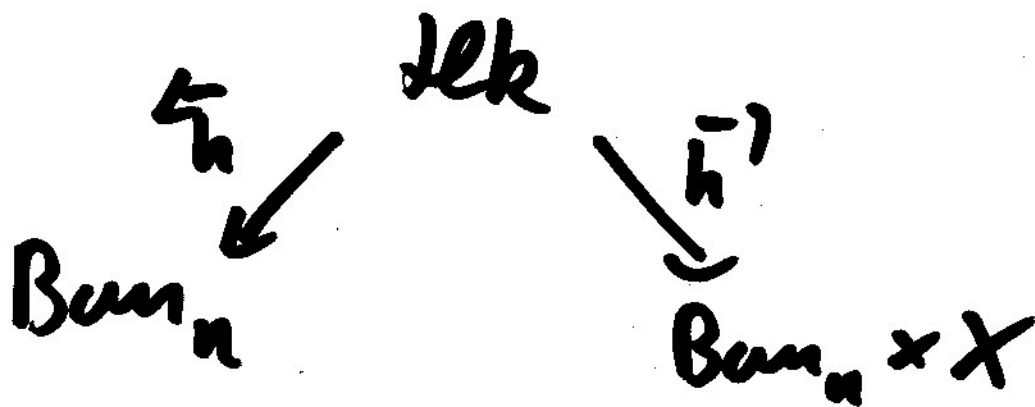
$$\text{Rep}_{\mathbb{Q}_e} \hat{G} \cong \text{Per}_{L^+G}(\text{Gr}_n)$$

$V \mapsto \text{Sat}(V)$ ind-scheme
sheaves prof.

" \cong k fltz on $L^+G \setminus L^+G/L^+G$
with cpet support "

=) Get Hecke operators T_V , $V \in \text{Rep } \bar{G}$ (13)

Consider



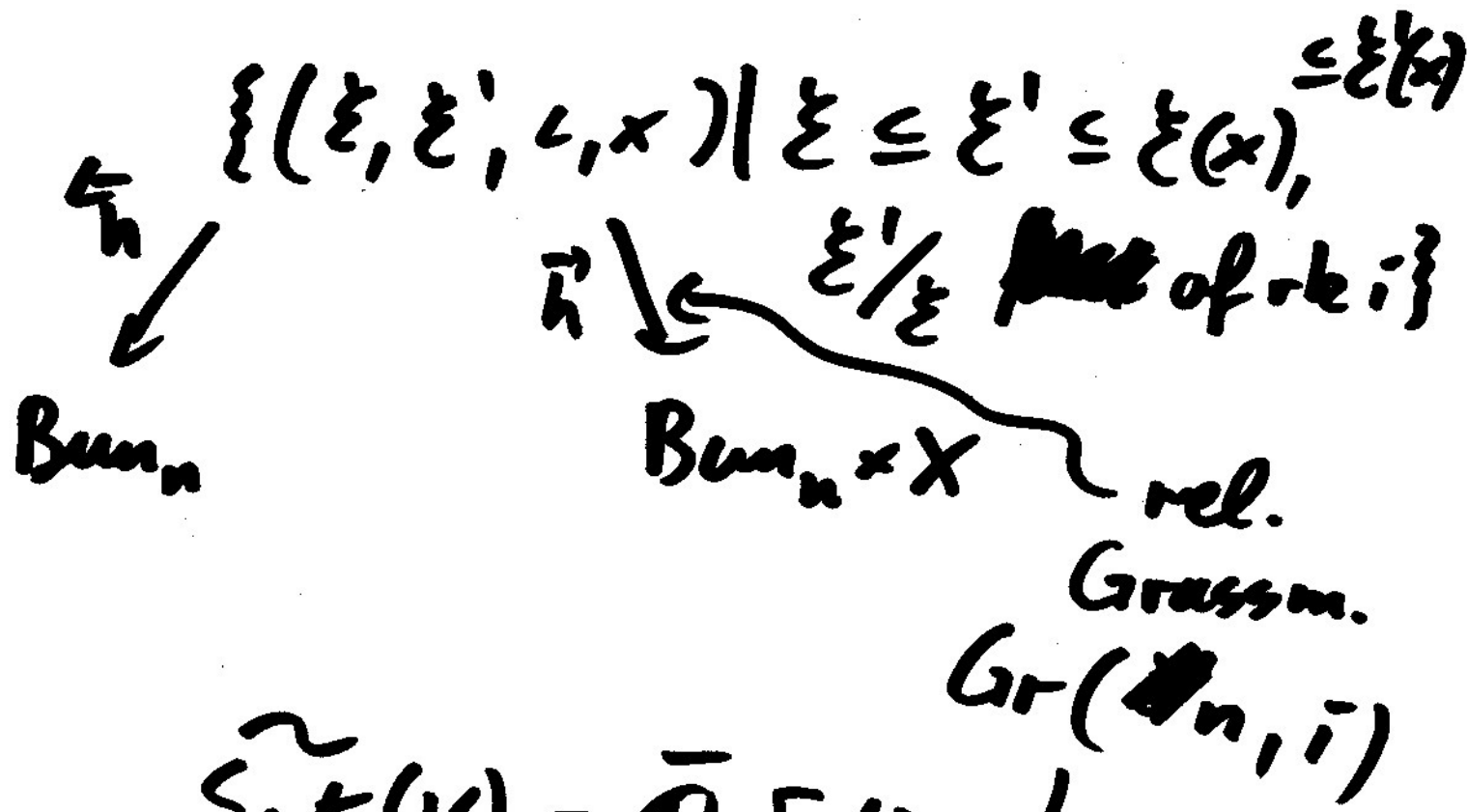
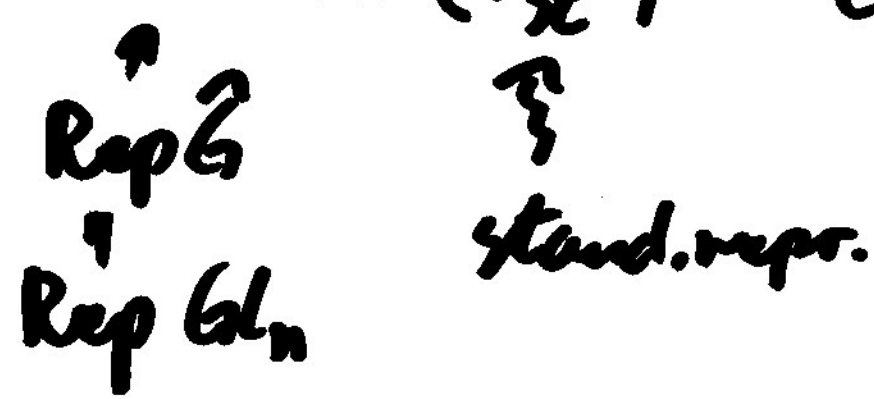
$$F \in D(\text{Bun}_G, \bar{Q}_e)$$

$$T_V(F) = \bar{h}'_! (\zeta_h^* F \otimes \underbrace{\text{Sat}(V)}_F)$$

$$\begin{array}{l}
 \uparrow \\
 D(\text{Bun}_n \times X, \bar{Q}_e) \quad \text{relative version} \\
 \text{of } \text{Sat}(V)
 \end{array}$$

(\sim analogs for $T_{V,x}$ by restr. to a $\text{Bun}_n \times \{x\}$)

For $V = \mathcal{L}^i(V_{st})$ this spec.



$$\begin{aligned} \tilde{\text{Sat}}(V) &= \overline{\mathbb{Q}}_e[\dim d] \\ &= \overline{\mathbb{Q}}_e[i(n-1-i)] \end{aligned}$$

Q: What is an Hecke eigensheaf? (15)

$$\text{Want } T_{V,x} f = \text{tr}(\rho(F_x)|V) \cdot f$$

$$\rho: \pi_1^{\text{ét}}(X,x) \rightarrow \bar{G} = \text{GL}_n(\bar{\mathbb{Q}}_l)$$

$$\bar{G} \xrightarrow{V} \text{GL}(V), V \in \text{Rep } \bar{G}$$

Under sheaf-fct dict.

$$\text{"tr}(\rho(F_x)|V) \cdot f \text{"} \hat{=} V \otimes_{F_x} F_x$$

$\begin{matrix} \cup \\ F_x \xrightarrow{\text{via } \rho} \end{matrix}$

Note:

$$\rho \hat{=} \mathcal{L}_\rho \bar{\mathbb{Q}}_l\text{-loc. system}$$

Def: A Hecke eigensheaf

with eigenvalue \mathcal{L}_ρ

is an object $F \in \mathcal{D}(\text{Bun}_G, \bar{\mathbb{Q}}_l)$

together with isoms.

$$\Psi_V: T_V(\mathcal{F}) \cong \mathcal{F} \otimes \underbrace{r_{V \times U} \mathcal{G}}$$

in $D(\text{Ban}_{n \times n} \times X, \bar{\mathcal{O}}_e)$

sketch st. Ψ_V satisfy various comp.

$\bar{\mathcal{O}}_e$ -loc. sys. corresp. $\hat{G}(\bar{\mathcal{O}}_e) \cong \pi_1(X, x) \cong GL_n(\bar{\mathcal{O}}_e) \xrightarrow{r_V} GL(V)$

Thm (Frenkel/Gaitsgory-Vibron)

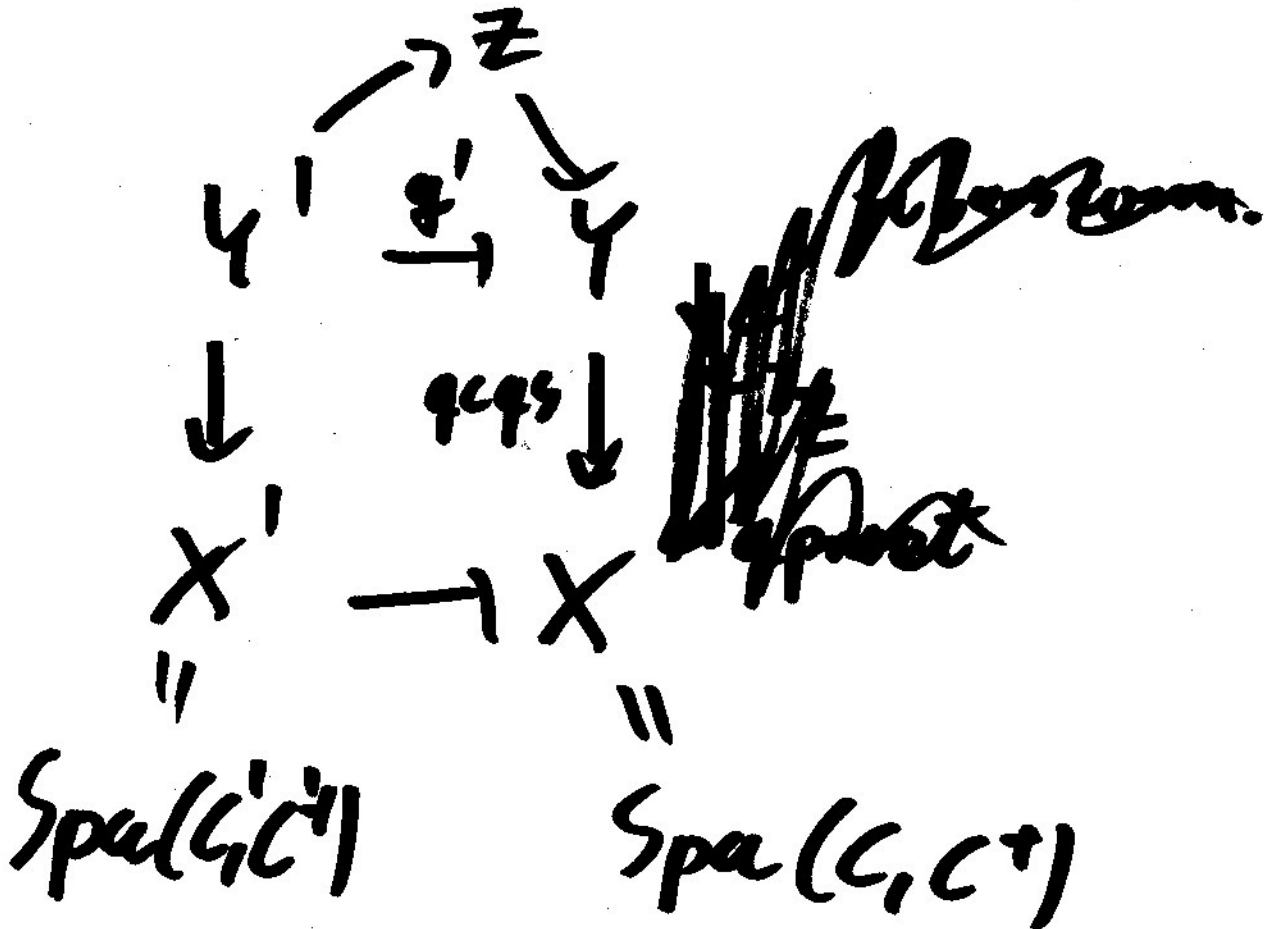
k of char $p > 0$, X/k sm. proj. curve, geom. conn.

\exists irred $\bar{\mathcal{O}}_e$ -loc. system of rank n

\exists non-zero perverse Hecke \wedge weak (unless comp. for p)

eigensheaf with eigenvalue λ . (17)
 Rad(Bun)

Actually, on



$$\begin{array}{ccc}
 Y'_v & \rightarrow & Y_v & & \mathbb{I} \times \text{Spa}(C, C) \\
 & & & & \downarrow \\
 \mathcal{F} & \rightarrow & R_{g' \times g'} \mathcal{F} & & \Sigma \times \text{Spa}(C, C) \\
 D^+(Y'_v, \mathcal{L}) & \subseteq & D^+(Y_v, \mathcal{L}) & &
 \end{array}$$

Y loc. spat. diam.

$$D^+(Y_{\acute{e}t}, \mathcal{L}) \xrightarrow{\cong} D^+(Y_v, \mathcal{L})$$

$$\xrightarrow{\cong} D^+_{\acute{e}t}(Y, \mathcal{L})$$

$$\pi: Y_v \rightarrow Y_{\acute{e}t}$$

$$\{ Y \xrightarrow{\text{étale}} Y \}$$



~~Q-lin system on F~~
covering \rightarrow X
X top. space, Q-loc. sys.

on X is a sheaf \mathcal{F} of Q-v.s.
s.t. \exists covering of X, s.t. $\mathcal{F}|_{U_i} \cong \mathcal{Q}^n$

\mathbb{Z}_ℓ -loc. system.

\cong comp. system.

of \mathbb{Z}/ℓ^n -loc. systems

\uparrow

étale sheaves of

\mathbb{Z}/ℓ^n -modules, loc. isom

to $(\mathbb{Z}/\ell^n)^m$

\mathcal{F}

\downarrow

X

\cong repr. of $\pi_1^{\text{ét}}(X, \kappa)$

on free, finite \mathbb{Z}/ℓ^n -modules

Best hope:

$$Q\text{coh}(LS_n) \xrightarrow{\cong} D(Bun_G, \bar{Q}_e)$$

comp. with Hecke
action