Recall last time: $F = k(X)$, $k = \mathbb{F}_q$, X smooth projective curve over k.

$\sigma : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell) \longleftrightarrow$ rank n $\overline{\mathbb{Q}}_\ell$-local system E on X.

geom. irreducible and everywhere unramified

$\rightsquigarrow$ $f'_\sigma \in C^\infty_{\mathrm{cusp}}\left(M(F)\backslash^{GL_n(\mathbb{A}_F)}/GL_n(\mathcal{O})\right)$, $M = \left\{\begin{pmatrix} \underbrace{*}_{n-1} & * \\ 0 & 1 \end{pmatrix}\right\} \subset GL_n$.

s.t. $\forall x \in |X|$, $\forall i = 1, \dots, n$

$$T_x^i f'_\sigma = q_x^{-i(i-1)/2}\, \mathrm{tr}\left(\Lambda^i \sigma(\mathrm{Frob}_x)\right) \cdot f'_\sigma$$

By definition, $f'_\sigma = \Phi(W_\sigma) := \sum_{\gamma \in N_{n-1}(F)\backslash GL_{n-1}(F)} W_\sigma\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} -\right)$.

with $W_\sigma \in C^\infty\left(GL_n(\mathbb{A}_F)/GL_n(\mathcal{O})\right)^{(N(\mathbb{A}_F), \psi)}$

Conj: f'_σ is constant along the fibers of $M(F)\backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O}) \longrightarrow GL_n(F)\backslash^{GL_n(\mathbb{A}_F)}/GL_n(\mathcal{O})$.

How to geometrize this?

Def: Let Bun'_n be the moduli stack of pairs $(\mathcal{E}, s)$, $\mathcal{E}$ rank n vector bundle on X, $s : \Omega^{n-1} \hookrightarrow \mathcal{E}$ embedding of $\mathcal{O}_X$-modules

$\Big($ $\mathcal{E}$ rank vector bundle on $X \times_k S$, $s : \underset{\substack{\| \\ (\Omega^1_X)^{\otimes n-1}}}{\Omega^{n-1}_X} \boxtimes \mathcal{O}_S \to \mathcal{E}$ embedding st. cokernel S-flat $\Big)$

One has $\mathrm{Bun}'_n(k) \simeq M(F) \backslash P(\mathbb{A}_F)^+ / P(\mathcal{O})$, where $P \subseteq GL_n$ $(n-1,1)$-parabolic and $P(\mathbb{A}_F)^+ = \prod'_{x \in |X|} P(K_x)^+$

$$P(K_x)^+ = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in P(K_x),\ |c| \leq 1 \right\}$$

Rk: $M(F) \backslash P(\mathbb{A}_F)^+ / P(\mathcal{O}) \subseteq M(F) \backslash P(\mathbb{A}_F) / P(\mathcal{O}) = M(F) \backslash GL_n(\mathbb{A}_F) / GL_n(\mathcal{O})$.

f'_σ, being an Hecke eigenfunction, is determined by its restriction to $M(F) \backslash P(\mathbb{A}_F)^+ / P(\mathcal{O})$.

Rk: The morphism $\mathrm{Bun}'_n \xrightarrow{r} \mathrm{Bun}_n$, $(\mathcal{E}, s) \mapsto \mathcal{E}$ induces the natural map $M(F) \backslash P(\mathbb{A}_F)^+ / P(\mathcal{O}) \to GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})$ on k-points.

Def: Let Q be the moduli stack of $(\mathcal{E}, (\mathcal{E}_i), (s_i))$

$\{$ a rank n vector bundle, $0 = \mathcal{E}_n \subseteq \mathcal{E}_{n-1} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E}$

full flag of subbundles of $\mathcal{E}$.

$s_i : \Omega^i \hookrightarrow \mathcal{E}_i / \mathcal{E}_{i+1}$.
$i = 0, \ldots, n-1$

$B \subseteq GL_n$ Borel

One has $Q(k) \simeq N(F) \backslash B(\mathbb{A}_F)^+ / B(\mathcal{O})$,

$B(\mathbb{A}_F)^+ = \prod'_{x \in |X|} B(K_x)^+$

$B(K_x)^+ = \left\{ \begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix}, \forall i \ |a_i| \leq 1 \right\}$

Rk: $N(F) \backslash B(\mathbb{A}_F)^+ / B(\mathcal{O}) \subseteq N(F) \backslash B(\mathbb{A}_F) / B(\mathcal{O})$

$= N(F) \backslash GL_n(\mathbb{A}_F) / GL_n(\mathcal{O})$.

W_σ as an Hecke eigenfunction is determined by its restriction to $N(F) \backslash B(\mathbb{A}_F)^+ / B(\mathcal{O})$.

Rk: The morphism $\nu : Q \to \mathrm{Bun}'_n$

$(\mathcal{E}, (\mathcal{E}_i), (s_i)) \longmapsto (\mathcal{E}, s_{n-1})$

induces the map $N(F) \backslash B(\mathbb{A}_F)^+ / B(\mathcal{O}) \longrightarrow M(F) \backslash P(\mathbb{A}_F)^+ / P(\mathcal{O})$.

on k-points.

Assume we found $\mathcal{W}_E \in D^b_c(Q, \overline{\mathbb{Q}}_\ell)$ s.t. $\mathrm{tr}_{\mathcal{W}_E} = W_\sigma|_{N(F)\backslash B(\mathbb{A}_F)^+/B(\mathbb{O})}$.

Then we can set $\mathrm{Aut}'_E := \nu_! \mathcal{W}_E \in D^b_c(\mathrm{Bun}'_n, \overline{\mathbb{Q}}_\ell)$.

We have $\mathrm{tr}_{\mathrm{Aut}'_E} = \sum_{x \in \nu^{-1}(-)} \mathrm{tr}_{\mathcal{W}_E}(x) = \sum_{\gamma \in N_{n-1}(F)\backslash GL_{n-1}(F)} W_\sigma\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} -\right)$.

i.e. $\mathrm{tr}_{\mathrm{Aut}'_E} = f'_\sigma$.

$\rightarrow$ Goal is to geometrize W_σ.

Recall $W_\sigma = \prod_{x \in |X|} W_{x,\sigma(\mathrm{Frob}_x)}$, $W_{x,\sigma(\mathrm{Frob}_x)} \in C^\infty(GL_n(K_x))$.

The Casselman - Shalika - Shintani formula

From now on, fix $x \in |X|$. and denote $K = K_x$, $\mathcal{O} = \mathcal{O}_x$, k residue field.

Let G split connected reductive gp over k, $\hat{G}$ dual gp $/\overline{\mathbb{Q}}$.

Let H = Hecke algebra = $C_c^\infty(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}), \overline{\mathbb{Q}}_\ell)$.

with convolution product: $h_1, h_2 \in H$ $\quad (h_1 * h_2)(g) = \int_{G(K)} h_1(g') h_2(g g'^{-1})\, dg'$.

<u>Th</u> (Satake isomorphism): There is a canonical isomorphism of $\overline{\mathbb{Q}}_\ell$-alg

$$\mathcal{S} : H \simeq R(\hat{G}) \quad \text{(representation ring of } \hat{G}\text{)}.$$

Let Λ^+ = set of dominant coweights of G. $\quad$ ($\Lambda^+ = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n,\ \lambda_1 \geq \dots \geq \lambda_n\}$ if $G = GL_n$)

For each $\lambda \in \Lambda^+$, get an irreducible rep $V(\lambda)$ of $\hat{G}$.

Via $\mathcal{S}$, give rise to a basis $(A_\lambda)_{\lambda \in \Lambda^+}$ of H.

<u>Rk</u>: For each $\lambda \in \Lambda^+$, A_λ is <u>not</u> the characteristic function of the double coset $G(\mathcal{O}) \cdot \lambda(t) \cdot G(\mathcal{O})$ (t uniformizer of K)

Let γ conjugacy class in $\hat{G}(\overline{\mathbb{Q}_\ell})$.

$\rightsquigarrow$ gives a morphism $\gamma : \mathrm{Rep}(\hat{G}) \to \overline{\mathbb{Q}_\ell}$

$$[V] \mapsto \mathrm{tr}(\gamma, V).$$

Denote also by γ the composite of this morphism with $\mathcal{S}$.

The Whittaker function attached to γ is the unique function on $G(K)$ s.t.:

Whitt $\subset C^\infty(G(K))$

- $W_\gamma(1) = 1$
- $W_\gamma(gg') = W_\gamma(g) \quad \forall g \in G(K),\ g' \in G(\mathcal{O})$.
- $W_\gamma(ng) = \psi(n) . W(g) \quad \forall g \in G(K), \forall n \in N(K)$. $\left[\psi = \sum \psi(u_i)\right.$ u_i coordinates on $N/[N,N]$ given by the simple coroots $\left.\right]$
- $\forall h \in H,\ W_\gamma * h = \gamma(h) . W_\gamma$

where $(f * h)(g) = \int_{G(K)} f(g'g^{-1}) h(g') dg'$.

Casselman-Shalika Shintani give an explicit formula for W_γ. Since $N(K)\backslash G(K)/G(\mathcal{O})$ is in bijection with Λ, enough to describe the value of W_γ on $\lambda(t)$, $\lambda \in \Lambda$.

Th: Let $\lambda \in \Lambda$. Then: $W_\gamma(\lambda(t)) = \begin{cases} q^{-\langle \lambda, \check{e} \rangle} \cdot \gamma(V(\lambda)) & \text{if } \lambda \in \Lambda^+ \\ 0 & \text{otherwise.} \end{cases}$

Observation/reformulation (Frenkel-Gaitsgory-Kazhdan-Vilonen)

Define $F : H \longrightarrow \text{Whitt}$

$$h \longmapsto \left(g \longmapsto \int_{N(K)} h(n^{-1}g)\, \psi(n)\, dn \right)$$

The above explicit formula can be rewritten as:

$$\text{Whitt} \ni W_\gamma = F\Big(\underbrace{\sum_{\lambda \in \Lambda^+} \gamma(V(\lambda)^*) \cdot A_\lambda}_{A_\gamma} \Big)$$

$$A_\lambda \overset{\varphi}{\longleftrightarrow} V(\lambda)$$

Enters geometric Satake

Recall $Gr_G = LG/L^+G$.

Th: (geometric Satake) $\left(Sat_{G,\bar{k}} := Perv_{L^+G}(Gr_{G,\bar{k}}), \star\right) \underset{\sim}{\overset{Sat_{\bar{k}}}{}} \left(Rep(\hat{G}), \otimes\right)$

Also, $\left(Sat_G = Perv_{L^+G}(Gr_G), \star\right) \simeq \left(Rep(\hat{G} \times Gal_k), \otimes\right)$.

Rk: Grothendieck function-sheaves dictionary gives an identification:

$$K_0(Sat_G^N) \underset{\sim}{} H$$

$$\text{in } Sat_G \qquad \star \mapsto *$$

s.t. pullback along $k \to \bar{k}$ induces an equivalence $Sat_G^N \simeq Sat_{G,\bar{k}}$

and φ: $H \simeq K_0(Sat_G^N) \simeq K_0(Sat_{G,\bar{k}}) \underset{K_0(Sat_{\bar{k}})}{\simeq} K_0(Rep(\hat{G})) = R(\hat{G})$.

Rk: Under $Sat_{\bar{k}}$, $V(\lambda)$ does not correspond to the constant sheaf on $Gr_{G,\lambda} = L^+G$-orbit of $\lambda(t)$.
but rather to $IC_{Gr_{G,\leq\lambda}}$ ($Gr_{G,\leq\lambda}$ = closure of $Gr_{G,\lambda}$).

Consider the following object:

$$\mathrm{Rep}(\hat{G} \times \mathrm{Gal}_k) \ni V_\gamma := \bigoplus_{\lambda \in \Lambda^+} \underset{\underset{\hat{G}}{\circlearrowleft}}{V(\lambda)} \otimes \underset{\underset{\mathrm{Gal}_k}{\circlearrowleft}}{E_\gamma(\lambda)^*}$$

$$\mathrm{Frob} \longmapsto \gamma$$

with $E_\gamma(\lambda)^* : \mathrm{Gal}_k \to \hat{G}(\overline{\mathbb{Q}}_\ell) \xrightarrow{V(\lambda)^*} GL(V(\lambda)^*)$

and define:

$$A_\gamma = \mathrm{Sat}^{-1}(V_\gamma) \in \mathrm{Sat}_G.$$

Then

$$\mathrm{tr}_{A_\gamma} = \sum_{\lambda \in \Lambda^+} \mathrm{tr}_{\mathrm{Sat}^{-1}(V(\lambda))} \cdot \mathrm{tr}\, \underbrace{\mathrm{Sat}^{-1}(E_\gamma(\lambda)^*)}$$

pulled back along $Gr_G \to \mathrm{Spec}(k)$

$$= \sum_{\lambda \in \Lambda^+} \gamma(V(\lambda)^*) \cdot A_\lambda = A_\gamma$$

It remains to geometrize F. F comes from:

$$\begin{tikzcd} & & {}^{L^-N}\backslash LN/L^+N \\ {}^{L^-N}\backslash Gr_{G,X^{(\infty)}} & {}^{L^-N}\backslash LN \overset{L^+N}{\times} Gr_{G,X^{(\infty)}} \arrow[l, "\mu"'] \arrow[ur, "\beta"] \arrow[dr, "\alpha"] & \\ & & L^+N\backslash Gr_G \arrow[r] & L^+G\backslash Gr_G \end{tikzcd}$$

" $F = \mu_! \left(\alpha^*(-) \otimes \beta^* \psi \right)$ ".

Need to geometrize / globalization.

Back to the global setting, $G = GL_n$.

<u>Def</u>: Let $\overline{Q}$ be the moduli stack of

$$\left(\mathcal{E}, \mathcal{E}', s, (\mathcal{E}_i'), (s_i) \right)$$

$\mathcal{E}, \mathcal{E}'$ rank n vector bundles, $s: \mathcal{E}' \hookrightarrow \mathcal{E}$, $0 = \mathcal{E}_n' \subset \mathcal{E}_{n-1}' \subseteq \dots \subseteq \mathcal{E}_1' \subseteq \mathcal{E}_0' = \mathcal{E}'$ full flag of subbundles

$\forall i$, $s_i : \Omega^i \simeq \mathcal{E}_i' / \mathcal{E}_{i+1}'$.

It comes with 3 maps:

- $\mu : \widetilde{Q} \longrightarrow Q$

$$(\mathcal{E}, \mathcal{E}', s, (\mathcal{E}'_i), (s'_i)) \longmapsto \left(\mathcal{E}, \ \mathcal{E}_i = \text{max sub vb of } \mathcal{E} \text{ containing } s(\mathcal{E}'_i) \ \ \ s_i : \Omega^i \simeq \mathcal{E}'_i/\mathcal{E}'_{i+1} \to \mathcal{E}_i/\mathcal{E}_{i+1}\right)$$

- $\beta : \widetilde{Q} \longrightarrow \mathbb{A}^1_k$.

$$(\mathcal{E}, \mathcal{E}', s, (\mathcal{E}'_i), (s'_i)) \longmapsto \text{sum of classes in } \operatorname{Ext}^1(\mathcal{E}'_i/\mathcal{E}'_{i+1}, \mathcal{E}'_{i-1}/\mathcal{E}'_i)$$

$$\operatorname{Ext}^1(\mathcal{E}'_i/\mathcal{E}'_{i+1}, \mathcal{E}'_{i-1}/\mathcal{E}'_i) \cong \operatorname{Ext}^1(\Omega^i, \Omega^{i-1}) = k$$

corresponding to $\mathcal{E}'_{i-1}/\mathcal{E}'_{i+1}$.

- $\alpha : \widetilde{Q} \longrightarrow Coh_0$ = moduli of torsion coh sheaves on X

$$(\mathcal{E}, \mathcal{E}', s, (\mathcal{E}'_i), (s'_i)) \longmapsto \operatorname{coker}(s).$$

Rk: For each $x \in |X|$, consider the ~~substack of~~ $\mathrm{Coh}_{0,x}$ of Coh_0 formed by torsion coherent sheaves supported at x. Have a map

$$\left\{\overset{\mathcal{E},\text{ plus}}{\mathcal{O}^n_{|X\setminus x|} \simeq \mathcal{E}_{X\setminus x}}\right\} = \mathrm{Gr}_{GL_n,x} \supseteq \mathrm{Gr}^+_{GL_n,x} = \left\{\mathcal{E},\ \underset{\text{cokernel supported at } x}{\mathcal{O}^n \hookrightarrow \mathcal{E}}\right\} \xrightarrow{\pi_x} \mathrm{Coh}_{0,x}.$$

$$\longmapsto \mathcal{E}/\mathcal{O}^n.$$

Fact: Can construct $\mathcal{L}_E \in D^b_c(\mathrm{Coh}_0, \overline{\mathbb{Q}}_\ell)$ s.t. $\forall x \in |X|$, $\pi_x^*(\mathcal{L}_E|_{\mathrm{Coh}_{0,x}}) \simeq$ restriction of $A_{x,r(\mathrm{Frob}_x)}$ to $\mathrm{Gr}^+_{GL_n,x}$.

Then set $\mathcal{W}_E = \mu_!(\alpha^*\mathcal{L}_E \otimes \beta^*\mathcal{L}_\psi)$

$\mathcal{L}_\psi$: rank 1 local system on $\mathbb{A}^1_k$ corresponding to ψ by Artin-Schreier theory.

i.e. $\mathrm{Aut}'_E = (\nu \circ \mu)_!(\alpha^*\mathcal{L}_E \otimes \beta^*\mathcal{L}_\psi)$.