Dr. A. Mihatsch
Dr. J. Anschütz **SS 2023**

Algebra I - Commutative algebra

11. Exercise sheet

Exercise 1 (4 points):
Let k be a field and let A, B be two finitely generated k-algebras. Show that

$$\dim(A \otimes_k B) = \dim(A) + \dim(B).$$

Exercise 2 (4 points):
Let k be a field, and consider the k-algebra morphism

$$\varphi \colon k[x,y]/(y^2 - x^3) \longrightarrow k[t],\ x \longmapsto t^2, y \longmapsto t^3.$$

Show that φ is finite, induces a bijection on Spec and is not an isomorphism.

Exercise 3 (4 points):
In this exercise we denote by $\mathrm{MinSpec}(A)$ the set of minimal prime ideals of a ring A.
1) Let $A_1, \ldots, A_n$ be rings and let B be their product. Show that

$$\mathrm{MinSpec}(B) = \bigcup_{i=1}^{n} \mathrm{MinSpec}(A_i).$$

2) Let $f \colon A \to B$ be an injective and integral ring homomorphism. Show the inclusion

$$\mathrm{MinSpec(A)} \subseteq \mathrm{Spec}(f)(\mathrm{MinSpec}(B))$$

and give an example where the inclusion is strict.

Exercise 4 (4 points):
Let k be an algebraically closed field and let $Z \subseteq k^4$ be the vanishing locus of the ideal $(xz, yz, xw, yw) \subseteq k[x,y,z,w]$. Determine the irreducible components of Z and their intersections.
Hint: Construct an injective ring homomorphism $k[x,y,z,w]/(xz,yz,xw,yw) \hookrightarrow k[r,s] \times k[u,v]$ *and use Exercise 3 to determine the irreducible components.*

To be handed in on: Thursday, 29.06.2023 (during the lecture, or via eCampus).