Dr. A. Mihatsch **SS 2023**
Dr. J. Anschütz

Algebra I - Commutative algebra

10. Exercise sheet

Exercise 1 (4 points):

Let k be a field and let $f\colon A \to B$ be a k-algebra homomorphism with B a finitely generated k-algebra. Let $\mathfrak{m} \subseteq B$ be a maximal ideal. Show that $f^{-1}(\mathfrak{m}) \subseteq A$ is a maximal ideal.

Exercise 2 (4 points):

Let $n \geq 0$ and $Z \subseteq k^n$ an algebraic subset. Show that $I(Z)$ is a prime ideal if and only if $Z = Z_1 \cup Z_2$ with Z_1, Z_2 algebraic subsets implies $Z = Z_1$ or $Z = Z_2$.

Exercise 3 (4 points):

A ring is called Jacobson if each prime ideal is the intersection of all maximal ideals containing it.
i) Show that a ring A is Jacobson if and only if for all primes $\mathfrak{p} \subseteq A$ and $a \notin \mathfrak{p}$ there exists a maximal ideal $\mathfrak{m} \subseteq A$ such that $a \notin \mathfrak{m}$ and $\mathfrak{p} \subseteq \mathfrak{m}$.
ii) Let $f\colon A \to B$ be an injective, integral morphism and assume that B is Jacobson. Show that A is Jacobson. Deduce from the lecture that for each field k and $n \geq 0$ the ring $k[X_1, \ldots, X_n]$ is Jacobson.

Exercise 4 (4 points):

Let A be a local ring and M a finitely presented, flat A-module. Show that M is free.
Hint: Let $\mathfrak{m} \subseteq A$ be the maximal ideal. Use exercise 4 from sheet 9 to construct a short exact sequence $0 \to K \to A^n \to M \to 0$ with K finitely generated and $(A/\mathfrak{m})^n \to M/\mathfrak{m}M$ an isomorphism. Now use flatness of M and the snake lemma to check that $0 \to K/\mathfrak{m}K \to (A/\mathfrak{m})^n \to M/\mathfrak{m} \to 0$ is again short exact.

To be handed in on: Thursday, 22.06.2023 (during the lecture, or via eCampus).