Dr. A. Mihatsch **SS 2023**
Dr. J. Anschütz

Algebra I - Commutative algebra

9. Exercise sheet

Exercise 1 (4 points):

Assume that $d \in \mathbb{Z}$ is not a square. Determine all $x, y, z \in \mathbb{Z}$ with $\gcd(x, y, z) = 1$ and $x^2 - dy^2 = z^2$.
Hint: Follow the arguments in the lecture and consider lines through the point $(-1, 0) \in \mathbb{Q}^2$.

Exercise 2 (4 points):

Let k be an algebraically closed field and let $f(x) \in k[x]$ be a polynomial. Determine the set $\operatorname{Spec}(k[x, y]/(y^2 - f(x)))$ and the cardinality of all fibers of the map

$$\operatorname{Spec}(k[x, y]/(y^2 - f(x))) \to \operatorname{Spec}(k[x])$$

that is induced by the k-algebra homomorphism $k[x] \to k[x, y]/(y^2 - f(x)),\ x \mapsto x$.

Exercise 3 (4 points):

Let $m, n \geq 1$ and let $\zeta_m = e^{2\pi i/m} \in \mathbb{C}$ be a primitive m-th root of unity. Set $G := \langle \zeta_m \rangle \subseteq \mathbb{C}^\times$. We let G act on $A := \mathbb{C}[T_1, \dots, T_n]$ via $(g, f(T_1, \dots, T_n)) \mapsto g \cdot f := f(gT_1, \dots, gT_n)$.
1) Determine the ring of invariants $A^G := \{f \in A \mid g \cdot f = f \text{ for all } g \in G\}$.
2) Set $m = n = 2$. Find a presentation $A^G \cong \mathbb{C}[X_1, \dots, X_k]/(h_1, \dots, h_l)$.

Exercise 4 (4 points):

Let A be a ring and M a finitely presented A-module. Let $n \geq 1$ and let $f \colon A^n \to M$ be a surjection. Show that $K := \ker(f)$ is finitely generated.
Hint: Let $0 \to Q \to A^m \to M \to 0$ *be a short exact sequence of* A*-modules with* Q *finitely generated. Construct a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r] \arrow[d] & A^m \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & K \arrow[r] & A^n \arrow[r, "f"] & M \arrow[r] & 0 \end{tikzcd}$$

and use the snake lemma.

To be handed in on: Thursday, 15.06.2023 (during the lecture, or via eCampus).