Dr. A. Mihatsch
Dr. J. Anschütz **SS 2023**

Algebra I - Commutative algebra

8. Exercise sheet

Exercise 1 (4 points):

Let A be a ring and $\mathfrak{a} \subseteq A$ an ideal. Show that $A/\mathfrak{a}$ is a finitely presented A-algebra if and only if $\mathfrak{a}$ is a finitely generated ideal.

Exercise 2 (4 points):

Let k be a field. Show that the ring extensions $k[X+Y] \to k[X,Y]/XY$ and $k[X^2-1] \to k[X]$ are integral.

Exercise 3 (4 points):

Let $A \to B$ be a finite morphism of rings, i.e., $A \to B$ is a ring homomorphism, which makes B into a finite A-module. Show that the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has finite fibers.
Remark: The morphism $\mathbb{Z} \to \mathbb{Q}$ shows that the converse is not true.

Exercise 4 (4 points):

Let A be a ring and let

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2", "\cong"'] & M_3 \arrow[r, "f_3"] \arrow[d, "\alpha_3"] & M_4 \arrow[r, "f_4"] \arrow[d, "\alpha_4", "\cong"'] & M_5 \arrow[d, "\alpha_5"] \\ N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r, "g_3"] & N_4 \arrow[r, "g_4"] & N_5 \end{tikzcd}$$

be a commutative diagram of A-modules with exact rows and α_2, α_4 isomorphisms.
1) Assume that α_1 is surjective. Show that α_3 is injective.
2) Assume that α_5 is injective. Show that α_3 is surjective.

To be handed in on: Thursday, 08.06.2023 (during the lecture, or via eCampus).