Dr. A. Mihatsch **SS 2023**
Dr. J. Anschütz

Algebra I - Commutative algebra

7. Exercise sheet

Exercise 1 (4 points):
Let $A \to B$ be a homomorphism of rings, let M be an A-module and let N be a B-module.
i) Show that the map

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(B \otimes_A M, N),\ \varphi \mapsto (b \otimes m \mapsto b\varphi(m))$$

is a well-defined isomorphism.
ii) Show that the map

$$M \otimes_A N \to (M \otimes_A B) \otimes_B N,\ m \otimes n \mapsto (m \otimes 1) \otimes n$$

is a well-defined isomorphism.
ii) Deduce that $S^{-1}M_1 \otimes_A S^{-1}M_2 \cong S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2$ for two A-modules M_1, M_2 and a multiplicative subset $S \subseteq A$.

Exercise 2 (4 points):
Let A be a ring. We define the *support* of an A-module M as $\mathrm{supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_\mathfrak{p} \neq 0\}$.
i) Assume M is finitely generated. Show that $\mathrm{supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M \otimes_A k(\mathfrak{p}) \neq 0\}$, where $k(\mathfrak{p}) = \mathrm{Quot}(A/\mathfrak{p})$.
ii) Assume M, N are finitely generated A-modules. Show $\mathrm{supp}(M \otimes_A N) = \mathrm{supp}(M) \cap \mathrm{supp}(N)$.

Exercise 3 (4 points):
Let A be a ring, let $S \subseteq A$ be a multiplicative subset and let M, N be A-modules.
i) Assume that M is a finitely presented A-module. Show that the map

$$S^{-1}\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N),\ \varphi/s \mapsto (m/t \mapsto \varphi(m)/st)$$

is a well-defined isomorphism.
ii) Construct a counterexample to i) if M is only assumed to be finitely generated.

Exercise 4 (4 points):
Let A be a principal ideal domain and let $f \in A \setminus \{0\}$ be a non-unit. Show that the $A[T]$-module $(f, T) \subseteq A[T]$ is not flat.

To be handed in on: Thursday, 25.05.2023 (during the lecture, or via eCampus).