Dr. A. Mihatsch **SS 2023**
Dr. J. Anschütz

Algebra I - Commutative algebra

6. Exercise sheet

Exercise 1 (4 points):

Let A be a ring, $f \in A$ a non-zero divisor, $\mathfrak{a} = (f)$ and $\mathfrak{b} \subseteq A$ an ideal. Show that the natural map

$$\mathfrak{a} \otimes_A \mathfrak{b} \longrightarrow \mathfrak{a} \cdot \mathfrak{b},\ a \otimes b \longmapsto a \cdot b$$

is an isomorphism.

Exercise 2 (4 points):

Let A be a ring, let I be a set and let $M, N_i, i \in I$, be A-modules.
1) Assume that M is finitely generated (resp. finitely presented). Show that the natural map

$$M \otimes_A \prod_{i \in I} N_i \longrightarrow \prod_{i \in I} M \otimes_A N_i$$

is surjective (resp. bijective).
2) Take $A = \mathbb{Z}[X_0, X_1, \ldots]$, $J = (X_0, X_1, \ldots)$. Show that the natural map $A/J \otimes_A A[[T]] \to A/J[[T]]$ is not injective.

Exercise 3 (4 points):

Let k be a field, K/k an algebraic field extension, and $\overline{k}$ an algebraic closure of k.
1) If $V \to W$ is a k-linear injection of k-vector spaces, show that $V \otimes_k \overline{k} \to W \otimes_k \overline{k}$ is a $\overline{k}$-linear injection.
2) Show that K/k is separable if and only if the ring $K \otimes_k \overline{k}$ is reduced.

Exercise 4 (4 points):

Let A be a ring and let I be an *invertible* A-module, i.e., there exists an A-module J such that $I \otimes_A J \cong A$. Let $\varphi\colon M \to N$ be a homomorphism of A-modules.
a) Show that φ is zero (resp. injective, resp. surjective) if and only if $\varphi \otimes_A I\colon M \otimes_A I \to N \otimes_A I$ is so.
b) Show that I is finitely generated.

To be handed in on: Friday, 19.05.2023 (till 12:00h via the box in front of 4.027, or via eCampus).