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SS 2023

Algebra I - Commutative algebra

5. Exercise sheet

Exercise 1 (4 points):
Let A be a ring and let $\mathfrak{a}_1, \dots, \mathfrak{a}_n \subseteq A$ be ideals such that $\bigcap_{i=1}^n \mathfrak{a}_i = \{0\}$. Assume that each ring $A/\mathfrak{a}_i$ is noetherian. Show that A is noetherian.

Exercise 2 (4 points):
Consider the matrix

$$S := \begin{pmatrix} -36 & 14 & -24 \\ 18 & 6 & 12 \end{pmatrix}.$$

Determine its elementary divisors and the kernel/cokernel of the map $\mathbb{Z}^3 \xrightarrow{S} \mathbb{Z}^2$ (up to isomorphy).

Exercise 3 (4 points):
Let A be a ring, let $\mathfrak{a} \subseteq A$ be an ideal and let $M, N_i, i \in I$, be A-modules for some set I.
i) Show that there exists a unique isomorphism

$$\Phi\colon \bigoplus_{i\in I}(N_i \otimes_A M) \to (\bigoplus_{i\in I} N_i) \otimes_A M$$

such that $\Phi((\dots, 0, n_i \otimes m, 0, \dots)) = (\dots, 0, n_i, 0, \dots) \otimes m$ for all $n_i \in N_i, i \in I$, $m \in M$.
ii) Show that there exists a unique isomorphism

$$\Psi\colon A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$$

such that $\Psi((a + \mathfrak{a}) \otimes m) \mapsto am + \mathfrak{a}M$ for all $a \in A$, $m \in M$.

Exercise 4 (4 points):
Let A be a ring and let M, N be A-modules. A bilinear map $(-,-)\colon M \times M \to N$ is called symmetric if $(m_1, m_2) = (m_2, m_1)$ for all $m_1, m_2 \in M$. It is called alternating if $(m, m) = 0$ for all $m \in M$.
i) Show that there exists an A-module $\mathrm{Sym}^2_A(M)$ and a symmetric bilinear map $\iota\colon M \times M \to \mathrm{Sym}^2_A(M)$ with the following universal property: for every A-modules N and for every symmetric bilinear map $(-,-)\colon M \times M \to N$ there exists a unique A-linear map $\Phi\colon \mathrm{Sym}^2_A(M) \to N$ such that for all $m_1, m_2 \in M$

$$(m_1, m_2) = \Phi(\iota(m_1, m_2)).$$

Construct similarly an A-module $\Lambda^2_A(M)$ with a universal alternating bilinear map $\gamma\colon M \times M \to \Lambda^2_A(M)$.
ii) Show that $\mathrm{Sym}^2_A(A^n)$ and $\Lambda^2_A(A^n)$ are free A-modules of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$.

To be handed in on: Thursday, 11.05.2023 (during the lecture, or via eCampus).