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SS 2023

Algebra I - Commutative algebra

4. Exercise sheet

Exercise 1 (4 points):

Let A be a ring.
1) Assume that $f_n \in A[[T]]$, $n \geq 0$, is a sequence of elements such that $f_n \in (T)^n$ for all $n \geq 0$. Show that there exists a unique element $f \in A[[T]]$ such that $f - \sum_{k=0}^{n} f_k \in (T)^{n+1}$ for all $n \geq 0$.
2) Assume that A is noetherian. Show that $A[[T]]$ is noetherian.

Exercise 2 (4 points):

1) Let A be the ring of power series in $\mathbb{C}[[z]]$ with a positive radius of convergence. Show that A is noetherian.
2) Show that the ring of holomorphic functions $\mathbb{C} \to \mathbb{C}$ is not noetherian.
One possible approach is to use the relation $\sin(2x) = 2\sin(x)\cos(x)$.

Exercise 3 (4 points):

Let $n \geq 1$. For an $n \times n$-matrix M over some ring A denote by $\chi_M(T) := \det(T \cdot \mathrm{Id} - M)$ its characteristic polynomial.
1) Let $A = \mathbb{Z}[a_{i,j} \mid 1 \leq i, j \leq n]$ and $M := (a_{i,j})_{i,j} \in \mathrm{Mat}_n(A)$. Show that

$$\chi_M(M) = 0.$$

Hint: You may use the Cayley–Hamilton theorem from linear algebra.
2) Deduce a general form of the theorem of Cayley–Hamilton: Let A be a ring and let M be any $n \times n$-matrix over A. Then $\chi_M(M) = 0$.

Exercise 4 (4 points):

Let A be a principal ideal domain.
1) Let $a \in A \setminus \{0\}$ and let $\pi \in A$ prime. Set $B := A/a$. For any $n \geq 0$ show that

$$\dim_{A/\pi} \pi^n B/\pi^{n+1}B = \begin{cases} 0 & \text{if } \nu_\pi(a) \leq n, \\ 1 & \text{if } \nu_\pi(a) \geq n+1. \end{cases}$$

Here, $\nu_\pi(a) = \max\{m \mid \pi^m | a\}$ is the so-called π-adic valuation on A.
2) Assume that $M = A^r \oplus A/a_1 \oplus \ldots \oplus A/a_k$, $N = A^s \oplus A/b_1 \oplus \ldots \oplus A/b_l$ with $a_1, \ldots, a_k, b_1, \ldots, b_l \in A$ non-zero and $a_1 | a_2 | \ldots | a_n$, $b_1 | b_2 | \ldots | b_l$. Show that if $M \cong N$ as A-modules, then $r = s$, $k = l$ and $a_i = u_i b_i$ for some units $u_i \in A$.

To be handed in on: Thursday, 04.05.2023 (during the lecture, or via eCampus).