Dr. A. Mihatsch **SS 2023**
Dr. J. Anschütz

Algebra I - Commutative algebra

3. Exercise sheet

Exercise 1 (4 points):

Let A be a principal ideal domain. The arguments for $A = \mathbb{Z}$ from the lecture work verbatim to show that the prime ideals of $A[T]$ are

(i) (0),

(ii) (f), $f \in A[T]$ irreducible,

(iii) (π, g) with $\pi \in A$ prime and $g \in A[T]$ a polynomial whose image in $(A/\pi)[T]$ is irreducible.

1) Assume that A has infinitely many prime ideals. Prove that the heights of the primes in (i), (ii), (iii) are $0, 1, 2$ respectively and that each maximal ideal of $A[T]$ has height 2.
2) Let k be a field and set $A := k[[u]]$. Show that $A[u^{-1}]$ is a field. Deduce that, in contrast to 1), the height 1 ideal $(uT - 1) \subseteq A[T]$ is maximal.

Exercise 2 (4 points):

Let k be an algebraically closed field and let

$$\varphi\colon k[x, y] \to k[u, v],\ x \mapsto u,\ y \mapsto uv.$$

1) Use exercise 1 to show that the maximal ideals of $k[x, y]$ are precisely the ideals

$$\mathfrak{m}_{\lambda,\mu} := (x - \lambda, y - \mu), \quad \lambda, \mu \in k.$$

2) Show that φ induces an isomorphism $k[x, y][x^{-1}] \to k[u, v][u^{-1}]$.
3) For each $(\lambda, \mu) \in k^2$ calculate $\mathrm{Spec}(\varphi)^{-1}(\mathfrak{m}_{\lambda,\mu})$.

Exercise 3 (4 points):

Let A be a ring of Krull dimension $n := \dim A$. Show that

$$n + 1 \leq \dim A[T] \leq 2n + 1.$$

Exercise 4 (4 points):

Let A be a ring and $S, T \subseteq A$ multiplicative subsets with $S \subseteq T$.
1) Let $\iota_S\colon A \to S^{-1}A$ be the natural ring homomorphism. Show that $\iota_S^{-1}((S^{-1}A)^\times)$ is the saturation $\overline{S}$ of S.
2) Show that there exists a unique ring homomorphism $\iota\colon S^{-1}A \to T^{-1}A$ such that $\iota \circ \iota_S = \iota_T$.
3) Deduce that ι is an isomorphism if and only if $\overline{T} = \overline{S}$.

To be handed in on: Thursday, 27.04.2023 (during the lecture, or via eCampus).