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Dr. J. Anschütz **SS 2023**

Algebra I - Commutative algebra

1. Exercise sheet

All rings are assumed to be commutative and unital.

Exercise 1 (4 points):

Determine the nilradical, the Jacobson radical and the units for each ring A below:
1) k a field and $A = k[T]$,
2) k a field and $A = k[\varepsilon, T]/(\varepsilon^2)$,
3) $n \geq 1$, k a field and $A = k[[T_1, \ldots, T_n]]$.

Exercise 2 (4 points):

Prove the *Chinese remainder theorem*: Let A be a ring and $\mathfrak{a}, \mathfrak{b} \subseteq A$ two ideals, such that $\mathfrak{a}+\mathfrak{b} = A$. Then the map

$$A/\mathfrak{a} \cap \mathfrak{b} \to A/\mathfrak{a} \times A/\mathfrak{b},\ r + \mathfrak{a} \cap \mathfrak{b} \mapsto (r + \mathfrak{a}, r + \mathfrak{b})$$

is an isomorphism. Moreover, show $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b}$, where $\mathfrak{a} \cdot \mathfrak{b}$ is the smallest ideal in A containing all products $a \cdot b$ with $a \in \mathfrak{a}$, $b \in \mathfrak{b}$.

Exercise 3 (4 points):

Recall that an element $e \in A$ in a ring A is called idempotent if $e^2 = e$.
1) Let A be a ring. Show that the map $e \mapsto (A_1 := eA, A_2 := (1-e)A)$ induces a bijection between the set $\mathrm{Idem}(A)$ of idempotents in A and decompositions $A = A_1 \times A_2$ as rings.
2) Let $A = \mathbb{Z}/133$. Determine $\mathrm{Idem}(A)$.

Exercise 4 (4 points):

Let k be a field and $k \to A$ a ring homomorphism, such that A is finite dimensional over k.
1) Show that A is a field if A is an integral domain.
2) Deduce that each prime ideal in A is maximal.
3) Deduce that if A is reduced, then A is isomorphic to a finite product of finite field extensions l/k.

To be handed in during the lecture on: Thursday, 13.04.2023.