Prof. Dr. P. Scholze **SS 2017**
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Algebraic Geometry II

12. Exercise sheet

Exercise 1 (4 points):
Let $f\colon Y \to X$ be a morphism of schemes. Let $E \in D(X)$ be a bounded complex of locally free $\mathcal{O}_X$-modules of finite rank and let $C \in D^+(Y)$ be a complex of $\mathcal{O}_Y$-modules.

i) Construct a natural morphism

$$\Phi_{C,E}\colon Rf_*(C) \otimes^{\mathbb{L}}_{\mathcal{O}_X} E \to Rf_*(C \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Lf^*(E)).$$

ii) Prove that $\Phi_{C,E}$ is an isomorphism in $D^+(Y)$.

Hint: In i) use that, for sheaves, a local section $e \in E$ defines a morphism

$$C \to C \otimes_{\mathcal{O}_Y} f^*(E),\ c \mapsto c \otimes f^*(e).$$

You may want to use then that tensoring with locally free sheaves of finite rank preserves injective sheaves. For ii) reduce the statement to a local statement and then to $E \cong \mathcal{O}_X$.

Exercise 2 (4 points):
Let k be a field and let X, Y be two quasi-compact and separated schemes over k. Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $p\colon X \times_k Y \to X$ resp. $q\colon X \times_k Y \to Y$ be the projections. Prove the Künneth formula

$$H^n(X \times_k Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_k Y}} q^*\mathcal{G}) \cong \bigoplus_{i+j=n} H^i(X, \mathcal{F}) \otimes_k H^j(Y, \mathcal{G})$$

for $n \geq 0$.
Hint: Compute $R\Gamma(X \times_k Y, -) \cong R\Gamma(X, Rp_*(-))$ using the projection formula from Exercise 1 and flat base change. Then use or prove that every complex of k-vector spaces is quasi-isomorphic to its cohomology groups.

Exercise 3 (4 points):
Let k be an algebraically closed field and let X be an elliptic curve over k, i.e., X is a proper smooth curve over k of genus 1 together with a distinguished base point $x_0 \in X(k)$. Prove that X can be embedded into $\mathbb{P}^2_k$ as a plane curve defined by the affine Weierstraß equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

such that x_0 maps to the point $(x : y : z) = (0 : 1 : 0)$.
Hint: Using Riemann-Roch and Serre duality prove that $\dim_k H^0(X, \mathcal{O}_X(nx_0)) = n$ for $n \geq 1$. Then pick $x \in H^0(X, \mathcal{O}_X(2x_0)) \setminus H^0(X, \mathcal{O}_X(x_0)) \subseteq H^0(X, \mathcal{O}_X(3x_0))$ and $y \in H^0(X, \mathcal{O}_X(3x_0)) \setminus H^0(X, \mathcal{O}_X(2x_0))$.

Exercise 4 (4 points):
Let R be a ring. We set

$$\Omega^{1,\mathrm{cont}}_{R((t))/R} := R((t)) \otimes_{R[t]} \Omega^1_{R[t]/R} = R((t))dt.$$

i) Let $f(t) = a_1 t + a_2 t^2 + \ldots \in R((t))$ such that $a_1 \in R^\times$ is a unit. Prove that

$$\alpha_f \colon R((t)) \to R((t)),\ t \mapsto f(t)$$

is an automorphism of $R((t))$.

ii) Prove that the residue

$$\mathrm{res}\colon \Omega^{1,\mathrm{cong}}_{R((t))/R} \to R,\ \sum_{n \gg -\infty} b_n t^n dt \mapsto b_{-1}$$

is invariant under the induced automorphisms on $\Omega^{1,\mathrm{cong}}_{R((t))/R}$ for α_f as in *i)*.

Hint: For $m \geq 0$ use the known statement for $R = \mathbb{C}$ from the lecture to conclude that ii) holds for the automorphism $\alpha_{f^{\mathrm{univ}}}$ with $f^{\mathrm{univ}}(t) = a_1 t + a_2 t^2 + \ldots$ over $R = \mathbb{Z}[a_1^{\pm 1}, a_2, \ldots, b_{-m}, b_{-m+1}, \ldots]$ and the differential $\sum_{n \geq -m} b_n t^n dt$. Then conclude the statement in general.

To be handed in on: Monday, 17. Juli 2017.