Prof. Dr. P. Scholze **SS 2017**
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Algebraic Geometry II

11. Exercise sheet

Exercise 1 (4 points):

Let A be a discrete valuation ring with residue field k and fraction field K. Let $s = \mathrm{Spec}(k)$ resp. $\eta = \mathrm{Spec}(K)$ be the special resp. generic point of $S := \mathrm{Spec}(A)$. Let $f\colon X \to S$ be a proper, smooth morphism whose fibers X_s, X_η are irreducible curves of genus $g \geq 1$. Let $\sigma_1, \sigma_2\colon \mathrm{Spec}(R) \to X$ be two sections of f and assume that $\sigma_{1|s} = \sigma_{2|s}$ while $\sigma_{1|\eta} \neq \sigma_{2|\eta}$. Let $\Gamma_i = \sigma_i(X) \subseteq X = X \times_S S$ be the graph of σ_i. Finally, define the line bundle $\mathcal{L} := \mathcal{O}_X(\Gamma_1 - \Gamma_2)$ on X.

i) Prove that the base change morphism $f_*(\mathcal{L}) \otimes_A K \to H^0(X_\eta, \mathcal{L}_\eta)$ is an isomorphism, but the base change morphism $f_*(\mathcal{L}) \otimes_A k \to H^0(X_s, \mathcal{L}_s)$ not.
Hint: Use Exercise 4 from Exercise sheet 10.

ii) Construct an example for A, X, σ_i.
Hint: Find X inside $\mathbb{P}^2_A$.

Exercise 2 (4 points):

Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes with f a closed immersion with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$ and g, $g \circ f$ smooth of relative dimension n resp. m. Let $\omega_{X/S} := \Lambda^n \Omega^1_{X/S}$ and $\omega_{Y/S} := \Lambda^m \Omega^1_{Y/S}$ be the canonical bundles. Define the normal bundle of f as $\mathcal{N}_{X/Y} := f^*(\mathcal{I})^\vee = (\mathcal{I}/\mathcal{I}^2)^\vee$. Prove the adjunction formula

$$\omega_{X/S} \cong f^*(\omega_{Y/S}) \otimes_{\mathcal{O}_X} \Lambda^{m-n} \mathcal{N}_{X/Y}.$$

Exercise 3 (4 points):

Let k be a field and let $X \subseteq \mathbb{P}^n_k$ be a projective, geometrically connected smooth curve with $\dim_k H^1(X, \mathcal{O}_X) = g \geq 2$ which is a complete intersection, i.e., X is the vanishing locus of $n-1$ polynomials $f_i \in H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d_i))$. Prove that $\Omega^1_{X/\mathrm{Spec}(k)}$ is very ample. Using $\dim_k H^0(X, \Omega^1_{X/k}) = g$ (to be proven in the lecture) conclude that X has genus $g > 2$.
Hint: Use Exercise sheet 7, Exercise 1, the adjunction formula from Exercise 2 and perhaps Exercise sheet 10, Exercise 3.

Exercise 4 (4 points):

i) Let A be a ring and let $C \in D(A)$ be a perfect complex. Prove that the function

$$\mathfrak{p} \in \mathrm{Spec}(A) \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \dim_{k(\mathfrak{p})} H^i(C \otimes^{\mathbb{L}}_A k(s))$$

is locally constant on $\mathrm{Spec}(A)$.

ii) Let $f\colon X \to S$ be a proper, flat morphism of schemes. Prove that the map

$$s \in S \mapsto \chi_a(X_s) := \sum_{i \geq 0} (-1)^i \dim_{k(s)} H^i(X_s, \mathcal{O}_{X_s}),$$

is locally constant, where $X_s := X \times_S \mathrm{Spec}(k(s))$ for $s \in S$.

To be handed in on: Monday, 10. Juli 2017.