Prof. Dr. P. Scholze **SS 2017**
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Algebraic Geometry II

8. Exercise sheet

Exercise 1 (4 points):
Let $\mathcal{A}$ be an abelian category and let

$$\begin{array}{ccccccccc} & & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & & \end{array}$$

be a commutative diagram with exact rows. Prove the snake lemma, i.e., that there exists a natural exact sequence

$$\ker(d_1) \to \ker(d_2) \to \ker(d_3) \xrightarrow{\delta} \mathrm{coker}(d_1) \to \mathrm{coker}(d_2) \to \mathrm{coker}(d_3),$$

and deduce that a short exact sequence of complexes in $\mathcal{A}$ induces a long exact sequence in cohomology.
Hint: To construct δ let $K \subseteq A_2$ be the preimage $\alpha_2^{-1}(\ker(d_3))$. Then d_2 factors over a morphism $d_2'\colon K \to B_1$. Deduce that the composition $K \to B_1 \to \mathrm{coker}(d_1)$ factors over $K/\alpha_1(A_1) \cong \ker(d_3)$.

Exercise 2 (4 points):
i) Let $\mathcal{A} = (\mathrm{Ab})$ be the category of abelian groups. Prove that every bounded above complex $A^\bullet \in \mathcal{C}^-(\mathcal{A})$ is quasi-isomorphic to the sum of its cohomology groups.
Hint: Take a projective resolution of $A^\bullet$ and use that for abelian groups submodules of free modules are again free.
ii) Construct an example of an abelian category $\mathcal{A}$ and two complexes $A^\bullet, B^\bullet \in \mathcal{C}(\mathcal{A})$ having isomorphic cohomology in each degree, but which are not quasi-isomorphic.
Hint: Set $\mathcal{A}$ for example as the category of R-modules with $R = k[x,y]$ or $k[x]/(x^2)$.

Exercise 3 (4 points):
Let $\mathcal{A}$ be an abelian category and let $f\colon A^\bullet \to B^\bullet$ be a morphism of complexes of $\mathcal{A}$. We define the mapping cone $C(f)$ of f as $C(f)^i := B^i \oplus A^{i+1}$ with differential given by

$$C(f)^i \to C(f)^{i+1},\ (b,a) \mapsto (d_{B^\bullet}(b) + f(a), -d_{A^\bullet}(a))$$

i) Prove that there exists a short exact sequence

$$0 \to B^\bullet \xrightarrow{\iota} C(f) \to A^\bullet[1] \to 0$$

where $A^\bullet[1]$ denotes the shifted complex with $(A^\bullet[1])^i = A^{i+1}$ and differential $d_{A^\bullet[1]} = -d_{A^\bullet}$. Prove that the associated connecting morphism $\delta\colon H^i(A^\bullet[1]) = H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet)$ is given by $H^{i+1}(f)$.
ii) Construct a canonical null homotopy h_0 of $\iota \circ f$. Let $g\colon B^\bullet \to C^\bullet$ be a second morphism of complexes and let h be a null homotopy of the composition $g \circ f$. Construct a canonical morphism $k\colon C(f) \to C^\bullet$ such that $k \circ \iota = g$ and $k \circ h_0 = h$.

Exercise 4 (4 points):

Let X be a spectral space, let I be a filtered category and let $\mathcal{F}_i, i \in I$, be a direct system of abelian sheaves on X. For $U \subseteq X$ open let

$$\Psi_U \colon \varinjlim_I F_i(U) \to (\varinjlim_I F_i)(U)$$

be the canonical morphism.

i) Assume $U \subseteq X$ is open and quasi-compact. Prove that Ψ_U is injective.

ii) Assume that $U \subseteq X$ is open and qcqs. Prove that Ψ_U is bijective.

iii) Prove that for any $n \geq 0$

$$\varinjlim_I H^n(X, \mathcal{F}_i) \cong H^n(X, \varinjlim_I \mathcal{F}_i).$$

Hint: You may assume, or prove, that there exists functorial injective resolutions for abelian sheaves on X. Using this there exists a direct system $\mathcal{G}_i, i \in I$, of complexes of injective abelian sheaves and a quasi-isomorphism $\{\mathcal{F}_i\} \to \{G_i\}$ of direct systems, i.e., each $\mathcal{F}_i \to \mathcal{G}_i$ is a quasi-isomorphism. Prove $H^n(X, \varinjlim_I \mathcal{G}_i) = 0$ for any $n > 0$ using Cech cohomology and conclude by induction on n.

To be handed in on: Monday, 19. June 2017.