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# Algebraic Geometry I

## 14. Exercise sheet

**Exercise 1:**

Let $k$ be a field.
i) Show that the group $\mathrm{GL}_{n+1}(k)$ acts naturally on $\mathbb{P}^n_k$ and that this action factors through

$$\mathrm{PGL}_{n+1}(k) := \mathrm{GL}_{n+1}(k)/k^\times.$$

ii) Show that any automorphism of $\mathbb{P}^n_k$ can be lifted to an automorphism of the pair $(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$.
*Hint: Use that* $\mathrm{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$.
iii) Show

$$\mathrm{Aut}_k(\mathbb{P}^n_k) \cong \mathrm{PGL}_{n+1}(k),$$

where $\mathrm{Aut}_k(\mathbb{P}^n_k)$ denotes the group of $k$-automorphisms of $\mathbb{P}^n_k$.

**Exercise 2:**

Let $X$ be a qcqs scheme and set $A := \Gamma(X, \mathcal{O}_X)$. Show that the following are equivalent:
i) There exists an open immersion $X \hookrightarrow Y$ with $Y$ affine.
ii) The canonical morphism $X \to \mathrm{Spec}(A)$ is an open immersion.
iii) The open subsets $D(f)$ for $f \in A$ form a basis of the topology of $X$.
iv) The line bundle $\mathcal{O}_X$ is ample.
*Remark: A qcqs scheme $X$ satisfying these properties is called quasi-affine.*

**Exercise 3:**

Let $X$ be a scheme admitting an ample line bundle $\mathcal{L}$.
i) Prove that for all $m > 0$ and $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ the open set $D(s) \subseteq X$ is quasi-affine.
*Hint: Use Exercise 2.*
ii) Let $x_1, \ldots, x_n \in X$ be points. Prove that there exists an $m > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $D(s)$ is affine and $x_1, \ldots, x_n \in D(s)$.
*Hint: Define prime ideals $\mathfrak{p}_i$ in the (graded) ring $R := \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$ by*

$$f \in \mathfrak{p}_i \Leftrightarrow f(x_i) = 0$$

*and use prime avoidance in the ring $R$ to find $s' \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $x_1, \ldots, x_n \in D(s')$. Using i) reduce to the case that $X$ is quasi-affine.*

**Exercise 4:**

Let $X$ be a scheme admitting an ample line bundle. Prove that $X$ is separated.
*Hint: Use Exercise 3 and the valuative criterion for separatedness.*

To be handed in on: Tuesday, 7. February 2017.