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Algebraic Geometry I

11. Exercise sheet

Let X be a scheme of finite type over a field k. A k-rational point $x \in X(k)$ is called smooth if its *tangent space*

$$T_x X := \mathrm{Hom}_k(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, k)$$

has dimension equal to the Krull dimension of $\mathcal{O}_{X,x}$. Here $\mathfrak{m}_{X,x} \subseteq \mathcal{O}_{X,x}$ denotes the maximal ideal of $\mathcal{O}_{X,x}$.

Exercise 1 (4 Points):

Let k be a field and let X be a normal scheme of finite type over k with $\dim X \leq 1$. Prove that every $x \in X(k)$ is a smooth point of X.

Exercise 2 (4 Points):

Let k be an algebraically closed field and let $f \in k[x, y]$ be a non-zero polynomial such that $f(0, 0) = 0$. Write

$$f = f_r + f_{r+1} + \ldots + f_n + \ldots$$

with f_n homogeneous of degree n and $f_r \neq 0$. Let $X := V(f) \subseteq \mathbb{A}_k^2$ and define $Z := V(f_r) \subseteq \mathbb{A}_k^2$.
i) Prove that $f_r = l_1 \cdots l_r$ with $l_i \in k[x, y]$ homogeneous of degree 1. Deduce that Z is a union of lines.
ii) Show that X is smooth in $(0, 0)$ if and only if $r = 1$.
iii) Solve exercise I.5.1 in Hartshorne's "Algebraic Geometry". (A non-smooth point $z \in X(k)$ is also called a singular point.)
Remark: The variety Z is called the "tangent cone" of X in $(0, 0)$.

Exercise 3 (4 Points):

i) Let R be a normal domain and let G be a group acting on R by ring homomorphisms. Show that the ring

$$R^G := \{r \in R \mid gr = r \text{ for all } g \in G\}$$

of G-invariants is again a normal domain.
ii) Let k be a field of characteristic $\neq 2$. Prove that the cone $X := V(f) \subseteq \mathbb{A}_k^3$ with

$$f(x, y, z) := xy - z^2 \in k[x, y, z]$$

is a normal domain and that the point $(0, 0, 0) \in X(k)$ is not smooth.

Exercise 4 (4 Points):

Let k be a field, let X be a scheme of finite type over k and let $x \in X(k)$ be a k-rational point.
i) Show that $T_x X$ is in natural bijection to the set of morphisms

$$g\colon \mathrm{Spec}(k[\varepsilon]/\varepsilon^2) \to X$$

such that $g_{|\mathrm{Spec}(k)} = x$.
ii) Assume $X = V(f_1, \ldots, f_r) \subseteq \mathbb{A}^n_k$ and write $x = (x_1, \ldots, x_n) \in X(k)$. We define the Jacobi matrix $J_x \in k^{r \times n}$ at x as the $r \times n$-matrix

$$J_x := \begin{pmatrix} \frac{\partial f_1}{\partial X_1}(x_1, \ldots, x_n) & \ldots & \frac{\partial f_1}{\partial X_n}(x_1, \ldots, x_n) \\ \ldots & \ldots & \ldots \\ \frac{\partial f_r}{\partial X_1}(x_1, \ldots, x_n) & \ldots & \frac{\partial f_r}{\partial X_n}(x_1, \ldots, x_n) \end{pmatrix}$$

where

$$\frac{\partial f}{\partial X_i} := \sum j_i a_{j_1, \ldots, j_n} X_1^{j_1} \cdots X_i^{j_i - 1} \cdots X_n^{j_n}$$

denotes the i-th partial derivative of a polynomial

$$f = \sum a_{j_1, \ldots, j_n} X_1^{j_1} \cdots X_n^{j_n} \in k[X_1, \ldots, X_n].$$

Let $d := \dim \mathcal{O}_{X,x}$ be the Krull dimension of the local ring $\mathcal{O}_{X,x}$. Prove that x is a smooth point of X if and only if J_x has rank $n - d$.

To be handed in on: Tuesday, 17. January 2017.