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WS 2016/17

Algebraic Geometry I

10. Exercise sheet

Exercise 1 (4 Points):
Let $n, m \geq 0$. Prove (without using properties of resultants) that there exist polynomials

$$P_1, \ldots, P_t \in \mathbb{Z}[A_0, \ldots, A_n, B_0, \ldots, B_m]$$

such that for all algebraically closed fields k and all homogeneous polynomials

$$f = \sum a_i X^{n-i} Y^i,\ g = \sum b_i X^{m-i} Y^i$$

with $a_i, b_j \in k$ there exists $(x, y) \in k^2 \setminus \{(0,0)\}$ with $f(x,y) = g(x,y) = 0$ if and only if

$$P_1(a_0, \ldots, a_n, b_0, \ldots, b_m) = \ldots = P_t(a_0, \ldots, a_n, b_0, \ldots, b_m) = 0.$$

Hint/Remark: Look at a suitable subscheme $X \subseteq \mathbb{P}^1_{\mathbb{Z}} \times \mathbb{A}^{n+m+2}_{\mathbb{Z}}$. Using resultants one can even take $t = 1$ with P_1 the resultant.

Exercise 2 (4 Points):
Let A be a noetherian normal domain, let $U \subseteq X := \operatorname{Spec}(A)$ be an open subset with complement $Z := X \setminus U$. Assume that for every $z \in Z$ the local ring $\mathcal{O}_{X,z}$ has Krull dimension ≥ 2. Show that for every vector bundle $\mathcal{E}$ on X the morphism

$$\mathcal{E}(X) \to \mathcal{E}(U),\ s \mapsto s_{|U}$$

is an isomorphism.
Hint: Proposition B.70 in the book "Algebraic Geometry I" by Görtz-Wedhorn.

Exercise 3 (4 Points):
Let $f\colon X \to S$ be a morphism. The schematic image $\operatorname{Im}(f)$ of f is defined as the minimal closed subscheme $Z \subseteq S$ such that f factors through the inclusion $Z \to S$.
i) Prove that the schematic image $\operatorname{Im}(f)$ of f exists. If f is quasi-compact, show that the corresponding quasi-coherent ideal is given by the kernel of $f^\#\colon \mathcal{O}_S \to f_*(\mathcal{O}_X)$.
ii) Let k be a field and let $f \in k[x_1, \ldots, x_n]$ be a polynomial of degree m. Define

$$j\colon \mathbb{A}^n_k \to \mathbb{P}^n_k,\ (x_1, \ldots, x_n) \mapsto (1 : x_1 : \ldots : x_n).$$

Prove that the schematic image of the morphism $V(f) \hookrightarrow \mathbb{A}^n_k \xrightarrow{j} \mathbb{P}^n_k$ is defined by the homogenization $\tilde{f}(x_0, \ldots, x_n) := x_0^m f(x_1/x_0, \ldots, x_n/x_0) \in \mathcal{O}_{\mathbb{P}^n_k}(m)(\mathbb{P}^n_k)$.

Exercise 4 (4 Points):
Let k be a field and let $0 \to \mathcal{F}_n \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \xrightarrow{(x_0,\ldots,x_n)} \mathcal{O}_{\mathbb{P}^n}(1) \to 0$ be the canonical exact sequence on the projective space $\mathbb{P}^n_k$. Prove that $\mathcal{F}_n$ is a direct sum of line bundles if and only $n = 1$. If $n = 1$ determine $\mathcal{F}_n$ and prove that the above sequence is non-split.

To be handed in on: Tuesday, 10. January 2017.