Prof. Dr. P. Scholze **WS 2016/17**
Dr. J. Anschütz

Algebraic Geometry I

6. Exercise sheet

Exercise 1 (4 Points):

i) Let R be a local ring. Show that the set $\mathbb{P}^n_{\mathbb{Z}}(R)$ is in natural bijection to the set of tuples $(x_0, \ldots, x_n)$ with $x_i \in R$ and some $x_j \in R^\times$ modulo the equivalence relation

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \Leftrightarrow \exists \alpha \in R^\times : x_i = \alpha y_i \ \forall i.$$

ii) Let $n, m \geq 0$ be two integers. Show that the schemes $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{P}^m_{\mathbb{Z}}$ and $\mathbb{P}^{n+m}_{\mathbb{Z}}$ are isomorphic if and only if $n = 0$ or $m = 0$.
Hint: Count k-valued points for k a finite field.

Exercise 2 (4 Points):

i) Prove that there exists a unique morphism $\sigma_{m,n} \colon \mathbb{P}^m_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{P}^n_{\mathbb{Z}} \to \mathbb{P}^{mn+m+n}_{\mathbb{Z}}$, called the Segre embedding, which induces for every ring B the map

$$(B^{m+1} \overset{\alpha}{\twoheadrightarrow} L, B^{n+1} \overset{\beta}{\twoheadrightarrow} L') \mapsto (\alpha \otimes \beta \colon B^{mn+m+n+1} \cong B^{m+1} \otimes_B B^{n+1} \twoheadrightarrow L \otimes_B L')$$

on B-valued points.
ii) Let B be a local A-algebra. Show that

$$\sigma_{m,n}((x_0, \ldots, x_m), (y_0, \ldots, y_n)) = (x_i y_j)_{i,j}$$

under the bijection from exercise 1.

Exercise 3 (4 Points):

Let A be a ring, $X = \mathrm{Spec}(A)$.
i) Show that a sequence $M \to N \to P$ of A-modules is exact if and only if the associated sequence $\tilde{M} \to \tilde{N} \to \tilde{P}$ of $\mathcal{O}_X$-modules is exact.
ii) Let $x \in X$ be a point and let $i \colon \{x\} \to X$ be the inclusion. Show that if $i_*(\mathcal{O}_{X,x})$ is quasi-coherent, then x does not admit a non-trivial generalization.

Exercise 4 (4 Points):

A module M over a ring A is called invertible if the functor $M \otimes_A -$ is an equivalence. A module M is called finite locally free if there exists $f_1, \ldots, f_n \in A$ generating the unit ideal such that $M[f_i^{-1}]$ is a free $A[f_i^{-1}]$-module of finite rank.
i) Prove that if M is invertible, then M is a direct summand of a finite free A-module.
ii) Prove that a module M is finite locally free if and only if it is flat and finitely presented.
iii) Prove that a module M is invertible if and only if it is locally free of rank 1.

To be handed in on: Tuesday, 29. November 2016.