1. Collection of previous results

These notes are a detailed exposition of a talk I have given at a workshop in Neckarbischofsheim about the Galois group of $\mathbb{Q}_p$ as a geometric fundamental group. We will, building on the work of previous talks, introduce the algebraic Fargues-Fontaine curve $X_{E,F}$. For its construction we have to choose two fields $E$ and $F$. We fix $E/\mathbb{Q}_p$ a finite extension with residue field $\mathbb{F}_q$ and an algebraically closed non-archimedean extension $F/\mathbb{F}_q$. In particular, $F$ is perfectoid. We also fix a uniformizer $\pi \in E$.

Let $Y_{ad} := Y_{E,F} := \lim_{I \subseteq [0,1]} \text{Spa}(B_I)$ be the adic space associated with $E$ and $F$, which was constructed in talk FF II, also see [Far, Definition 2.5].

Fact 1.1 (talk FF II). $Y_{ad}$ has global sections $H^0(Y_{ad}, \mathcal{O}_{Y_{ad}}) = B$ and $B$ is an integral domain. [Far, Definition 2.5]

The Frobenius $\varphi: F \to F: x \mapsto x^q$ induces an automorphism $\varphi : Y_{ad} \to Y_{ad}$ such that $\varphi^\mathbb{Z}$ acts properly discontinuously on $Y_{ad}$. In fact, for $\varpi \in F^\times$ with $|\varpi|_F < 1$ there exists a continuous map $\delta : Y_{ad} \to [0, \infty] : y \mapsto \frac{\log |\pi(\tilde{y})|}{\log |\varpi(\tilde{y})|}$ satisfying $\delta(\varphi(y)) = \delta(y)^{1/q}$ for $y \in Y_{ad}$, where $\tilde{y}$ denotes the maximal generalization of the point $y$ in $Y_{ad}$ (compare with [Wei, Proposition 3.3.5]). We can conclude that the quotient space

$$X_{ad} := X_{E,F} := Y_{ad}/\varphi^\mathbb{Z}$$

is naturally provided with a structure sheaf making $X_{ad}$ an adic space, the so-called adic Fargues-Fontaine curve $X_{ad} = X_{E,F}$. We denote by

$$\text{pr} : Y_{ad} \to X_{ad}$$

the natural morphism of adic spaces.

It is a formal consequence of the properly discontinuous action of $\varphi^\mathbb{Z}$ on $Y_{ad}$ that the pullback $\text{pr}^*$ induces an equivalence of the category of $\mathcal{O}_{X_{ad}}$-modules with the category of $\varphi$-modules over $\mathcal{O}_{Y_{ad}}$, i.e. $\mathcal{O}_{Y_{ad}}$-modules carrying a $\varphi^\mathbb{Z}$-equivariant
action. For example, the structure sheaf \( \mathcal{O}_{X^{ad}} \) corresponds to the \( \varphi \)-module \( \mathcal{O}_{Y^{ad}} \) with its canonical isomorphism \( \varphi_{\mathcal{O}_{Y^{ad}}} \colon \varphi^* \mathcal{O}_{Y^{ad}} \cong \mathcal{O}_{Y^{ad}}. \) More generally, for \( d \in \mathbb{Z} \) we denote by \( \mathcal{O}_{X^{ad}}(d) \) or just \( \mathcal{O}(d) \) the line bundle on \( X^{ad} \) corresponding to the \( \varphi \)-module \( \mathcal{O}_{Y^{ad}}(d) \) consisting of the sheaf \( \mathcal{O}_{Y^{ad}} \) with the twisted \( \varphi \)-action

\[
\varphi_{\mathcal{O}_{Y^{ad}}}(f) := \pi^{-d} \varphi_{\mathcal{O}_{Y^{ad}}}(f)
\]

for \( f \in \mathcal{O}_{Y^{ad}}. \) The global sections \( P_d := H^0(X^{ad}, \mathcal{O}_{X^{ad}}(d)) \) are thus given by

\[
P_d = B^{d \varphi_{\mathcal{O}_{Y^{ad}}}(d) = 1} = B^{d = \pi^d}.
\]

For example, \( P_0 = E \) and \( P_d = 0 \) for \( d < 0 \) (\cite{FFb} Corollary 1.15). Elements in \( P_1 = B^{d = \pi} \) can be constructed explicitly. Namely, let \( \mathcal{G} \) be the formal group over \( \mathcal{O}_E \) associated to a Lubin-Tate law \( \mathcal{L} \) over \( \mathcal{O}_E. \) Then \( \mathcal{G} \) comes equipped with a logarithm \( \log_{\mathcal{L}}(-) \in T \cdot E[[T]] \) and a twisted Teichmüller lift

\[
[-]_{Q} : \mathcal{G}(\mathcal{O}_E) \rightarrow \mathcal{G}(W_{\mathcal{O}_E}(\mathcal{O}_E))
\]

\[
\varepsilon \mapsto \lim_{n \to \infty} \pi^n |_{\mathcal{L}} ((\varepsilon^q)^n),
\]

(\cite{FFb} Proposition 2.11) where \( \pi |_{\mathcal{L}} (-) \) denotes multiplication with respect to the Lubin-Tate law.

**Fact 1.2.** The map

\[
\mathcal{G}(\mathcal{O}_E) = (m_F, +_{\mathcal{L}}) \rightarrow P_1 = B^{d = \pi}
\]

\[
\varepsilon \mapsto \log_{\mathcal{L}}((\varepsilon |_Q)
\]

is an isomorphism of \( E \)-vector spaces (\cite{FFb} Theorem 4.6.).

We will however just use the existence of the map \( \mathcal{G}(\mathcal{O}_E) \rightarrow B^{d = \pi}. \) Up to convergence issues (see \cite{FFb} Remark 4.8) its well-definedness can be deduced as follows

\[
\varphi(\log_{\mathcal{L}}((\varepsilon |_Q)) = \log_{\mathcal{L}}((\varepsilon^q |_Q) = \log_{\mathcal{L}}((\pi |_{\mathcal{L}} (\varepsilon |_Q)) = \pi \log_{\mathcal{L}}((\varepsilon |_Q).
\]

By definition, a point \( y \in Y^{ad} \) is called classical, if its support

\[
\text{supp}(y) := \{ f \in B \mid f(y) = 0 \} \subseteq B
\]

is a maximal ideal. Similarly, define classical points in the open sets \( \text{Spa}(B_I) \subseteq Y^{ad}, \) \( I \subseteq [0,1[ \) with extremities in \( F^\times |_F \subseteq \mathbb{R}_{>0}, \) as the points whose support is a maximal ideal. Let \( Y^{ad}_{cl} \subseteq Y^{ad} \) be the subset of classical points of \( Y^{ad}. \) By \cite{FFb} Theorem 3.9, \( Y^{ad}_{cl} = \lim_{I \subseteq [0,1[} \text{Spa}(B_I)_{cl}. \) We want to point out, that for a classical point \( y \in Y^{ad}_{cl} \) the valuation on \( k(y) \) is of rank one, i.e. \( y \) is the only point in \( Y^{ad} \) with support \( \text{supp}(y). \) In fact, by \cite{FFb} Theorem 4.3, \( \) and \cite{FFb} Corollary 3.11 each closed maximal ideal of \( B \) is generated by a primitive element of degree 1. Then by \cite{FFb} Theorem 2.4, the image of \( W_{\mathcal{O}_E}(\mathcal{O}_E) \subseteq H^0(Y^{ad}, \mathcal{O}_{Y^{ad}+}) \) in \( k(y) \) is already a valuation ring of rank one, and hence \( \text{Spa}(k(y), k(y)+) = \{ y \}. \) In particular, we obtain a bijection

\[
Y^{ad}_{cl} \xrightarrow{1:1} \{ m \subseteq B \text{ closed maximal ideal} \}.
\]

**Fact 1.3** (talks FF I, FF III). If \( y \in Y^{ad}_{cl} \) is classical, then the residue field \( k(y) \) is perfectoid with a canonical identification \( k(y)^{\flat} \cong F \) of its tilt with the field \( F \) (\cite{FFb} Theorem 2.4). In particular, \( k(y) \) is algebraically closed. Moreover, the local ring \( \mathcal{O}_{Y^{ad},y} \) is a discrete valuation ring whose \( \mathfrak{m}_{Y^{ad},y} \)-adic completion is Fontaine’s ring \( B_{dR,y}^{+} \) associated to the perfectoid field \( k(y). \) (\cite{FFb} Theorem 3.9, \) and \cite{FFb} Definition 3.1)
Let $\text{Div}(Y^{\text{ad}})$ be the group of divisors on $Y^{\text{ad}}$, i.e. locally finite sums of classical points in $Y^{\text{ad}}$.

**Fact 1.4** (talk FF III). *The map*

$$\{ a \subseteq B \text{ non-zero closed ideal} \} \rightarrow \text{Div}^+(Y^{\text{ad}})$$

$$a \mapsto V(a)$$

*is an isomorphism* ([FFb, Theorem 3.8.]).

The fact 1.4 was used to analyse the multiplicative structure of the graded $E$-algebra

$$P := P_{E,\pi} := \bigoplus_{d=0}^{\infty} B^{\varphi^d} = \bigoplus_{d=0}^{\infty} B^{\varphi^d}.$$

Define the set of classical points in $X^{\text{ad}}$ as $X^{\text{ad}}_{\text{cl}} := \text{pr}(Y^{\text{ad}}_{\text{cl}}) \subseteq X^{\text{ad}}$ and let $\text{Div}(X^{\text{ad}})$ be the group of divisors on $X^{\text{ad}}$, i.e. locally finite sums of classical points on $X^{\text{ad}}$. As $X^{\text{ad}}$ is quasi-compact, being the image of the quasi-compact set $\text{Spa}(B_I)$ for some compact interval $I \subseteq [0,1]$, divisors on $X^{\text{ad}}$ are actually finite sums of classical points on $X^{\text{ad}}$. By definition, divisors on $X^{\text{ad}}$ are in bijection with $\varphi$-invariant divisors on $Y^{\text{ad}}$

$$\text{Div}(X^{\text{ad}}) \cong \text{Div}(Y^{\text{ad}})^{\varphi=1}$$

as $\text{pr}^{-1}(X^{\text{ad}}_{\text{cl}}) = Y^{\text{ad}}_{\text{cl}}$.

**Fact 1.5** (talk FF III). *The algebra $P$ is graded factorial with irreducible elements of degree 1, i.e. every non-zero homogeneous element can be written uniquely (up to the units $E^* \times P = P_1^* \times 0$) as the product of homogeneous elements of degree 1. More precisely, the divisor map*

$$\text{div} : \left( \bigcup_{d \geq 0} P_d \setminus \{ 0 \} \right)/E^* \rightarrow \text{Div}^+(X^{\text{ad}})$$

$$f \mapsto \text{div}(f)$$

*is an isomorphism* ([FFb, Theorem 4.3]). In particular, there is a bijection

$$\text{div} : (P_1 \setminus \{ 0 \})/E^* \xrightarrow{1:1} X^{\text{ad}}_{\text{cl}}.$$

### 2. The Algebraic Fargues-Fontaine Curve

We now define the algebraic Fargues-Fontaine curve.

**Definition 2.1.** *The algebraic Fargues-Fontaine curve* (for given $E$, $F$ and $\pi$) is defined as the $E$-scheme

$$X := X_{E,F} = \text{Proj}(P),$$

with $P = P_{E,F,\pi} := \bigoplus_{d \geq 0} B^{\varphi^d}$. Note, the ring $B$ depends on $E$ and $F$, but not on $\pi$.

The curve $X_{E,F}$ is independent of $\pi$ in the sense that the choice of another uniformizer $\pi'$ yields a curve $X'$ canonically isomorphic to $X$ as the following lemma shows. (see also [FFa, Section 7.1.4.])

**Lemma 2.2.** *Let $\pi_1, \pi_2 \in E$ be uniformizers with corresponding algebras* 

$$P_{\pi_i} = \bigoplus_{d \geq 0} B^{\varphi^d}$$

The curve $X_{E,F}$ is independent of $\pi$ in the sense that the choice of another uniformizer $\pi'$ yields a curve $X'$ canonically isomorphic to $X$ as the following lemma shows. (see also [FFa, Section 7.1.4.])
for \( i = 1, 2 \). Then
\[
\text{Proj}(P_{\pi_1}) \cong \text{Proj}(P_{\pi_2}),
\]
canonically and \( P_{\pi_1} \cong P_{\pi_2} \) non-canonically.

Proof. The field \( F \) is algebraically closed, hence the closure \( L := \mathbb{F}_q \subseteq \mathcal{O}_F \) lies in \( F \). As the ring \( W_{\mathcal{O}_E}(L) \) is henselian with algebraically closed residue field there exists \( u \in W_{\mathcal{O}_E}(L)^{\times} \) with
\[
\frac{\varphi(u)}{u} = \frac{\pi_1}{\pi_2}.
\]
Note that \( W_{\mathcal{O}_E}(L) \subseteq B \). In particular, the multiplications
\[
B^{\varphi=\pi_2^d} \to B^{\varphi=\pi_1^d},
\]
\[
f \mapsto u^d f
\]
for \( d \in \mathbb{Z} \) combine to an isomorphism \( \alpha_u : P_{\pi_2} \to P_{\pi_1} \). The element \( u \) is unique up to invertible elements \( v \in W_{\mathcal{O}_E}(L)^{\varphi=1} = \mathcal{O}_F \). For \( v \in \mathcal{O}_E^{\times} \) the isomorphisms \( \alpha := \alpha_u \) and \( \beta := \alpha_{vu} \) satisfy
\[
v^d \alpha(f) = \beta(f)
\]
for \( f \in P_{\pi_2,d} \) homogenous of degree \( d \). It is easy to see that two morphisms
\[
\alpha, \beta : A \to A'
\]
between non-negatively graded algebras, satisfying the above equation for some unit \( v \in A_0^{\times} \) and every \( d \geq 0 \) induce the same morphism on Proj. This proves the lemma. \( \square \)

We will see that \( X \) is indeed a “curve”, i.e. one-dimensional. In some respect, \( X \) behaves like the curve \( \mathbb{P}^1_E \) over the field \( E \) although \( X \) is not of finite type over \( E \). As \( X \) is defined via the Proj construction there are natural line bundles on \( X \) obtained by the shifted graded \( P \)-modules \( P[d] \) for \( d \in \mathbb{Z} \). Let
\[
\mathcal{O}(d) := \mathcal{O}_X(d) := \check{P}[d].
\]
Then the \( \mathcal{O}(d) \) are line bundles on \( X \) as \( P \) is generated by \( P_1 \). The global sections of \( \mathcal{O}(d) \) can be computed, using that \( P \) is graded factorial \([1.3] \) as
\[
P_d = H^0(X, \mathcal{O}_X(d)).
\]
In fact, \( P_d \) injects into \( H^0(X, \mathcal{O}_X(d)) \) as \( P \) is an integral domain. Let conversly, \( a \in H^0(X, \mathcal{O}_X(d)) \) be a global section. For \( t \in P_1 \) there exists \( d_t \geq 0 \) and \( g_t \in P_d \) with \( a_{|D^+(t)} = \frac{g_t}{t^{d_t}} \). We may assume that \( g_t \) is not divisible by \( t \) as \( P \) is graded factorial. Choose some \( t' \notin E^\times \). Then restricting to the intersection \( D^+(t) \cap D^+(t') = D^+(t \cdot t') \) yields \( \frac{g_{tt'}}{t^{d_t}} = \frac{g_{tt'}}{t'^{d_t}} \) as \( P \) is an integral domain. As \( P \) is graded factorial and \( t, t' \) are relatively prime, we can conclude \( d_t = d_{t'} = 0 \) and hence \( g := g_t = g_{t'} \) so that \( a \) is induced by the section \( g \in P_d \) as \( t \) was arbitrary.

For completeness we introduce a proof of the following lemma. To proof it we will use the adjunction
\[
\text{Hom}(Z, \text{Spec}(A)) \cong \text{Hom}(A, \Gamma(Z, \mathcal{O}_Z))
\]
for a ring \( A \) and an arbitrary locally ringed space \( Z \) (\([3.1] \text{ Proposition 1.6.3}] \)).

\footnote{If such a \( t' \) does not exists, the claim is trivial, as then \( P = E[t] \). But actually such a \( t' \) exists: by \([3.1] \) the \( E \)-vector space \( P_1 \) is infinite dimensional.}
Lemma 2.3. Let $S = \text{Spec}(R)$ be an affine scheme and

$$A = \bigoplus_{d \geq 0} A_d$$

be a graded $R$-algebra, generated by $A_1$. Let $h : \text{Proj}(A) \to S$ be the canonical morphism. Then for any locally ringed space $g : Z \to S$ the map

$$\eta : \text{Hom}_S(Z, \text{Proj}(A)) \to \{(L \in \text{Pic}(Z), \gamma : g^*A \to \bigoplus_{d \geq 0} L^\otimes d \text{ surjective}) / \cong\}$$

is a bijection, where $O(1) \in \text{Proj}(A)$ denotes the canonical line bundle $O(1) = \tilde{A}[1]$ and $\gamma_{\text{can}} : h^*(\tilde{A}) \to \bigoplus_{d \geq 0} O(d)$ the canonical surjection.

Proof. We first prove that the morphism $\gamma_{\text{can}}$, which is induced by the canonical morphism

$$A \to H^0(\text{Proj}(A), \bigoplus_{d \geq 0} O(d)),$$

is indeed surjective. As the open sets $D^+(t)$ for $t \in A_1$ cover $\text{Proj}(A)$ and the question is local, we may restrict to $D^+(t)$ for some $t \in A_1$. Then the morphism $\gamma_{\text{can}}$ is given by the multiplication

$$A[1/t]_\alpha \otimes_R A \to \bigoplus_{d \geq 0} A[1/t]_d,$$

which is easily seen to be surjective. We denote by $F(Z)$ the target of $\eta$. Then $F$ is a sheaf with respect to local isomorphisms. We define for $t \in A_1 \setminus \{0\}$ the subfunctor

$$F_t(Z) := \{(L, \gamma) \in F(Z) \mid \gamma(t) \text{ generates } L\}$$

of $F$. The inclusion $F_1 \to F$ is represented by open immersions. Indeed, for a morphism $(L, \gamma) : Z \to F$ the fiber product $Z \times_F F_1$ is represented by the open subset

$$D(\gamma(t)) := \{z \in Z \mid \gamma(t) \text{ generates } L_z\}.$$

We claim that $F_t$ is represented by the scheme $\text{Spec}(A[1/t]_\alpha)$ by sending a morphism $f : Z \to \text{Spec}(A[1/t]_\alpha)$ corresponding to the morphism $f : A[1/t]_\alpha \to \Gamma(Z, O_Z)$ to the pair

$$(O_Z, \gamma : \tilde{A}|_Z \to \bigoplus_{d \geq 0} O_Z)$$

where $\gamma$ maps a local section represented by $a \in A_d$ to $f(a/\alpha^d) \in O_Z$. As $\gamma(t^d) = 1$ for $d \geq 0$ the morphism $\gamma$ is surjective. Let conversely, $(L, \gamma) \in F_t(Z)$ be given. Define $f(a/t^d) \in \Gamma(Z, O_Z)$ for $a \in A_d$ by the formula

$$\gamma(a) = f(a/t^d) \gamma(t^d) \in L^\otimes d(Z).$$

Then $f : A[1/t]_\alpha \to \Gamma(Z, O_Z)$ is well-defined and a homomorphism of rings. It can be checked that these morphisms $\text{Spec}(A[1/t]_\alpha) \to F_t$ and $F_t \to \text{Spec}(A[1/t]_\alpha)$ are mutually inverse. Moreover, the $F_t$ for $t \in A_1$ cover $F$ as $A$ is generated by $A_1$ and $\gamma : g^*A_1 \to L$ surjective. We can conclude that $\eta$ is an isomorphism of functors as for every $t \in A_1$ the pullback

$$\text{Spec}(A[1/t]_\alpha) = D^+(t) = \text{Proj}(A) \times_F F_t \to F_t$$

is an isomorphism. □
As \( H^0(X^\text{ad}, \bigoplus_{d \geq 0} \mathcal{O}(d)) = p \) we obtain by \ref{2.3} a morphism
\[
\alpha : X^\text{ad} \to X
\]
of locally ringed spaces satisfying \( \alpha^*(\mathcal{O}_X(d)) \cong \mathcal{O}_{X^\text{ad}}(d) \). More precisely, it has to be checked, that the open sets
\[
D(t) := \{ x \in X^\text{ad} | t \text{ generates } \mathcal{O}_{X^\text{ad}}(1) \}
\]
for \( t \in P_1 \) cover \( X^\text{ad} \). We first show that for \( t \in P_1 \setminus \{0\} \) the vanishing locus
\[
V(t) := \{ x \in X^\text{ad} | t(x) = 0 \}
\]
consists of classical points. This property can be checked on \( Y^\text{ad} \) and because \( Y^\text{ad} = \lim_{t \in [0,1]} \text{Spa}(B_t) \), we may restrict to \( U := \text{Spa}(B_I) \subseteq Y^\text{ad} \) for some interval \( I \subseteq [0,1) \) whose extremities lie in \( |F^\times| \). By \cite[Theorem 3.9.]{FFb} the ring \( B_I \) is a principal ideal domain. Assume \( y \in V(t) \) for \( t \in P_1 \subseteq B_I \). If \( t \neq 0 \), then \( t \) does not vanish at the generic point of \( U \), and hence \( V(t) \) consists of points, whose support is maximal. In other words, \( V(t) \subseteq X^\text{ad} \) consists of classical points. By \ref{1.5} there is the bijection
\[
\text{div} : (P_1 \setminus \{0\})/E^\times \to X^\text{ad}.
\]
For \( t, t' \in P_1 \setminus \{0\} \) with \( t' \notin E^\times t \) (such \( t, t' \) exist as \( P_1 \) is infinite-dimensional over \( E \)), see \ref{3.1} we therefore get
\[
V(t) \cap V(t') = \emptyset,
\]
which was our claim.

3. THE FUNDAMENTAL EXACT SEQUENCE

In order to understand \( X \) we need the fundamental exact sequence. Fix an effective divisor
\[
D = \sum_{i=1}^{n} a_i y_i \in \text{Div}^+(Y^\text{ad})
\]
of degree \( d := \sum_{i=1}^{n} a_i \). Assume that \( y_i \notin \{ y_j \}^{\neq \text{x}} \) for \( i \neq j \) and let \( x_i := \text{pr}(y_i) \in X^\text{ad} \).

By \ref{1.5} we know that \( \{ x_i \} = V(t_i) \) for some \( t_i \in P_1 \setminus \{0\} = H^0(X^\text{ad}, \mathcal{O}_{X^\text{ad}}(1)) \), which is unique up to multiplication by \( E^\times = P_1^\times \). Let \( t := \prod_{i=1}^{n} t_i^{a_i} \). Then the divisor of \( t \in H^0(X^\text{ad}, \mathcal{O}_{X^\text{ad}}(d)) \) is precisely \( \sum_{i=1}^{n} a_i x_i \).

**Theorem 3.1** (Fundamental exact sequence). For \( r \geq 0 \) the sequence
\[
\begin{array}{cccccc}
0 & \to & H^0(X^\text{ad}, \mathcal{O}(r)) & \overset{t}{\to} & H^0(X^\text{ad}, \mathcal{O}(d + r)) & \overset{u}{\to} & \prod_{i=1}^{n} \mathcal{O}_{X^\text{ad}, x_i} / m_{X^\text{ad}, x_i}^{a_i} \\
& & \cong & & \cong & & \cong \\
0 & \to & P_r & \overset{t}{\to} & P_{d+r} & \overset{u}{\to} & \prod_{i=1}^{n} B_{\text{dir}, y_i}^{+} / m_{Y^\text{ad}, y_i}^{a_i} B_{\text{dir}, y_i}^{+} \\
\end{array}
\]
is exact, where \( u \) is the canonical evaluation morphism
\[
P_{d+r} \subseteq B = H^0(Y^\text{ad}, \mathcal{O}_{Y^\text{ad}}) \to \mathcal{O}_{Y^\text{ad}, y_i} / m_{Y^\text{ad}, y_i}^{a_i} \cong B_{\text{dir}, y_i}^{+} / m_{Y^\text{ad}, y_i}^{a_i} B_{\text{dir}, y_i}^{+}.
\]
Proof. We first show \( \ker(u) = tP_r \). Let \( f \in P_{d+r} \) be an element with \( u(f) = 0 \). We consider \( f \) as a function on \( Y^{ad} \) and look at its divisor \( \text{div}(f) \in \text{Div}^+(Y^{ad}) \). As \( u(f) = 0 \) we get

\[
\text{div}(f) \geq \sum_{i=1}^{n} a_i y_i.
\]

But \( \text{div}(f) \) is \( \varphi \)-invariant because \( \varphi(f) = \pi^d f \), and hence

\[
\text{div}(f) \geq \sum_{i=1}^{n} a_i \sum_{n \in \mathbb{Z}} \varphi(y_i) = \text{div}(t)
\]

where \( t \) is considered as a function on \( Y^{ad} \). Hence, by fact 1.3

\[
f = gt
\]

for some \( g \in B \). We get \( \varphi(g)\pi^d t = \pi^{d+r} gt \) and thus \( g \in P_r \) as \( B \) is an integral domain.

Factoring \( t = t_1 \cdot t' \) and considering for \( r \geq 0 \) the diagram

\[
\begin{array}{ccc}
P_r & \xrightarrow{t_1} & P_{r+1} \\
\downarrow & & \downarrow \quad t' \\
P_r & \xrightarrow{t} & P_{r+d}
\end{array}
\]

reduces the proof for surjectivity to the case \( d = 1 \) and \( t = t_1 \). Furthermore, we may assume \( r = 0 \). In fact, if \( a \in C := k(y_1) \) and \( u(t) = a^{1/r+1} \) for some \( t \in P_1 \), then \( u(t^{r+1}) = a \). We thus have to show that the map

\[
u : B^{p=\pi} \to C = k(y)
\]

is surjective. By fact 1.3 \( C \) is perfectoid and algebraically closed with tilt \( F \). In particular, \( \mathcal{O}_C/\pi \cong \mathcal{O}_F/\pi^\flat \) for some \( \pi^\flat \in F \) with \( |\pi^\flat|_F = |\pi|_C \). We will use the description \( \mathcal{G}(\mathcal{O}_F) \cong B^{p=\pi} \) from fact 1.2. We get the sequence of maps

\[
\lim_{[\pi]_{LT}} \mathcal{G}(\mathcal{O}_C) \to \lim_{[\pi]_{LT}} \mathcal{G}(\mathcal{O}_C/\pi) \cong \lim_{[\pi]_{LT}} \mathcal{G}(\mathcal{O}_F/\pi^\flat) \cong \lim_{[\varphi]_{LT}} \mathcal{G}(\mathcal{O}_F) = \mathcal{G}(\mathcal{O}_F).
\]

We used that \( F \) is perfectoid to conclude

\[
\lim_{[\varphi]_{LT}} \mathcal{G}(\mathcal{O}_F/\pi^\flat) \cong \lim_{[\varphi]_{LT}} \mathcal{G}(\mathcal{O}_F) \cong \mathcal{G}(\mathcal{O}_F).
\]

Putting things together we get the map

\[
\Psi : \lim_{[\pi]_{LT}} \mathcal{G}(\mathcal{O}_C) \to C \\
(z_n)_n \mapsto \log_{LT}(z_0)
\]

More precisely, take \( (z_n)_n \in \lim_{[\pi]_{LT}} \mathcal{G}(\mathcal{O}_C/\pi) \) with reduction \( (\tau_n)_n \in \lim_{[\pi]_{LT}} \mathcal{G}(\mathcal{O}_C/\pi) \)

and \( \varepsilon \in \mathcal{G}(\mathcal{O}_F) \) with \( \varepsilon^{1/q^n} = \tau_n \in \mathcal{O}_F/\pi^\flat = \mathcal{O}_C/\pi \) for all \( n \). Then

\[
[z]_Q = \lim_{n \to \infty} [\pi^n]_{LT}([\varepsilon^{1/q^n}]) = \lim_{n \to \infty} [\pi^n]_{LT}(z_n) = z_0,
\]

showing that

\[
\Psi((z_n)_n) = \log_{LT}([\varepsilon]_Q) = \log_{LT}(z_0).
\]
The map $\Psi$ is surjective as $C$ is algebraically closed and we can conclude. Indeed, the formula
\[ \log_{LT}(\pi_{LT}(x)) = \pi \log_{LT}(x) \]
for $x \in G(O_C)$ and the surjectivity of $[\pi]_{LT}: m_C \to m_C$ ($C$ is algebraically closed) shows that the image of $\log_{LT}: m_C \to C$ contains elements of arbitrary large absolute value. But then the logarithm $\log_{LT}$ has to be surjective as it has the Artin-Hasse-exponential as a local inverse near 0. □

Theorem 4.1 yields the following corollary.

Corollary 3.2. Let $t \in P_1 \setminus \{0\}$ with vanishing locus $V(t) = \{x\} \subseteq X^{\text{ad}}$ and $y \in Y^{\text{ad}}$ a classical point over $x$. Then for $C := k(y)$ the map
\[ \theta : P/tP \to \{g \in C[T] \mid g(0) \in E\} \]
\[ \sum_{d \geq 0} f_d \mapsto \sum_{d \geq 0} f_d(y)T^d \]
is an isomorphism of graded algebras. In particular, $\text{Proj}(P/tP) = \{(0)\}$ has one element.

Proof. It is clear that $\theta$ is a morphism of graded algebras. Moreover, it is an isomorphism in degrees $d \geq 1$ by 3.1 and trivially for $d = 0$. Finally, let $p \neq 0$ be an homogenous prime ideal of the right hand side $\{g \in C[T] \mid g(0) \in E\}$. Then $cT^d \in p$ for some $d \geq 1$ and $c \in C^\times$. Multiplying by $c^{-1}T$ yields $T^{d+1} \in p$ such that $p = (T)$, a contradiction. □

4. Properties of the algebraic Fargues-Fontaine curve

Now we are ready to prove the main theorem of this talk.

Theorem 4.1. The scheme $X$ is noetherian, integral and regular of Krull dimension one. More precisely, for $t \in P_1 \setminus \{0\}$

- $D^+(t) = \text{Spec}(B_t)$ with $B_t := P[1/t]_0 = B[1/t]^{x=1}$ a principal ideal domain.
- $V^+(t) = \{\infty_t\}$ with $\infty_t \in X$ the closed point given by the homogenous prime ideal generated by $t$, so $\infty_t = (t) \subseteq P$.

The map
\[ \text{div} : (P_1 \setminus \{0\})/E^\times \to |X| := \{x \in X \text{ closed}\} \]
\[ t \mapsto \infty_t \]
is bijective.

Proof. As $B$ is an integral domain, the curve $X$ is integral. Pick $t \in P_1 \setminus \{0\}$. Then
\[ V^+(t) \cong \text{Proj}(P/tP) = \{tP\} \]
by 3.2 showing one claim. The description of $B_t$ is clear and we can conclude that $B_t$ is factorial as $P$ is graded factorial. Moreover, the irreducible elements in $B_t$ are exactly the fractions $t'/t$ with $t' \in P_1$ not lying in $E^\times t$. We now want to prove that the ideal $(t'/t) \subseteq B_t$ is maximal. For this we use the exact sequence
\[ 0 \to t' \cdot P_r \to P_{r+1} \overset{\theta}{\to} k(x') \to 0 \]
\[ 2 \text{as for } \mathbb{P}^1_k \]
coming from \[3.1\]. Here, \(x' \in X^{ad}_d\) denotes the unique point on \(X^{ad}_d\) with \(t'(x') = 0\) \([1.5]\). As \(\theta(t) \neq 0\), by \[3.1\] the morphism \(\theta\) factors over 
\[P_1[1/t] \to k(x')\]
showing that \(B_t/(t'/t) \to k(x')\) is surjective. Assume \(f \in B_t\) satisfies \(\theta(f) = f(x') = 0\). Then there exists \(d \geq 1\) with 
\[f = \frac{g}{t^d}\]
for some \(g \in P_d\) and \(g\) automatically satisfies \(g(x') = 0\). Hence \(g \in t'P_{d-1}\) by the fundamental exact sequence \[3.1\] showing 
\[B_t/(t'/t) \cong k(x').\]
We can conclude that \(B_t\) is a principal ideal domain as it is factorial with every irreducible element generating a maximal ideal. Covering \(X\) by two sets of the form \(D^+(t)\) with \(t \in P_1\) shows that \(X\) is noetherian and regular of Krull dimension one.
Because \(t\) generates the ideal \(\ker(P \xrightarrow{eval} k(\infty_t)[T]) \subseteq P\) by \[3.1\] resp. \[3.2\] and \(P\) has units \(E^\times\) the map 
\[\text{div} : (P_1 \setminus \{0\})/E^\times \to \{x \in X \text{ closed}\} \]  
\[t \mapsto \infty_t\]
is injective. But for some \(t \in P_1 \setminus \{0\}\) every irreducible element in \(B_t\) is of the form \(t'/t\) for some \(t' \in P_1\) and hence \(\text{div}\) is surjective as \(B_t\) is a PID. \(\square\)

For \(x \in \{x \in X \text{ closed}\}\) we define 
\[\deg : \text{Div}(X) \to \mathbb{Z} : \sum_{x \in \{x \in X \text{ closed}\}} n_x x \mapsto \sum_{x \in \{x \in X \text{ closed}\}} n_x.\]
In other words, \(\deg(x) := 1\) for \(x \in \{x \in X\}\). Then for every \(f \in k(X)^\times\) in the function field \(k(X)\) of \(X\) we have 
\[\deg(\text{div}(f)) = 0,\]
which can be reinterpreted as the statement that the curve \(X\) is “complete”. Indeed, as \(P\) is graded factorial the case for general \(f \in k(X)^\times\) is reduced to the case \(f = t/t'\) with \(t, t' \in P_1 \setminus \{0\}\), where it follows from \[4.1\] namely \(\text{div}(f) = \infty_t - \infty_{t'}\). All in all, we can conclude, as \(X \setminus \{\infty_t\} = \text{Spec}(B_t)\) with \(B_t\) a principal ideal domain, that similar to the case for \(\mathbb{P}_E^1\) the degree map yields an isomorphism 
\[\text{Pic}(X) \cong \text{Cl}(X) \xrightarrow{\deg} \mathbb{Z}\]
sending the line bundle \(\mathcal{O}_X(d)\) to \(d \in \mathbb{Z}\).
But not everything for \(X\) is similar to the projective line \(\mathbb{P}_E^1\). For example, if \(x \in \{x \in X\}\) is a closed point, then the sequence 
\[0 \to \mathcal{O}(-1) \to \mathcal{O} \to k(x) \to 0\]
is exact showing that the non-zero \(E\)-vector space \(k(x)/E\) embeds into the space \(H^1(X, \mathcal{O}(-1))\), which is therefore in particular not zero contrary to the case for \(\mathbb{P}_E^1\). But still \(H^1(X, \mathcal{O}_X(d)) = 0\) for \(d \geq 0\) (see \[FFb\] Proposition 6.5.).

We can now compare the algebraic curve \(X\) with the adic curve \(X^{ad}\). Recall that by \[2.3\] the identity \(P = H^0(X^{ad}, \bigoplus_{d \geq 0} \mathcal{O}_{X^{ad}}(d))\) corresponds to a morphism 
\[\alpha : X^{ad} \to X.\]
of locally ringed spaces such that $\alpha^*(\mathcal{O}_X(d)) \cong \mathcal{O}_{X^{ad}}(d)$.

**Theorem 4.2.** The morphism $\alpha : X^{ad} \to X$ induces bijections

$$
\alpha : X^{ad} \to |X| \\
\alpha : \tilde{\mathcal{O}}_{X,x} \to \tilde{\mathcal{O}}_{X^{ad},x^{ad}}
$$

for $x^{ad} \in X^{ad}$ with $x := \alpha(x^{ad}) \in X$. In particular, for $x \in |X|$ the residue field $k(x)$ is algebraically closed and perfectoid with tilt $k(x)^\flat \cong F$ canonically up to a power of the Frobenius $\varphi : F \to F$.

**Proof.** By 1.5 and 4.1 sending a section $t \in P_1 = H^0(X, \mathcal{O}_X(1)) = H^0(X^{ad}, \mathcal{O}_{X^{ad}})$ to its vanishing set $V(t) \subseteq X$ resp. $V(t) \subseteq X^{ad}$ induces bijections of $|X|$ resp. $X^{ad}$ with the set $(P_1 \setminus \{0\})/E^\times$. In the proof of 4.1 we have seen that $\alpha$ induces an isomorphism

$$
\alpha : k(x) \to k(x^{ad})
$$

for $x^{ad} \in X^{ad}$. Moreover, if $\{x\} = V(t)$ with $t \in P_1$, then $t$ is a uniformizer in $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X^{ad},x^{ad}}$ showing that the completions

$$
\tilde{\mathcal{O}}_{X,x} \cong \tilde{\mathcal{O}}_{X^{ad},x^{ad}}
$$

are isomorphic. \hfill \Box

**References**


