G-BUNDLES ON THE ABSOLUTE FARGUES-FONTAINE CURVE

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ABSTRACT. We prove that the category of vector bundles on the absolute Fargues-Fontaine curve is canonically equivalent to the category of isocrystals. We deduce a similar result for G-bundles for some arbitrary reductive group G over a p-adic local field.

1. INTRODUCTION

In [1] L. Fargues and J.-M. Fontaine associate to every p-adic local field E with residue field $\mathbb{F}_q$ and every perfectoid field $F/\mathbb{F}_q$ a remarkable scheme $X_{E,F}$ over E, the so called "Fargues-Fontaine curve". More precisely, the scheme $X_{E,F}$ is a connected, regular Noetherian scheme of dimension 1 over E, but it is *not of finite type over E*. The construction of the scheme $X_{E,F}$ was motivated by considerations in p-adic Hodge theory and various period rings in p-adic Hodge theory appear as rings of functions on the scheme $X_{E,F}$ (or closely related objects).

A fundamental theorem of Fargues and Fontaine is the classification of vector bundles on the scheme $X_{E,F}$. If F is algebraically closed, then the classification of vector bundles on $X_{E,F}$ reads as follows.

Theorem ([1, Théorème 8.2.10])**.** *Assume that F is algebraically closed. Then the isomorphism classes of vector bundles on $X_{E,F}$ are in bijection with the set*

$$\{(\lambda_1, \cdots, \lambda_n) \in \mathbb{Q}^n \mid n \in \mathbb{N}, \lambda_1 \geq \cdots \geq \lambda_n\}.$$

Remarkably, this result is uniform for any chosen F. Recall that an isocrystal over $\bar{\mathbb{F}}_q$ is a finite dimensional vector space D over the completion $L = \widehat{E^{\mathrm{un}}}$ of the maximal unramified extension E^{un} of E together with a Frobenius semilinear isomorphism $\varphi_D \colon D \xrightarrow{\sim} D$. Then the classification of vector bundles on $X_{E,F}$ looks exactly like the classification of isomorphism classes of isocrystals over $\bar{\mathbb{F}}_q$. Even stronger, there exists a functor, whose definition will be recalled in Definition 2.3,

$$\mathcal{E}_F(-) \colon \varphi - \mathrm{Mod}_L \to \mathrm{Bun}(X_{E,F})$$

from the category $\varphi - \mathrm{Mod}_L$ of isocrystals over $\bar{\mathbb{F}}_q$ to the category $\mathrm{Bun}_{X_{E,F}}$ of vector bundles on $X_{E,F}$ inducing a bijection on isomorphism classes. However, the functor $\mathcal{E}(-)$ is far away from being an equivalence. For example the space of homomorphisms between two vector bundles $\mathcal{E}(D)$, $\mathcal{E}(D')$ on $X_{E,F}$ associated to isocrystals D, D' over $\bar{\mathbb{F}}_q$ depends on F in general.

When trying to get rid of the dependence of the Fargues-Fontaine curve on F one is immediately let to the question whether there exists an "object" over $\bar{\mathbb{F}}_q$ (or $\mathbb{F}_q$) resembling an "absolute" Fargues-Fontaine curve "$X_{E,\bar{\mathbb{F}}_q}$" (or "$X_{E,\mathbb{F}_q}$"). We do not pursue this question in the paper, but we will instead classify "vector bundles" on

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this hypothetical absolute curve. One reason is that it is easily possible to make sense out these "absolute vector bundles" over $\bar{\mathbb{F}}_q$. Namely, we define the category $\mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ of absolute vector bundles as the inverse limit over all algebraically closed perfectoid fields $F/\bar{\mathbb{F}}_q$ of the categories of vector bundles on $X_{E,F}$:[1]

$$\mathrm{Bun}_{X_E}(\bar{\mathbb{F}}_q) := 2 - \varprojlim_F \mathrm{Bun}(X_{E,F}).$$

We then define a functor

$$\mathcal{E}(-)\colon \varphi - \mathrm{Mod}_L \to \mathrm{Bun}_{X_E}(\bar{\mathbb{F}}_q)$$

as the inverse limit of the functors $\mathcal{E}_F(-)$. Our main theorem about the classification of absolute vector bundles is the following.

Theorem (cf. Theorem 3.5)**.** *The functor*

$$\mathcal{E}(-)\colon \varphi - \mathrm{Mod}_L \xrightarrow{\sim} \mathrm{Bun}_{X_E}(\bar{\mathbb{F}}_q)$$

is an equivalence of categories.

In order to prove this theorem we first establish various properties of the category $\mathrm{Bun}_{X_E}(\bar{\mathbb{F}}_q)$, which are avatars of properties of the category $\varphi - \mathrm{Mod}_L$ of isocrystals. For example, we prove that the category $\mathrm{Bun}_{X_E}(\bar{\mathbb{F}}_q)$ is abelian and admits an Harder-Narasimhan formalism with splitting Harder-Narasimhan filtration (cf. Proposition 3.2 and Proposition 3.3). To proof these properties we use the action of the automorphism group

$$A := \mathrm{Aut}_{\bar{\mathbb{F}}_q}(F)$$

of the completed algebraic closure $F := \widehat{\overline{\mathbb{F}_q((t))}}$ of the local field $\mathbb{F}_q((t))$ on the Fargues-Fontaine curve $X_{E,F}$. The crucial property of this action is that it does not have any non-trivial closed orbit (cf. Proposition 3.1). Having proved the above properties of the category of absolute vector bundles, the proof of their classification is rather straightforward using the classification theorems for isocrystals and vector bundles on the Fargues-Fontaine curve $X_{E,F}$ for F algebraically closed.
Finally we apply our classification theorem to extend the classification of vector bundles to arbitrary G-bundles on the "absolute" Fargues-Fontaine curve, where G/E is an arbitrary reductive group G (cf. Corollary 4.6).

The author wants to express his gratitude to L. Fargues for the discussions culminating in the question answered in this paper.

2. Vector bundles on the absolute Fargues-Fontaine curve

Let $E/\mathbb{Q}_p$ be a p-adic local field with residue field $\mathbb{F}_q$ and denote by L the completion of the maximal unramified extension of E with respect to some fixed algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$. Let $\sigma\colon L \to L$ be the Frobenius of L. We recall the definition of isocrystals over $\bar{\mathbb{F}}_q$.

Definition 2.1. An isocrystal (D, φ_D) over $\bar{\mathbb{F}}_q$ is a finite-dimensional L-vector space D with a σ-semilinear automorphism $\varphi_D\colon D \to D$. The category of isocrystals is denoted by $\varphi - \mathrm{Mod}_L$.

[1] Actually, this definition is equivalent to Definition 2.6 by Proposition 2.8 and pro-étale descent for vector bundles.

The category $\varphi-\mathrm{Mod}_L$ of isocrystals over $\bar{\mathbb{F}}_q$ admits a very concrete description due to Dieudonné and Manin. For an isocrystal D define its height

$$\mathrm{ht}(D)=\dim_L D$$

as the dimension of D over L. Let D be an isocrystal D of height one and let $d\in D$ be a generator of D. Then $\varphi_D(d)=ld$ for some uniquely determined $l\in L$. The (normalized) valuation

$$\deg(D):=\nu_L(l)\in\mathbb{Z}$$

of l is independent of the choice of d and called the degree of D. For an arbitrary isocrystal D set

$$\deg(D):=\deg(\Lambda^{\mathrm{ht}(D)}D)$$

as the degree of its top exterior power. Finally define the slope of an isocrystal D by

$$\mu(D):=\frac{\deg(D)}{\mathrm{ht}(D)}\in\mathbb{Q}\cup\{\infty\}.$$

The classification of isocrystals is then given by the following well-known theorem of Dieudonné and Manin.

Theorem 2.2. *The category of isocrystals is semisimple abelian. For every slope $\lambda\in\mathbb{Q}$ there exists up to isomorphism a unique simple isocrystal D_λ. The endomorphism algebra of D_λ is isomorphic to the unique central division algebra $A_{-\lambda}$ over E of invariant $-\lambda\in\mathbb{Q}/\mathbb{Z}$. For different slopes $\lambda\neq\lambda'$*

$$\mathrm{Hom}(D_\lambda,D_{\lambda'})=0.$$

Rephrased differently Theorem 2.2 states that there is an equivalence of categories

$$\varphi-\mathrm{Mod}_L\cong\prod_{\lambda\in\mathbb{Q}}A_\lambda^{\mathrm{op}}-\mathrm{Mod},$$

where A_λ^{op} denotes the opposite of the central division algebra A_λ over E of invariant $\lambda\in\mathbb{Q}/\mathbb{Z}$. The equivalence is given by sending an isocrystal D to the tuple

$$(\mathrm{Hom}(D_\lambda,D))_{\lambda\in\mathbb{Q}},$$

where D_λ is a fixed simple isocrystal of slope $\lambda\in\mathbb{Q}$.

We recall the definition of the adic Fargues-Fontaine curve $X_S^{\mathrm{ad}}=X_{E,S}^{\mathrm{ad}}$ associated with the local field E and some perfectoid space $S/\mathbb{F}_q$ (cf. [1] and [4]). If $S=\mathrm{Spa}(R,R^+)$ is affinoid perfectoid then define

$$Y_S:=Y_{E,S}:=\mathrm{Spa}(W_{\mathcal{O}_\mathcal{E}}(R^+),W_{\mathcal{O}_\mathcal{E}}(R^+))\setminus V(\varpi[\varpi_R])$$

where

$$W_{\mathcal{O}_E}(R^+)=\{\ \sum_{i=0}^{\infty}[a_i]\pi^i\mid a_i\in R^+\ \}$$

denotes the ramified Witt vectors of R^+, $\varpi\in E$ a uniformizer and $\varpi_R\in R^\times\cap R^{\circ\circ}$ a topologically nilpotent unit in R. The space Y_S does not depend on the choice of the elements ϖ and ϖ_R. Moreover, it is functorial in R. In particular, the Frobenius of R induces an automorphism

$$\varphi\colon Y_S\xrightarrow{\sim}Y_S.$$

The action of $\varphi^{\mathbb{Z}}$ on Y_S can be shown to be properly discontinuous. The adic Fargues-Fontaine curve $X_S^{\mathrm{ad}} := X_{S,E}^{\mathrm{ad}}$ is defined as the quotient

$$X_S^{\mathrm{ad}} := X_{E,S}^{\mathrm{ad}} := Y_S/\varpi^{\mathbb{Z}}.$$

If $S' \to S$ is an open immersion of affinoid perfectoid spaces over $\mathbb{F}_q$ then the induced morphisms $Y_{S'} \to Y_S$ and $X_{S'}^{\mathrm{ad}} \to X_S^{\mathrm{ad}}$ are open immersions, too. This allows to extend the definition of Y_S and X_S^{ad} to all perfectoid spaces S over $\mathbb{F}_q$. As the action of φ on Y_S is properly discontinuous the category of vector bundles on X_S^{ad} is equivalent to the category of φ-bundles on Y_S, i.e., to vector bundles $\mathcal{F}$ on Y_S together with an isomorphism $\varphi_{\mathcal{F}}\colon \varphi^*\mathcal{F} \cong \mathcal{F}$ with their Frobenius pullback. In particular, for every $d \in \mathbb{Z}$ there exists a line bundle $\mathcal{O}_{X_S^{\mathrm{ad}}}(d)$ on X_S^{ad} associated to the φ-bundle $\mathcal{O}_{Y_S}(d) := \mathcal{O}_{Y_S}$ with isomorphism

$$\varphi_{\mathcal{O}_{Y_S}(d)} := \varpi^{-d} \cdot \varphi_{\mathcal{O}_{Y_S}}$$

where $\varphi_{\mathcal{O}_{Y_S}}\colon \varphi^*\mathcal{O}_{Y_S} \xrightarrow{\sim} \mathcal{O}_{Y_S}$ denotes the canonical isomorphism. The schematic Fargues-Fontaine curve $X_S = X_{E,S}$ is then finally defined as the Proj

$$X_S := X_{E,S} = \mathrm{Proj}(\bigoplus_{d=0}^{\infty} H^0(X_S^{\mathrm{ad}}, \mathcal{O}_{X_S^{\mathrm{ad}}}(d))).$$

The line bundles $\mathcal{O}_{X_S^{\mathrm{ad}}}(d)$ depend on the choice of the uniformizer $\varpi \in E$, but the schematic Fargues-Fontaine curve not. As X_S is defined as a Proj of some algebra generated in degree 1 it comes equipped with canonical line bundles $\mathcal{O}_{X_S}(d)$ for $d \in \mathbb{Z}$. There exists a canonical morphism

$$\alpha\colon X_S^{\mathrm{ad}} \to X_S$$

of locally ringed spaces from the adic Fargues-Fontaine curve to the schematic one. The pullback of vector bundles along this morphisms induces an equivalence

$$\mathrm{Bun}(X_S) \xrightarrow{\sim} \mathrm{Bun}(X_S^{\mathrm{ad}})$$

of the categories of vector bundles [3, Theorem 6.3.12]. For example, there exists canonical isomorphisms

$$\alpha^*(\mathcal{O}_{X_S}(d)) \cong \mathcal{O}_{X_S^{\mathrm{ad}}}(d)$$

for $d \in \mathbb{Z}$.

Assume now that S contains the fixed algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Then the functoriality of the ramified Witt vectors define a canonical embedding $L = W_{\mathcal{O}_E}(\bar{\mathbb{F}}_q)[\frac{1}{\varpi}] \to \mathcal{O}_{Y_S}(Y_S)$.

Definition 2.3. Let (D, φ_D) be an isocrystal over $\bar{\mathbb{F}}_q$. We denote by

$$\mathcal{E}_S(D) \in \mathrm{Bun}(X_S)$$

the vector bundle on X_S whose associated φ-bundle on Y_S is given by

$$(\mathcal{O}_{Y_S} \otimes_L D, \varphi \otimes \varphi_D).$$

Sending D to $\mathcal{E}_S(D)$ defines a functor

$$\mathcal{E}_S(-)\colon \varphi - \mathrm{Mod}_L \to \mathrm{Bun}(X_S).$$

For example, if $d \in \mathbb{Z}$ is an integer then the isocrystal $(L, \varpi^d \cdot \sigma)$ is sent to the line bundle $\mathcal{O}_{X_S}(-d)$.

If $S = \mathrm{Spa}(F, F^\circ)$ is the spectrum of an algebraically closed perfectoid field over $\bar{\mathbb{F}}_q$ then vector bundles on X_S have been classified by the following theorem of L. Fargues and J.-M. Fontaine.

Theorem 2.4. *Assume that $F/\bar{\mathbb{F}}_q$ is an algebraically closed perfectoid field. Set $S = \mathrm{Spa}(F, F^\circ)$. Then the functor*

$$\mathcal{E}_F(-)\colon \varphi - \mathrm{Mod}_L \to \mathrm{Bun}(X_F)$$

is essentially surjective. If D_λ is a simple isocrystal of slope $-\lambda \in \mathbb{Q}$, then

$$\mathrm{End}(D_\lambda) \cong \mathrm{End}_{X_F}(\mathcal{E}_F(D_\lambda)).$$

Proof. This is proven in [1, Theoreme 8.2.10] and [1, Proposition 8.2.8]. □

We want to remark that the functor $\mathcal{E}_S(-)$ in Theorem 2.4 is far away from being an equivalence. For isocrystals D, D' of slopes $\mu(D) \leq \mu(D')$ the space

$$\mathrm{Hom}_{X_S}(\mathcal{E}_S(D), \mathcal{E}_S(D'))$$

is huge. For example, if $D = (L, \sigma)$ and $D' = (L, \varpi^{-d}\sigma)$ for some uniformizer $\varpi \in L$, then

$$\mathrm{Hom}_{X_S}(\mathcal{E}_S(D), \mathcal{E}_S(D')) = (B^+_{\mathrm{cris},E})^{\varphi = \varpi^d},$$

is infinite dimensional over $\mathbb{Q}_p$.

Let $(\mathrm{Perf}\,/\overline{\mathbb{F}}_q)$ be the category of perfectoid spaces over $\overline{\mathbb{F}}_q$. This category will be considered as a site by equipping it with the pro-étale topology (cf. [5, Chapter 8]).

Definition 2.5. We denote by Bun_X the fibered category on $(\mathrm{Perf}\,/\bar{\mathbb{F}}_q)$ sending a perfectoid space $S/\bar{\mathbb{F}}_q$ to the category $\mathrm{Bun}(X_S)$ of vector bundles on the scheme X_S (or equivalently on the adic space X_S^{ad}).

The fibered category Bun_X is even a stack for the pro-étale topology (cf. [6]). We will be interested in the sections of Bun_X over $\mathrm{Spa}(\bar{\mathbb{F}}_q)$, i.e., "vector bundles on the Fargues-Fontaine curve $X_{\mathrm{Spa}(\bar{\mathbb{F}}_q)}$". However, the discrete ring $\bar{\mathbb{F}}_q$ is not perfectoid[2] as $\bar{\mathbb{F}}_q$ is not a Tate ring. Therefore we have to extend the stack Bun_X to be able to speak about these sections. Let

$$(\widetilde{\mathrm{Perf}\,/\bar{\mathbb{F}}_q})$$

be the topos of sheaves of sets on the site $(\mathrm{Perf}\,/\bar{\mathbb{F}}_q)$. Let $\mathfrak{X}$ be a stack on $(\mathrm{Perf}\,/\bar{\mathbb{F}}_q)$ and let $F \in (\widetilde{\mathrm{Perf}\,/\bar{\mathbb{F}}_q})$ be a sheaf. Then we define

$$\mathfrak{X}(F) := \varprojlim_{S \to F} \mathfrak{X}(S)$$

where the (2-)limit is taken over perfectoid spaces $S/\bar{\mathbb{F}}_q$ with a morphism to F. With this definition the stack $\mathfrak{X}$ on $(\mathrm{Perf}\,/\bar{\mathbb{F}}_q)$ extends to a stack, still denoted $\mathfrak{X}$, on $(\widetilde{\mathrm{Perf}\,/\bar{\mathbb{F}}_q})$.

Let $R/\bar{\mathbb{F}}_q$ be a perfect ring equipped with the discrete topology and denote by $\mathrm{Spa}(R)$ the adic spectrum of the affinoid ring (R, R). Then the functor

$$(\mathrm{Perf}\,/\bar{\mathbb{F}}_q) \to (\mathrm{Sets}), \quad S \mapsto \mathrm{Hom}_{\bar{\mathbb{F}}_q}(S, \mathrm{Spa}(R)) = \mathrm{Hom}_{\bar{\mathbb{F}}_q}(R, \mathcal{O}_S(S))$$

[2] although it is perfect

is a pro-étale sheaf and will again be denoted by $\mathrm{Spa}(R)$. In particular, $\mathrm{Spa}(\bar{\mathbb{F}}_q)$ defines a terminal object in $(\widetilde{\mathrm{Perf}\,/\bar{\mathbb{F}}_q})$.

Definition 2.6. The category of vector bundles on the absolute Fargues-Fontaine curve is defined as the global sections $\mathrm{Bun}_X(\bar{\mathbb{F}}_q) := \mathrm{Bun}_X(\mathrm{Spa}(\bar{\mathbb{F}}_q))$ of the stack Bun_X.

We remark that we did not define an absolute Fargues-Fontaine curve and then the category of vector bundles on it, but directly this category. In particular, we do not claim that there exists an "absolute Fargues-Fontaine curve" over $\bar{\mathbb{F}}_q$. However, driven by the intuition that objects in $\mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ are vector bundles on "$X_{\bar{\mathbb{F}}_q}$" we also call them "absolute vector bundles".

We will now define a functor from the category of isocrystals over $\bar{\mathbb{F}}_q$ to the category of absolute vector bundles. Let $S' \to S$ be a morphism of perfectoid spaces over $\bar{\mathbb{F}}_q$. It induces a morphism $f\colon X_{S'} \to X_S$ of Fargues-Fontaine curves. We get a diagram

$$\begin{array}{ccc} \varphi-\mathrm{Mod}_L & \xrightarrow{\mathcal{E}_S(-)} & \mathrm{Bun}_X(S) \\ & \searrow^{\mathcal{E}_{S'}(-)} & \downarrow f^* \\ & & \mathrm{Bun}_X(S'), \end{array}$$

which is easily seen to be 2-commutative, i.e., there is a natural isomorphism

$$f^* \circ \mathcal{E}_S(-) \cong \mathcal{E}_{S'}(-)$$

of functors.

Definition 2.7. We denote by

$$\mathcal{E}(-)\colon \varphi-\mathrm{Mod}_L \to \mathrm{Bun}_X(\bar{\mathbb{F}}_q)$$

the inverse limit of the functors

$$\mathcal{E}_S(-)\colon \varphi-\mathrm{Mod}_L \to \mathrm{Bun}_X(S).$$

Our main theorem will be that the functor $\mathcal{E}(-)$ is an equivalence of categories.

We define

$$F := \widehat{\overline{\mathbb{F}_q((t))}}$$

as a completed algebraic closure of $\mathbb{F}_q((t))$ and set

$$S := \mathrm{Spa}(F, F^\circ).$$

We remark that the sheaf $S \times_{\mathrm{Spa}(\bar{\mathbb{F}}_q)} S$ on $(\mathrm{Perf}\,/\bar{\mathbb{F}}_q)$ is represented by a perfectoid space. Indeed, this product is isomorphic to a connected pro-étale cover of the punctured perfectoid open unit disc

$$\mathbb{D}_F^{*,1/p^\infty} \cong S \times_{\bar{\mathbb{F}}_q} \mathrm{Spa}(\bar{\mathbb{F}}_q((t^{1/p^\infty})))$$

over F. Here $\bar{\mathbb{F}}_q((t^{1/p^\infty}))$ denotes the completed perfection of $\bar{\mathbb{F}}_q((t))$.

Proposition 2.8. *The category $\mathrm{Bun}_X(\overline{\mathbb{F}}_q)$ of absolute vector bundles is equivalent to the 2-limit of the diagram*

$$\mathrm{Bun}_X(S) \rightrightarrows \mathrm{Bun}_X(S \times_{\overline{\mathbb{F}}_q} S) \rightrightarrows \cdots .$$

In particular, the pullback functor $\mathrm{Bun}_X(\bar{\mathbb{F}}_q) \to \mathrm{Bun}_X(S)$ is faithful.

Proof. The object S is a cover of $\mathrm{Spa}(\bar{\mathbb{F}}_q)$ in the site $(\widetilde{\mathrm{Perf}\,/\bar{\mathbb{F}}_q})$. Namely, the morphism

$$S \to \mathrm{Spa}(\bar{\mathbb{F}}_q((t^{1/p^\infty})))$$

is a pro-étale cover and moreover every perfectoid space $T/\bar{\mathbb{F}}_q$ admits locally (in the analytic topology) an invertible, topologically nilpotent global section, i.e., locally a morphism $T \to \mathrm{Spa}(\bar{\mathbb{F}}_q((t^{1/p^\infty})))$. This shows that $\mathrm{Spa}(\bar{\mathbb{F}}_q((t^{1/p^\infty})))$ is a cover of $\mathrm{Spa}(\bar{\mathbb{F}}_q)$. Then the statement is the defining property for stacks: If $\mathfrak{X}$ is a stack on some site $\mathcal{C}$ and $T' \to T$ a covering in $\mathcal{C}$, then the category $\mathfrak{X}(T)$ is equivalent to the category of descent data along $T' \to T$ which in turn is equivalent to the 2-limit of $\mathfrak{X}$ over the Čech nerve

$$T \rightleftarrows T' \times_T T' \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \cdots .$$

This finishes the proof. ☐

For a topological group G we denote by

$$\underline{G} = \mathrm{Hom}_{\mathrm{cts}}(\pi_0(-), G)$$

the sheaf on $(\mathrm{Perf}\,/\bar{\mathbb{F}}_q)$ represented by G and by $B\underline{G}$ the stack of $\underline{G}$-torsors on $(\mathrm{Perf}\,/\bar{\mathbb{F}}_q)$.

We will need the following statement.

Proposition 2.9. *Let G be a topological group. Then every $\underline{G}$-torsor over $\bar{\mathbb{F}}_q$ is trivial, i.e.,*

$$B\underline{G}(\bar{\mathbb{F}}_q) = *.$$

Proof. Let as before

$$F := \widehat{\overline{\mathbb{F}_q((t))}}$$

be a completed algebraic closure of $\bar{\mathbb{F}}_q((t))$. As in Proposition 2.8 we know that $\underline{G}$-torsors over $\bar{\mathbb{F}}_q$ are equivalent to $\underline{G}$-torsors over $S := \mathrm{Spa}(F)$ together with a descent datum along $S \to \mathrm{Spa}(\bar{\mathbb{F}}_q)$. But every pro-étale $\underline{G}$-torsor over S is trivial and hence any descent datum along $S \to \mathrm{Spa}(\bar{\mathbb{F}}_q)$ is equivalent to some element $\sigma \in \underline{G}(S \times_{\bar{\mathbb{F}}_q} S)$. But $S \times_{\bar{\mathbb{F}}_q} S$ is connected (it is represented by the inverse limit of connected covers of the punctured open unit disc over F) and therefore $\sigma \in \underline{G}(S \times_{\bar{\mathbb{F}}_q} S) = \underline{G}(S)$ is already defined over S. The cocycle condition over $S \times_{\bar{\mathbb{F}}_q} S \times_{\bar{\mathbb{F}}_q} S$ implies that

$$\sigma = \sigma\sigma,$$

i.e., $\sigma = 1$. This implies that the trivial torsor $\underline{G}$ over S admits only the trivial descent datum. Hence, every $\underline{G}$-torsor over $\mathrm{Spa}(\bar{\mathbb{F}}_q)$ is trivial. ☐

3. Classification of vector bundles

In this section we prove the main theorem about vector bundles on the absolute Fargues-Fontaine curve. We keep the notation from the last section, in particular

$$F = \widehat{\overline{\mathbb{F}_q((t))}}$$

denotes a completed algebraic closure of $\bar{\mathbb{F}}_q((t))$. Moreover, let

$$A := \mathrm{Aut}_{\overline{\mathbb{F}}_q}(F)$$

be the group of continuous $\overline{\mathbb{F}}_q$-linear automorphisms of F. Let $\mathbb{C}_p = \mathbb{C}_E$ be a completed algebraic closure of E, i.e., we fix an embedding $E \hookrightarrow \mathbb{C}_p$ into the p-adic complex numbers. Then F is non-canonically isomorphic to the tilt $\mathbb{C}_p^\flat$ of $\mathbb{C}_p$. This shows that (non-canonically) the absolute Galois group $\mathrm{Gal}(\mathbb{C}_p/L)$ of the completed maximal unramified extension L/E injects into A. But the group A is larger: For any positive rational $m \in \mathbb{Q}_{>0}$ there exists an $\bar{\mathbb{F}}_q$-automorphism of F sending t to t^m.

By transport of structure the group A acts on the Fargues-Fontaine curve $X_{E,F}$ associated to E and F. This action has the following crucial property.

Proposition 3.1. *The action of the group A on the closed points of $X_{E,F}$ has no finite orbit.*

Proof. In the case $E = \mathbb{Q}_p$ this is proven in [1, Proposition 10.1.1] in the following way. Fix an isomorphism $\mathbb{C}_p^\flat \cong F$ sending $p^\flat = (p, p^{1/p}, ...) \in \mathbb{C}_p^\flat$ to $t \in F$. We will identify F and $\mathbb{C}_p^\flat$ via this isomorphism. In particular, this yields an embedding $\mathrm{Gal}(\mathbb{C}_p/L) \hookrightarrow A$. By [1, Proposition 10.1.1] the action of $\mathrm{Gal}(\mathbb{C}_p/L)$ on $X_{\mathbb{Q}_p,F}$ has at most one finite orbit given by the point ∞ corresponding to the untilt $\mathbb{C}_p$ of $F = \mathbb{C}_p^\flat$. In particular, A has at most the finite orbit $\infty \in X_{\mathbb{Q}_p,F}$. To show that the point $\infty \in X_{\mathbb{Q}_p,F}$ is not stable under A we use the uniformization of the adic Fargues-Fontaine curve

$$X^{\mathrm{ad}}_{\mathbb{Q}_p,F} = Y_{\mathbb{Q}_p,F}/\varphi^{\mathbb{Z}}$$

by the space $Y := Y_{\mathbb{Q}_p,F}$. The closed points of $X_{\mathbb{Q}_p,F}$ are then in bijection with $\varphi^{\mathbb{Z}}$-orbits of classical points in Y.[3] The $\varphi^{\mathbb{Z}}$-orbit corresponding to $\infty \in X_{\mathbb{Q}_p,F}$ is given by the points $y_n \in Y$, $n \in \mathbb{Z}$, whose vanishing ideal in $\mathcal{O}_Y(Y)$ is generated by $(p - [(p^\flat)^{1/p^n}])$. Now choose an element $\alpha \in A$ such that $\alpha(t) = t^m$ where $m \in \mathbb{Q}_{>0} \setminus \mathbb{Z}[1/p]$. Then

$$\alpha(p - [p^\flat]) = p - [(p^\flat)^m]$$

is not contained in any of the vanishing ideals of the points y_n, $n \in \mathbb{Z}$, because for every $n \in \mathbb{Z}$ the canonical morphism

$$\theta_n \colon \mathcal{O}_Y(Y) \to k(y_n) = \mathbb{C}_p$$

onto the residue field $k(y_n)$ of y_n, i.e., Fontaine's θ composed with the n-th power of Frobenius, sends the Teichmüller lift $[(p^\flat)^m]$ to some element with absolute value $|p|_{\mathbb{C}_p}^{mp^n} \neq |p|_{\mathbb{C}_p}$. This shows that α does not preserve the $\varphi^{\mathbb{Z}}$-orbit $\{y_n \mid n \in \mathbb{Z}\}$. Hence the point $\infty \in X_{\mathbb{Q}_p,F}$ is not fixed by A proving the claim in the case $E = \mathbb{Q}_p$. If $E/\mathbb{Q}_p$ is a general local field, then there exists an A-equivariant morphism $X_{E,F} \to X_{\mathbb{Q}_p,F}$. Hence every finite orbit of A on $X_{E,F}$ must lie over some finite orbit of $X_{\mathbb{Q}_p,F}$ of which there are none. This proves the proposition also for general E. □

Proposition 3.1 has the following consequence.

Proposition 3.2. *The category $\mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ of absolute vector bundles is abelian.*

Proof. Clearly, the category $\mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ is additive. Let

$$f^* \colon \mathrm{Bun}_X(\bar{\mathbb{F}}_q) \to \mathrm{Bun}_X(\mathrm{Spa}(F))$$

be the canonical pullback functor. By Proposition 2.8 an absolute vector bundle is equivalently given by a vector bundle $\mathcal{E}$ on $X_{E,F}$ together with a descent datum

$$\sigma_{\mathcal{E}} \colon \mathrm{pr}_1^* \mathcal{E} \cong \mathrm{pr}_2^* \mathcal{E}$$

[3] A point $y \in Y$ is classical if and only if the ideal $\{f \in \mathcal{O}_Y(Y) \mid f(y) = 0\}$ is maximal.

on $\mathrm{Spa}(F) \times_{\bar{\mathbb{F}}_q} \mathrm{Spa}(F)$. In particular, every vector bundle with descent datum admits a canonical A-equivariant structure by pulling back the isomorphism $\sigma_{\mathcal{E}}$ along the morphism

$$\mathrm{Spa}(F) \times A \to \mathrm{Spa}(F) \times_{\bar{\mathbb{F}}_q} \mathrm{Spa}(F), \quad (x, g) \mapsto (x, gx).$$

Hence if $g \colon (\mathcal{E}, \sigma_{\mathcal{E}}) \to (\mathcal{F}, \sigma_{\mathcal{F}})$ is a morphism of bundles with descent datum, then its cokernel $\mathcal{C}$, which admits a canonical descent datum, admits an action by A. In particular, the torsion subsheaf of $\mathcal{C}$ is stable under A and hence trivial by Proposition 3.1. This shows that $\mathcal{C}$ is torsion-free, hence a vector bundle on $X_{E,F}$ as $X_{E,F}$ is a Dedekind scheme. Therefore also the kernel $\mathcal{K}$ of g, which is again a vector bundle, is canonically equipped with a descent datum. In particular, the kernel and cokernel of g descent to absolute vector bundles proving that $\mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ is abelian. □

Let $\mathcal{E} \in \mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ be an absolute vector bundle and let

$$f^* \colon \mathrm{Bun}_X(\bar{\mathbb{F}}_q) \to \mathrm{Bun}_X(F)$$

be the canonical pullback functor. We define the degree $\deg(\mathcal{E})$, the rank $\mathrm{rk}(\mathcal{E})$ and the slope $\mu(\mathcal{E})$ of $\mathcal{E}$ by

$$\begin{aligned} \deg(\mathcal{E}) &:= \deg(f^*\mathcal{E}) \\ \mathrm{rk}(\mathcal{E}) &:= \mathrm{rk}(f^*\mathcal{E}) \\ \mu(\mathcal{E}) &:= \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E}). \end{aligned}$$

It is immediate that the functions deg, rk on $\mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ satisfy the assumptions for the Harder-Narasimhan formalism of [1, Chapitre 5.5.] (taking as a "generic fiber functor" the identity of $\mathrm{Bun}_X(\bar{\mathbb{F}}_q)$, which is allowed by Proposition 3.2). Hence, the Harder-Narasimhan formalism is applicable and every absolute vector bundle admits a Harder-Narasimhan filtration whose associated graded pieces are semistable of strictly decreasing slope.

We are aiming to prove that the category of absolute vector bundles is equivalent to the category of isocrystals, hence the next proposition must hold.

Proposition 3.3. *Let $\mathcal{E} \in \mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ be an absolute vector bundle. Then the HN-filtration of $\mathcal{E}$ splits canonically. Moreover, there do not exist any non-trivial homomorphisms between semistable vector bundles of different slopes.*

Proof. We can argue as in the case of isocrystals (cf. [1, Exemple 5.5.2.3.]). Namely, also the functions $-\deg$ and rk satisfy the assumptions for the HN-formalism of [1, Chapitre 5.5] giving "reversed" HN-filtrations. More precisely, if

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_n = \mathcal{E}$$

is the HN-filtration with respect to the degree function deg and

$$0 = \mathcal{E}'_0 \subseteq \mathcal{E}'_1 \subseteq \cdots \subseteq \mathcal{E}'_m = \mathcal{E}$$

the one with respect to the degree function $-\deg$, then $n = m$ and, by comparing slopes,

$$\mathcal{E}'_{i+1}/\mathcal{E}'_i \cong \mathcal{E}_{n-i}/\mathcal{E}_{n-i-1}.$$

This defines inductively the canonical splitting of the HN-filtration. Now let $\mathcal{E}$ and $\mathcal{F}$ be semistable vector bundles of slopes $\mu(\mathcal{E}) \neq \mu(\mathcal{F})$. The general Harder-Narasimhan formalism implies that

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0$$

if $\mu(\mathcal{E}) > \mu(\mathcal{F})$. Considering the HN-formalism with deg replaced by $-\deg$, then also proves

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0$$

if $\mu(\mathcal{E}) < \mu(\mathcal{F})$. □

We record a small lemma concerning semistability of an absolute vector bundle and of its pullback to $\mathrm{Spa}(F)$.

Lemma 3.4. *Let $\mathcal{E} \in \mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ be an absolute vector bundle and let*

$$f^*\colon \mathrm{Bun}_X(\bar{\mathbb{F}}_q) \to \mathrm{Bun}_X(F)$$

be the canonical pullback. Then $\mathcal{E}$ is semistable if and only if $f^*\mathcal{E}$ is semistable.

Proof. The canonicity of the HN-filtration together with the fact that it works in families (cf. [3, Corollary 7.4.10]) implies that the HN-filtration of $f^*\mathcal{E}$ is stable under the descent datum, hence that it descents to a filtration of $\mathcal{E}$. Assume now that $\mathcal{E}$ is a semistable absolute vector bundle and let $\mathcal{F} \subseteq f^*\mathcal{E}$ be the first step in the HN-filtration of $f^*\mathcal{E}$. As argued above, $\mathcal{F}$ descents to a subbundle $\mathcal{E}' \subseteq \mathcal{E}$. Moreover by using semistability of $\mathcal{E}$ we can conclude

$$\mu(\mathcal{F}) = \mu(\mathcal{E}') \leq \mu(\mathcal{E}) = \mu(f^*\mathcal{F}),$$

i.e., that $f^*\mathcal{E}$ is semistable. Conversely, if $f^*\mathcal{E}$ is semistable and $\mathcal{E}' \subseteq \mathcal{E}$ a subbundle, then

$$\mu(\mathcal{E}') = \mu(f^*\mathcal{E}') \leq \mu(f^*\mathcal{E}) = \mu(\mathcal{E}),$$

hence $\mathcal{E}$ is semistable. □

We are now ready to prove our main theorem.

Theorem 3.5. *Let E be a p-adic local field and denote by $L = \widehat{E^{\mathrm{un}}}$ the completion of the maximal unramified extension of E. Then the functor*

$$\mathcal{E}(-)\colon \varphi - \mathrm{Mod}_L \to \mathrm{Bun}_X(\bar{\mathbb{F}}_q)$$

is an equivalence.

Proof. We first prove fully faithfulness of $\mathcal{E}(-)$. By Proposition 3.3 and the similar result for isocrystals (cf. Theorem 2.2), we may reduce the assertion about fully faithfulness of $\mathcal{E}(-)$ to the claim that

$$\mathrm{Hom}(D, D) \cong \mathrm{Hom}_{X_{E,\bar{\mathbb{F}}_q}}(\mathcal{E}(D), \mathcal{E}(D))$$

for D a simple isocrystal of slope λ (cf. Theorem 2.2). If D is a simple isocrystal of slope λ, then both groups $\mathrm{Hom}(D, D)$ and $\mathrm{Hom}_{X_{E,\bar{\mathbb{F}}_q}}(\mathcal{E}(D), \mathcal{E}(D))$ are isomorphic to the central division algebra over E of invariant $-\lambda$ (cf. Theorem 2.2 and Theorem 2.4). This proves fully faithfulness of $\mathcal{E}(-)$. In order to prove that the functor $\mathcal{E}(-)$ is essentially surjective, i.e., that every absolute vector bundle $\mathcal{E} \in \mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ is isomorphic to some $\mathcal{E}(D)$ with $D \in \varphi - \mathrm{Mod}_L$ an isocrystal over $\bar{\mathbb{F}}_q$ we may by decomposing the HN-filtration reduce to the case that $\mathcal{E}$ is semistable. We know by Theorem 2.4 that after pullback to F the vector bundle $\mathcal{E}$ is isomorphic to $\mathcal{E}_F(D)$ for some isoclinic isocrystal D, i.e., that it is contained in the residual gerbe of the stack Bun_X at the point $\mathcal{E}(D) \in \mathrm{Bun}_X(\bar{\mathbb{F}}_q)$. Then we invoke a result of Kedlaya and Liu (cf. [6, Theorem 2.26]), namely that this residual gerbe is isomorphic to the classifying stack $B(\underline{J(E)^\times})$ for the sheaf represented by the group of units of the endomorphism algebra $J(E) = \mathrm{End}(D)$ of the isocrystal D. By Proposition 2.9

the sections of this gerbe over $\bar{\mathbb{F}}_q$ are trivial. Hence, $\mathcal{E} \cong \mathcal{E}(D)$ and thus the functor $\mathcal{E}(-)$ is essentially surjective. □

4. G-bundles on the absolute Fargues-Fontaine curve

Let G be an arbitrary reductive group over E. In this section we want to generalize our main theorem 3.5 from GL_n-bundles to arbitrary G-bundles on the absolute Fargues-Fontaine curve.

First we recall the definition of a G-bundle in our setting.

Definition 4.1. Let $S/\mathbb{F}_q$ be a non-empty perfectoid space. A G-bundle (or G-torsor) over X_S is defined to be an exact, faithful tensor functor

$$\mathcal{T} \colon \mathrm{Rep}_E(G) \to \mathrm{Bun}(X_S)$$

from the category of representations of G on finite-dimensional E-vector spaces to the category of vector bundles on X_S.

For $S = \emptyset$ the groupoid of G-bundles on $X_S = \emptyset$ is defined to be the terminal category. For a perfectoid space $S/\bar{\mathbb{F}}_q$ we denote by $\mathrm{Bun}_G(S)$ the groupoid of G-bundles on X_S and by Bun_G the fibered category associating to each S the groupoid $\mathrm{Bun}_G(S)$. As in the case of vector bundles the fibered category Bun_G turns out to be a stack.

In fact definition 4.1 is equivalent to the usual notion of a G-torsor in the étale topology (cf. [2]). Using GAGA for the curve it follows directly from our definition that G-bundles on X_S are equivalent to G-bundles on X_S^{ad}.

Definition 4.2. The category of G-bundles (or G-torsors) on the absolute Fargues-Fontaine curve is defined as

$$\mathrm{Bun}_G(\bar{\mathbb{F}}_q),$$

i.e., as the global sections of the stack Bun_G on $(\mathrm{Perf}/\bar{\mathbb{F}}_q)$.

Recall that L denotes the completion of the maximal unramified extension of E and $\sigma \colon L \to L$ its Frobenius.

Definition 4.3. The Kottwitz category for G is defined as follows. It has as objects the elements in

$$G(L).$$

For $b, b' \in G(L)$ the set of homomorphisms from b to b' is defined to be the set

$$\{c \in G(L) \mid cb\sigma(c)^{-1} = b'\}.$$

Finally, composition is defined by multiplication in $G(L)$. In other words, the Kottwitz category is the quotient groupoid

$$[G(L)/\sigma - \mathrm{conj.}]$$

of $G(L)$ modulo σ-conjugacy. The set of isomorphism classes in the Kottwitz category, i.e., the quotient set of $G(L)$ modulo σ-conjugation, is denoted by $B(G)$.

Note that in the case $G = \mathrm{GL}_n$ the Kottwitz category is equivalent to the groupoid of isocrystals of height n together with their isomorphisms. Similarly to the case $G = \mathrm{GL}_n$ there is a functor from the Kottwitz category to the category of G-bundles on the absolute Fargues-Fontaine curve for arbitrary reductive groups G/E.

Definition 4.4. Let $S/\bar{\mathbb{F}}_q$ be an arbitrary perfectoid space. Let $b \in G(L)$ be an element. We define $\mathcal{T}_S(b)$ to be the G-bundle over X_S which is given by the functor

$$\begin{array}{ccc} \mathrm{Rep}_E(G) & \to & \varphi - \mathrm{Bun}(Y_S) \cong \mathrm{Bun}(X_S) \\ (V, \rho\colon G \to \mathrm{GL}(V)) & \mapsto & (\mathcal{O}_{Y_S} \otimes_L V_L, \varphi_{\mathcal{O}_{Y_S}} \otimes \rho(b)). \end{array}$$

Sending b to $\mathcal{T}_S(b)$ defines a functor

$$\mathcal{T}_S(-)\colon G(L) \to \mathrm{Bun}_G(S)$$

from the Kottwitz category to the category of G-bundles on the Fargues-Fontaine curve over S. As in the case of vector bundles the functors $\mathcal{T}_S(-)$ for varying S define a functor

$$\mathcal{T}(-)\colon G(L) \to \mathrm{Bun}_G(\bar{\mathbb{F}}_q) \cong 2 - \varprojlim_S \mathrm{Bun}_G(S)$$

In [2] L. Fargues classified G-torsors on the Fargues-Fontaine curve $X_{E,F}$ where F is an arbitrary algebraically closed perfectoid field. As in the case of vector bundles the result says that the functor $\mathcal{T}_F(-)$ is essentially surjective. Moreover, contrary to the case of vector bundles, the functor $\mathcal{T}_F(-)$ is even fully faithful, i.e., an equivalence.

In the spirit of the Tannaka style definition of a G-bundle we could have defined a G-bundle on the absolute Fargues-Fontaine curve as an exact, faithful tensor functor from representations of G to absolute vector bundles. This definition turns out to be equivalent.

Proposition 4.5. *There is an equivalence of categories, induced by the universal property of* 2*-limits, between the category* $\mathrm{Bun}_G(\bar{\mathbb{F}}_q)$ *of absolute* G*-bundles and the category of exact faithful tensor functors*

$$\mathcal{T}\colon \mathrm{Rep}_E(G) \to \mathrm{Bun}_X(\bar{\mathbb{F}}_q).$$

from the category of representations $\mathrm{Rep}_E(G)$ *of* G *to the category* $\mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ *of absolute vector bundles.*

Proof. The category $\mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ is by definition equivalent to the limit

$$2 - \varprojlim_S \mathrm{Bun}_X(S)$$

of Bun_X over all perfectoid spaces $S/\bar{\mathbb{F}}_q$. In particular, there is an equivalence of the category of arbitrary functors

$$\mathcal{T}'\colon \mathrm{Rep}_E(G) \to \mathrm{Bun}_X(\bar{\mathbb{F}}_q)$$

with the 2-limit of the categories of arbitrary functors

$$\mathcal{T}'_S\colon \mathrm{Rep}_E(G) \to \mathrm{Bun}_X(S).$$

We want to show that under this equivalence the exact, faithful tensor functors are matched up. On the one hand the pullback functor $\mathrm{Bun}_X(\bar{\mathbb{F}}_q) \to \mathrm{Bun}_X(S)$ is a tensor functor and preserves exactness and faithfulness (for S non-empty). Hence if an exact faithful tensor functor $\mathcal{T}'\colon \mathrm{Rep}_E(G) \to \mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ is given, then the corresponding functors $\mathcal{T}'_S$ are again exact, faithful tensor functors (for S non-empty). On the other hand, a collection of exact, faithful tensor functors $\mathcal{T}_S$ induce a functor $\mathcal{T}\colon \mathrm{Rep}_E(G) \to \mathrm{Bun}_X(\bar{\mathbb{F}}_q)$ which is easily seen to be an exact, faithful tensor functor. Indeed, to see this it suffices to compose $\mathcal{T}$ with the exact, faithful pullback functor $\mathrm{Bun}_X(\bar{\mathbb{F}}_q) \to \mathrm{Bun}_X(\mathrm{Spa}(\bar{\mathbb{F}}_q((t^{1/p^\infty}))))$. ∎

Theorem 3.5 and Proposition 4.5 imply the following corollary.

Corollary 4.6. *Let E be a p-adic local field and let G be any reductive group over E. Let $L = \widehat{E^{\mathrm{un}}}$ be the completed maximal unramified extension of E. Then the functor*

$$\mathcal{T}(-)\colon G(L) \to \mathrm{Bun}_G(\bar{\mathbb{F}}_q)$$

from the Kottwitz category to the category of G-bundles on the absolute Fargues-Fontaine curve is an equivalence.

Proof. By Proposition 4.5 and Theorem 3.5 the groupoid of G-bundles on the absolute Fargues-Fontaine curve is equivalent to the groupoid of exact and faithful tensor functors

$$\mathrm{Rep}_E(G) \to \varphi - \mathrm{Mod}_L$$

to the category of isocrystals. This category is known to be equivalent to the Kottwitz category $G(L)$. □

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