The smooth locus in infinite-level Rapoport-Zink spaces

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Abstract

Rapoport-Zink spaces are deformation spaces for p-divisible groups with additional structure. At infinite level, they become preperfectoid spaces. Let $\mathscr{M}_\infty$ be an infinite-level Rapoport-Zink space of EL type, and let $\mathscr{M}_\infty^\circ$ be one connected component of its geometric fiber. We show that $\mathscr{M}_\infty^\circ$ contains a dense open subset which is cohomologically smooth in the sense of Scholze. This is the locus of p-divisible groups which do not have any extra endomorphisms. As a corollary, we find that the cohomologically smooth locus in the infinite-level modular curve $X(p^\infty)^\circ$ is exactly the locus of elliptic curves E with supersingular reduction, such that the formal group of E has no extra endomorphisms.

1 Main theorem

Let p be a prime number. Rapoport-Zink spaces [RZ96] are deformation spaces of p-divisible groups equipped with some extra structure. This article concerns the geometry of Rapoport-Zink spaces of EL type (endomorphisms + level structure). In particular we consider the infinite-level spaces $\mathscr{M}_{\mathcal{D},\infty}$, which are preperfectoid spaces [SW13]. An example is the space $\mathscr{M}_{H,\infty}$, where $H/\overline{\mathbf{F}}_p$ is a p-divisible group of height n. The points of $\mathscr{M}_{H,\infty}$ over a nonarchimedean field K containing $W(\overline{\mathbf{F}}_p)$ are in correspondence with isogeny classes of p-divisible groups $G/\mathcal{O}_K$ equipped with a quasi-isogeny $G\otimes_{\mathcal{O}_K}\mathcal{O}_K/p \to H\otimes_{\overline{\mathbf{F}}_p}\mathcal{O}_K/p$ and an isomorphism $\mathbf{Q}_p^n \cong VG$ (where VG is the rational Tate module).

The infinite-level space $\mathscr{M}_{\mathcal{D},\infty}$ appears as the limit of finite-level spaces, each of which is a smooth rigid-analytic space. We would like to investigate the question of smoothness for the space $\mathscr{M}_{\mathcal{D},\infty}$ itself, which is quite a different matter. We need the notion of cohomological smoothness [Sch17], which makes sense for general morphisms of analytic adic spaces, and which is reviewed in Section 4. Roughly speaking, an adic space is cohomologically smooth over C (where $C/\mathbf{Q}_p$ is complete and algebraically closed) if it satisfies local Verdier duality. In particular, if U is a quasi-compact adic space which is cohomologically smooth over $\mathrm{Spa}(C,\mathcal{O}_C)$, then the cohomology group $H^i(U,\mathbf{F}_\ell)$ is finite for all i and all primes $\ell\neq p$.

Our main theorem shows that each connected component of the geometric fiber of $\mathscr{M}_{\mathcal{D},\infty}$ has a dense open subset which is cohomologically smooth.

Theorem 1.0.1. *Let $\mathcal{D}$ be a basic EL datum (cf. Section 2). Let C be a complete algebraically closed extension of the field of scalars of $\mathscr{M}_{\mathcal{D},\infty}$, and let $\mathscr{M}_{\mathcal{D},\infty}^\circ$ be a connected component of the base change $\mathscr{M}_{\mathcal{D},\infty,C}$. Let $\mathscr{M}_{\mathcal{D},\infty}^{\circ,\text{non-sp}} \subset \mathscr{M}_{\mathcal{D},\infty}^\circ$ be the non-special locus (cf. Section 3.5), corresponding to p-divisible groups without extra endomorphisms. Then $\mathscr{M}_{\mathcal{D},\infty}^{\circ,\text{non-sp}}$ is cohomologically smooth over C.*

We remark that outside of trivial cases, $\pi_0(\mathscr{M}_{\mathcal{D},\infty,C})$ has no isolated points, which implies that no open subset of $\mathscr{M}_{\mathcal{D},\infty,C}$ can be cohomologically smooth. (Indeed, the H^0 of any quasi-compact open fails to be finitely generated.) Therefore it really is necessary to work with individual connected components of the geometric fiber of $\mathscr{M}_{\mathcal{D},\infty}$.

Theorem 1.0.1 is an application of the perfectoid version of the Jacobian criterion for smoothness, due to Fargues–Scholze [FS]; cf. Theorem 4.2.1. The latter theorem involves the Fargues-Fontaine curve X_C (reviewed in Section 3). It asserts that a functor $\mathscr{M}$ on perfectoid spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$ is cohomologically smooth, when $\mathscr{M}$ can be interpreted as global sections of a smooth morphism $Z \to X_C$, subject to a certain condition on the tangent bundle Tan_{Z/X_C}.

In our application to Rapoport-Zink spaces, we construct a smooth morphism $Z \to X_C$, whose moduli space of global sections is isomorphic to $\mathscr{M}^\circ_{\mathcal{D},\infty}$ (Lemma 5.2.1). Next, we show that a geometric point $x \in \mathscr{M}^\circ_{\mathcal{D},\infty}(C)$ lies in $\mathscr{M}^{\circ,\mathrm{non-sp}}_{\mathcal{D},\infty}(C)$ if and only if the corresponding section $s \colon X_C \to Z$ satisfies the condition that all slopes of the vector bundle $s^* \mathrm{Tan}_{Z/X_C}$ on X_C are positive (Theorem 5.5.1). This is exactly the condition on Tan_{Z/X_C} required by Theorem 4.2.1, so we can conclude that $\mathscr{M}^\circ_{\mathcal{D},\infty}$ is cohomologically smooth.

The geometry of Rapoport-Zink spaces is related to the geometry of Shimura varieties. As an example, consider the tower of classical modular curves $X(p^\infty)$, considered as rigid spaces over C. There is a perfectoid space $X(p^\infty)$ over C for which $X(p^\infty) \sim \varprojlim_n X(p^n)$, and a Hodge-Tate period map $\pi_{HT} \colon X(p^\infty) \to \mathbf{P}^1_C$ [Sch15], which is $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant. Let $X(p^\infty)^\circ \subset X(p^\infty)$ be a connected component.

Corollary 1.0.2. *The following are equivalent for a C-point x of $X(p^\infty)^\circ$.*

1. *The point x corresponds to an elliptic curve E, such that the p-divisible group $E[p^\infty]$ has $\mathrm{End}\, E[p^\infty] = \mathbf{Z}_p$.*
2. *The stabilizer of $\pi_{HT}(x)$ in $\mathrm{PGL}_2(\mathbf{Q}_p)$ is trivial.*
3. *There is a neighborhood of x in $X(p^\infty)^\circ$ which is cohomologically smooth over C.*

2 Review of Rapoport-Zink spaces at infinite level

2.1 The infinite-level Rapoport-Zink space $\mathscr{M}_{H,\infty}$

Let k be a perfect field of characteristic p, and let H be a p-divisible group of heignt n and dimension d over k. We review here the definition of the infinite-level Rapoport-Zink space associated with H.

First there is the formal scheme $\mathscr{M}_H$ over $\mathrm{Spf}\, W(k)$ parametrizing deformations of H up to isogeny, as in [RZ96]. For a $W(k)$-algebra R in which p is nilpotent, $\mathscr{M}_H(R)$ is the set of isomorphism classes of pairs (G, ρ), where G/R is a p-divisible group and $\rho \colon H \otimes_k R/p \to G \otimes_R R/p$ is a quasi-isogeny.

The formal scheme $\mathscr{M}_H$ locally admits a finitely generated ideal of definition. Therefore it makes sense to pass to its adic space $\mathscr{M}_H^{\mathrm{ad}}$, which has generic fiber $(\mathscr{M}_H^{\mathrm{ad}})_\eta$, a rigid-analytic space over $\mathrm{Spa}(W(k)[1/p], W(k))$. Then $(\mathscr{M}_H^{\mathrm{ad}})_\eta$ has the following moduli interpretation: it is the sheafification of the functor assigning to a complete affinoid $(W(k)[1/p], W(k))$-algebra (R, R^+) the set of pairs (G, ρ), where G is a p-divisible group defined over an open and bounded subring $R_0 \subset R^+$, and $\rho \colon H \otimes_k R_0/p \to G \otimes_{R_0} R_0/p$ is a quasi-isogeny. There is an action of $\mathrm{Aut}\, H$ on $\mathscr{M}_H^{\mathrm{ad}}$ obtained by composition with ρ.

Given such a pair (G, ρ), Grothendieck-Messing theory produces a surjection $M(H) \otimes_{W(k)} R \to \mathrm{Lie}\, G[1/p]$ of locally free R-modules, where $M(H)$ is the covariant Dieudonné module. There is a Grothendieck-Messing period map $\pi_{GM} \colon (\mathscr{M}_H^{\mathrm{ad}})_\eta \to \mathcal{F}\ell$, where $\mathcal{F}\ell$ is the rigid-analytic space parametrizing rank d locally free quotients of $M(H)[1/p]$. The morphism π_{GM} is equivariant for the action of $\mathrm{Aut}\, H$. It has open image $\mathcal{F}\ell^a$ (the admissible locus).

We obtain a tower of rigid-analytic spaces over $(\mathscr{M}_H^{\mathrm{ad}})_\eta$ by adding level structures. For a complete affinoid $(W(k)[1/p], W(k))$-algebra (R, R^+), and an element of $(\mathscr{M}_H^{\mathrm{ad}})_\eta(R, R^+)$ represented locally on $\mathrm{Spa}(R, R^+)$ by a pair (G, ρ) as above, we have the Tate module $TG = \varprojlim_m G[p^m]$, considered as an adic space over $\mathrm{Spa}(R, R^+)$ with the structure of a $\mathbf{Z}_p$-module [SW13, (3.3)]. Finite-level spaces $\mathscr{M}_{H,m}$ are obtained by trivializing the $G[p^m]$; these are finite étale covers of $(\mathscr{M}_H^{\mathrm{ad}})_\eta$. The infinite-level space is obtained by trivializing all of TG at once, as in the following definition.

Definition 2.1.1 ([SW13, Definition 6.3.3]). Let $\mathscr{M}_{H,\infty}$ be the functor which sends a complete affinoid $(W(k)[1/p], W(k))$-algebra (R, R^+) to the set of triples (G, ρ, α), where (G, ρ) is an element of $(\mathscr{M}_H)_\eta^{\mathrm{ad}}(R, R^+)$, and $\alpha\colon \mathbf{Z}_p^n \to TG$ is a $\mathbf{Z}_p$-linear map which is an isomorphism pointwise on $\mathrm{Spa}(R, R^+)$.

There is an equivalent definition in terms of *isogeny* classes of triples (G, ρ, α), where this time $\alpha\colon \mathbf{Q}_p^n \to VG$ is a trivialization of the rational Tate module. Using this definition, it becomes clear that $\mathscr{M}_{H,\infty}$ admits an action of the product $\mathrm{GL}_n(\mathbf{Q}_p) \times \mathrm{Aut}^0 H$, where Aut^0 means automorphisms in the isogeny category. Then the period map $\pi_{GM}\colon \mathscr{M}_{H,\infty} \to \mathscr{F}\ell$ is equivariant for $\mathrm{GL}_n(\mathbf{Q}_p) \times \mathrm{Aut}^0 H$, where $\mathrm{GL}_n(\mathbf{Q}_p)$ acts trivially on $\mathscr{F}\ell$.

We remark that $\mathscr{M}_{H,\infty} \sim \varprojlim_m \mathscr{M}_{H,m}$ in the sense of [SW13, Definition 2.4.1].

One of the main theorems of [SW13] is the following.

Theorem 2.1.2. *The adic space $\mathscr{M}_{H,\infty}$ is a preperfectoid space.*

This means that for any perfectoid field K containing $W(k)$, the base change $\mathscr{M}_{H,\infty} \times_{\mathrm{Spa}(W(k)[1/p],W(k))} \mathrm{Spa}(K, \mathcal{O}_K)$ becomes perfectoid after p-adically completing.

We sketch here the proof of Theorem 2.1.2. Consider the "universal cover" $\tilde{H} = \varprojlim_p H$ as a sheaf of $\mathbf{Q}_p$-vector spaces on the category of k-algebras. This has a canonical lift to the category of $W(k)$-algebras [SW13, Proposition 3.1.3(ii)], which we continue to call $\tilde{H}$. The adic generic fiber $\tilde{H}_\eta^{\mathrm{ad}}$ is a preperfectoid space, as can be checked "by hand": it is a product of the d-dimensional preperfectoid open ball $(\mathrm{Spa}\, W(k)[\![T_1^{1/p^\infty}, \dots, T_d^{1/p^\infty}]\!])_\eta$ by the constant adic space $VH^{\text{ét}}$, where $H^{\text{ét}}$ is the étale part of H. Given a triple (G, ρ, α) representing an element of $\mathscr{M}_{H,\infty}(R, R^+)$, the quasi-isogeny ρ induces an isomorphism $\tilde{H}_\eta^{\mathrm{ad}} \times_{\mathrm{Spa}(W(k)[1/p],W(k))} \mathrm{Spa}(R, R^+) \to \tilde{G}_\eta^{\mathrm{ad}}$; composing this with α gives a morphism $\mathbf{Q}_p^n \to \tilde{H}_\eta^{\mathrm{ad}}(R, R^+)$. We have therefore described a morphism $\mathscr{M}_{H,\infty} \to (\tilde{H}_\eta^{\mathrm{ad}})^n$.

Theorem 2.1.2 follows from the fact that the morphism $\mathscr{M}_{H,\infty} \to (\tilde{H}^{\mathrm{ad}})_\eta^n$ presents $\mathscr{M}_{H,\infty}$ as an open subset of a Zariski closed subset of $(\tilde{H}^{\mathrm{ad}})_\eta^n$. We conclude this subsection by spelling out how this is done. We have a *quasi-logarithm* map $\mathrm{qlog}_H\colon \tilde{H}_\eta^{\mathrm{ad}} \to M(H)[1/p] \otimes_{W(k)[1/p]} \mathbf{G}_a$ [SW13, Definition 3.2.3], a $\mathbf{Q}_p$-linear morphism of adic spaces over $\mathrm{Spa}(W(k)[1/p], W(k))$.

Now suppose (G, ρ) is a deformation of H to (R, R^+). The logarithm map on G fits into an exact sequence of $\mathbf{Z}_p$-modules:

$$0 \to G_\eta^{\mathrm{ad}}[p^\infty](R, R^+) \to G_\eta^{\mathrm{ad}}(R, R^+) \to \mathrm{Lie}\, G[1/p].$$

After taking projective limits along multiplication-by-p, this turns into an exact sequence of $\mathbf{Q}_p$-vector spaces,

$$0 \to VG(R, R^+) \to \tilde{G}_\eta^{\mathrm{ad}}(R, R^+) \to \mathrm{Lie}\, G[1/p].$$

On the other hand, we have a commutative diagram

$$\begin{tikzcd}
\tilde{H}_\eta(R, R^+) \arrow[r, "\cong"] \arrow[d, "\mathrm{qlog}_H"'] & \tilde{G}_\eta(R, R^+) \arrow[d, "\log_G"] \\
M(H) \otimes_{W(k)} R \arrow[r] & \mathrm{Lie}\, G[1/p].
\end{tikzcd}$$

The lower horizontal map $M(H) \otimes_{W(k)} R \to \mathrm{Lie}\, G[1/p]$ is the quotient by the R-submodule of $M(H) \otimes_{W(k)} R$ generated by the image of $VG(R, R^+) \to \tilde{G}_\eta^{\mathrm{ad}}(R, R^+) \cong \tilde{H}_\eta^{\mathrm{ad}}(R, R^+) \to M(H) \otimes_{W(k)} R$.

Thus if we have a triple (G, ρ, α) representing an element of $\mathscr{M}_{H,\infty}(R, R^+)$, then we have a map $\mathbf{Q}_p^n \to \tilde{H}_\eta^{\mathrm{ad}}(R, R^+)$, such that the cokernel of $\mathbf{Q}_p^n \to \tilde{H}_\eta^{\mathrm{ad}}(R, R^+) \to M(H) \otimes_{W(k)} R$ is a projective R-module of rank d, namely $\operatorname{Lie} G[1/p]$. This condition on the cokernel allows us to formulate an alternate description of $\mathscr{M}_{H,\infty}$ which is independent of deformations.

Proposition 2.1.3. *The adic space $\mathscr{M}_{H,\infty}$ is isomorphic to the functor which assigns to a complete affinoid $(W(k)[1/p], W(k))$-algebra (R, R^+) the set of n-tuples $(s_1, \dots, s_n) \in \tilde{H}_\eta^{\mathrm{ad}}(R, R^+)^n$ such that the following conditions are satisfied:*

1. *The quotient of $M(H) \otimes_{W(k)} R$ by the span of the $\operatorname{qlog}(s_i)$ is a projective R-module W of rank d.*
2. *For all geometric points $\operatorname{Spa}(C, \mathcal{O}_C) \to \operatorname{Spa}(R, R^+)$, the sequence*

$$0 \to \mathbf{Q}_p^n \stackrel{(s_1,\dots,s_n)}{\to} \tilde{H}_\eta^{\mathrm{ad}}(C, \mathcal{O}_C) \to W \otimes_R C \to 0$$

   is exact.

2.2 Infinite-level Rapoport-Zink spaces of EL type

This article treats the more general class of Rapoport-Zink spaces of EL type. We review these here.

Definition 2.2.1. Let k be an algebraically closed field of characteristic p. A *rational EL datum* is a quadruple $\mathcal{D} = (B, V, H, \mu)$, where

- B is a semisimple $\mathbf{Q}_p$-algebra,
- V is a finite B-module,
- H is an object of the isogeny category of p-divisible groups over k, equipped with an action $B \to \operatorname{End} H$,
- μ is a conjugacy class of $\overline{\mathbf{Q}}_p$-rational cocharacters $\mathbf{G}_m \to \mathbf{G}$, where $\mathbf{G}/\mathbf{Q}_p$ is the algebraic group $\mathrm{GL}_B(V)$.

These are subject to the conditions:

- If $M(H)$ is the (rational) Dieudonné module of H, then there exists an isomorphism $M(H) \cong V \otimes_{\mathbf{Q}_p} W(k)[1/p]$ of $B \otimes_{\mathbf{Q}_p} W(k)[1/p]$-modules. In particular $\dim V = \operatorname{ht} H$.
- In the weight decomposition of $V \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p \cong \bigoplus_{i \in \mathbf{Z}} V_i$ determined by μ, only weights 0 and 1 appear, and $\dim V_0 = \dim H$.

The *reflex field* E of $\mathcal{D}$ is the field of definition of the conjugacy class μ. We remark that the weight filtration (but not necessarily the weight decomposition) of $V \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ may be descended to E, and so we will be viewing V_0 and V_1 as $B \otimes_{\mathbf{Q}_p} E$-modules.

The infinite-level Rapoport-Zink space $\mathscr{M}_{\mathcal{D},\infty}$ is defined in [SW13] in terms of moduli of deformations of the p-divisible group H along with its B-action. It admits an alternate description along the lines of Proposition 2.1.3.

Proposition 2.2.2 ([SW13, Theorem 6.5.4])**.** *Let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum. Let $\breve{E} = E \cdot W(k)$. Then $\mathscr{M}_{\mathcal{D},\infty}$ is isomorphic to the functor which inputs a complete affinoid $(\breve{E}, \mathcal{O}_{\breve{E}})$-algebra (R, R^+) and outputs the set of B-linear maps*

$$s \colon V \to \tilde{H}_\eta^{\mathrm{ad}}(R, R^+),$$

subject to the following conditions.

- *Let W be the quotient*
$$V \otimes_{\mathbf{Q}_p} R \overset{\mathrm{qlog}_H \circ s}{\longrightarrow} M(H) \otimes_{W(k)} R \to W \to 0.$$
Then W is a finite projective R-module, which locally on R is isomorphic to $V_0 \otimes_E R$ as a $B \otimes_{\mathbf{Q}_p} R$-module.

- *For any geometric point $x = \operatorname{Spa}(C, \mathcal{O}_C) \to \operatorname{Spa}(R, R^+)$, the sequence of B-modules*
$$0 \to V \to \tilde{H}(\mathcal{O}_C) \to W \otimes_R C \to 0$$
is exact.

If $\mathcal{D} = (\mathbf{Q}_p, \mathbf{Q}_p^n, H, \mu)$, where H has height n and dimension d and $\mu(t) = (t^{\oplus d}, 1^{\oplus(n-d)})$, then $E = \mathbf{Q}_p$ and $\mathscr{M}_{\mathcal{D},\infty} = \mathscr{M}_{H,\infty}$.

In general, we call $\breve{E}$ the field of scalars of $\mathscr{M}_{\mathcal{D},\infty}$, and for a complete algebraically closed extension C of $\breve{E}$, we write $\mathscr{M}_{\mathcal{D},\infty,C} = \mathscr{M}_{\mathcal{D},\infty} \times_{\operatorname{Spa}(\breve{E},\mathcal{O}_{\breve{E}})} \operatorname{Spa}(C, \mathcal{O}_C)$ for the corresponding geometric fiber of $\mathscr{M}_{\mathcal{D},\infty}$.

The space $\mathscr{M}_{\mathcal{D},\infty}$ admits an action by the product group $\mathbf{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$, where $J/\mathbf{Q}_p$ is the algebraic group $\operatorname{Aut}^\circ_B(H)$. A pair $(\alpha, \alpha') \in \mathbf{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$ sends s to $\alpha' \circ s \circ \alpha^{-1}$.

There is once again a Grothendieck-Messing period map $\pi_{GM} \colon \mathscr{M}_{\mathcal{D},\infty} \to \mathcal{F}\ell_\mu$ onto the rigid-analytic variety whose (R, R^+)-points parametrize $B \otimes_{\mathbf{Q}_p} R$-module quotients of $M(H) \otimes_{W(k)} R$ which are projective over R, and which are of type μ in the sense that they are (locally on R) isomorphic to $V_0 \otimes_E R$. The morphism π_{GM} sends an (R, R^+)-point of $\mathscr{M}_{\mathcal{D},\infty}$ to the quotient W of $M(H) \otimes_{W(k)} R$ as above. It is equivariant for the action of $\mathbf{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$, where $\mathbf{G}(\mathbf{Q}_p)$ acts trivially on $\mathcal{F}\ell_\mu$. In terms of deformations of the p-divisible group H, the period map π_{GM} sends a deformation G to $\operatorname{Lie} G$.

There is also a Hodge-Tate period map $\pi_{HT} \colon \mathscr{M}_{\mathcal{D},\infty} \to \mathcal{F}\ell'_\mu$, where $\mathcal{F}\ell'_\mu(R, R^+)$ parametrizes $B \otimes_{\mathbf{Q}_p} R$-module quotients of $V \otimes_{\mathbf{Q}_p} R$ which are projective over R, and which are (locally on R) isomorphic to $V_1 \otimes_E R$. The morphism π_{HT} sends an (R, R^+)-point of $\mathscr{M}_{\mathcal{D},\infty}$ to the image of $V \otimes_{\mathbf{Q}_p} R \to M(H) \otimes_{W(k)} R$. It is equivariant for the action of $\mathbf{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$, where this time $J(\mathbf{Q}_p)$ acts trivially on $\mathcal{F}\ell'_\mu(R, R^+)$. In terms of deformations of the p-divisible group H, the period map π_{HT} sends a deformation G to $(\operatorname{Lie} G^\vee)^\vee$.

3 The Fargues-Fontaine curve

3.1 Review of the curve

We briefly review here some constructions and results from [FF]. First we review the absolute curve, and then we cover the version of the curve which works in families.

Fix a perfectoid field F of characteristic p, with $F^\circ \subset F$ its ring of integral elements. Let $\varpi \in F^\circ$ be a pseudo-uniformizer for F, and let k be the residue field of F. Let $W(F^\circ)$ be the ring of Witt vectors, which we equip with the $(p, [\varpi])$-adic topology. Let $\mathcal{Y}_F = \operatorname{Spa}(W(F^\circ), W(F^\circ)) \backslash \{|p[\varpi]| = 0\}$. Then $\mathcal{Y}_F$ is an analytic adic space over $\mathbf{Q}_p$. The Frobenius automorphism of F induces an automorphism ϕ of $\mathcal{Y}_F$. Let $B_F = H^0(\mathcal{Y}_F, \mathcal{O}_{\mathcal{Y}_F})$, a $\mathbf{Q}_p$-algebra endowed with an action of ϕ. Let P_F be the graded ring $P_F = \bigoplus_{n \geqslant 0} B_F^{\phi = p^n}$. Finally, the Fargues-Fontaine curve is $X_F = \operatorname{Proj} P_F$. It is shown in [FF] that X_F is the union of spectra of Dedekind rings, which justifies the use of the word "curve" to describe X_F. Note however that there is no "structure morphism" $X_F \to \operatorname{Spec} F$.

If $x \in X_F$ is a closed point, then the residue field of x is a perfectoid field F_x containing $\mathbf{Q}_p$ which comes equipped with an inclusion $i \colon F \hookrightarrow F_x^\flat$, which presents $F_x^\flat$ as a finite extension of F. Such a pair (F_x, i) is called an untilt of F. Then $x \mapsto (F_x, i)$ is a bijection between closed points of X_F and isomorphism classes of untilts of F, modulo the action of Frobenius on i. Thus if $F = E^\flat$ is the tilt of a given perfectoid field $E/\mathbf{Q}_p$, then $X_{E^\flat}$ has a canonical closed point ∞, corresponding to the untilt E of $E^\flat$.

An important result in [FF] is the classification of vector bundles on X_F. (By a vector bundle on X_F we are referring to a locally free $\mathcal{O}_{X_F}$-module $\mathcal{E}$ of finite rank. We will use the notation $V(\mathcal{E})$ to mean the corresponding geometric vector bundle over X_F, whose sections correspond to sections of $\mathcal{E}$.) Recall that an *isocrystal* over k is a finite-dimensional vector space N over $W(k)[1/p]$ together with a Frobenius semi-linear automorphism ϕ of N. Given N, we have the graded P_F-module $\bigoplus_{n\geqslant 0}(N \otimes_{W(k)[1/p]} B_F)^{\phi=p^n}$, which corresponds to a vector bundle $\mathcal{E}_F(N)$ on X_F. Then the Harder-Narasimhan slopes of $\mathcal{E}_F(N)$ are negative to those of N. If F is algebraically closed, then every vector bundle on X_F is isomorphic to $\mathcal{E}_F(N)$ for some N.

It is straightforward to "relativize" the above constructions. If $S = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over k, one can construct the adic space $\mathcal{Y}_S$, the ring B_S, the scheme X_S, and the vector bundles $\mathcal{E}_S(N)$ as above. Frobenius-equivalences classes of untilts of S correspond to effective Cartier divisors of X_S of degree 1.

In our applications, we will start with an affinoid perfectoid space S over $\mathbf{Q}_p$. We will write $X_S = X_{S^\flat}$, and we will use ∞ to refer to the canonical Cartier divisor of X_S corresponding to the untilt S of $S^\flat$. Thus if N is an isocrystal over k, and $S = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over $W(k)[1/p]$, then the fiber of $\mathcal{E}_S(N)$ over ∞ is $N \otimes_{W(k)[1/p]} R$.

Let $S = \mathrm{Spec}(R, R^+)$ be as above and let ∞ be the corresponding Cartier divisor. We denote the completion of the ring of functions on $\mathcal{Y}_S$ along ∞ by $B^+_{\mathrm{dR}}(R)$. It comes equipped with a surjective homomorphism $\theta\colon B^+_{\mathrm{dR}}(R) \to R$, whose kernel is a principal ideal $\ker(\theta) = (\xi)$.

3.2 Relation to p-divisible groups

Here we recall the relationships between p-divisible groups and global sections of vector bundles on the Fargues-Fontaine curve. Let us fix a perfect field k of characteristic p, and write $\mathrm{Perf}_{W(k)[1/p]}$ for the category of perfectoid spaces over $W(k)[1/p]$. Given a p-divisible group H over k with covariant isocrystal N, if H has slopes $s_1, \dots, s_k \in \mathbb{Q}$, then N has the slopes $1 - s_1, \dots, 1 - s_k$. For an object S in $\mathrm{Perf}_{W(k)[1/p]}$ we define the vector bundle $\mathcal{E}_S(H)$ on X_S by

$$\mathcal{E}_S(H) = \mathcal{E}_S(N) \otimes_{\mathcal{O}_{X_S}} \mathcal{O}_{X_S}(1).$$

Under this normalization, the Harder-Narasimhan slopes of $\mathcal{E}_S(H)$ are (pointwise on S) the same as the slopes of H.

Let us write $H^0(\mathcal{E}(H))$ for the sheafification of the functor on $\mathrm{Perf}_{W(k)[1/p]}$, which sends S to $H^0(X_S, \mathcal{E}_S(H))$.

Proposition 3.2.1. *Let H be a p-divisible group over a perfect field k of characteristic p, with isocrystal N. There is an isomorphism $\tilde{H}^{\mathrm{ad}}_\eta \cong H^0(\mathcal{E}(H))$ of sheaves on $\mathrm{Perf}_{W(k)[1/p]}$ making the diagram commute:*

$$\begin{tikzcd} \tilde{H}^{\mathrm{ad}}_\eta \arrow[rr] \arrow[dr, "\mathrm{qlog}_H"'] & & H^0(\mathcal{E}(H)) \arrow[dl] \\ & N \otimes_{W(k)[1/p]} \mathbf{G}_a, & \end{tikzcd}$$

where the morphism $H^0(\mathcal{E}(H)) \to N \otimes_{W(k)[1/p]} \mathbf{G}_a$ sends a global section of $\mathcal{E}(H)$ to its fiber at ∞.

Proof. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $W(k)[1/p]$. Then $\tilde{H}^{\mathrm{ad}}_\eta(R, R^+) \cong \tilde{H}(R^\circ) \cong \tilde{H}(R^\circ/p)$. Observe that $\tilde{H}(R^\circ/p) = \mathrm{Hom}_{R^\circ/p}(\mathbf{Q}_p/\mathbf{Z}_p, H)[1/p]$, where the Hom is taken in the category of p-divisible groups over R°/p. Recall the crystalline Dieudonné functor $G \mapsto M(G)$ from p-divisible groups to Dieudonné crystals [Mes72]. Since the base ring R°/p is semiperfect, the latter category is equivalent to the category of finite projective modules over Fontaine's period ring $A_{\mathrm{cris}}(R^\circ/p) = A_{\mathrm{cris}}(R^\circ)$, equipped with Frobenius and Verschiebung.

Now we apply [SW13, Theorem A]: since R°/p is f-semiperfect, the crystalline Dieudonné functor is fully faithful up to isogeny. Thus

$$\mathrm{Hom}_{R^\circ/p}(\mathbf{Q}_p/\mathbf{Z}_p, H)[1/p] \cong \mathrm{Hom}_{A_{\mathrm{cris}}(R^\circ),\phi}(M(\mathbf{Q}_p/\mathbf{Z}_p), M(H))[1/p],$$

where the latter Hom is in the category of modules over $A_{\mathrm{cris}}(R^\circ)$ equipped with Frobenius. Recall that $B^+_{\mathrm{cris}}(R^\circ) = A_{\mathrm{cris}}(R^\circ)[1/p]$. Since H arises via base change from k, we have $M(H)[1/p] = B^+_{\mathrm{cris}}(R^\circ) \otimes_{W(k)[1/p]} N$. For its part, $M(\mathbf{Q}_p/\mathbf{Z}_p)[1/p] = B^+_{\mathrm{cris}}(R^\circ)e$, for a basis element e on which Frobenius acts as p. Therefore

$$\tilde{H}(R^\circ) \cong (B^+_{\mathrm{cris}}(R^\circ) \otimes_{W(k)[1/p]} N)^{\phi=p}.$$

On the Fargues-Fontaine curve side, we have by definition $H^0(X_S, \mathcal{E}_S(H)) = (B_S \otimes_{W(k)[1/p]} N)^{\phi=p}$. The isomorphism between $(B_S \otimes_{W(k)[1/p]} N)^{\phi=p}$ and $(B^+_{\mathrm{cris}}(R^\circ) \otimes_{W(k)[1/p]} N)^{\phi=p}$ is discussed in [LB18, Remarque 6.6].

The commutativity of the diagram in the proposition is [SW13, Proposition 5.1.6(ii)], at least in the case that S is a geometric point, but this suffices to prove the general case. □

With Proposition 3.2.1 we can reinterpret the infinite-level Rapoport Zink spaces as moduli spaces of *modifications* of vector bundles on the Fargues-Fontaine curve. First we do this for $\mathscr{M}_{H,\infty}$. In the following, we consider $\mathscr{M}_{H,\infty}$ as a sheaf on the category of perfectoid spaces over $W(k)[1/p]$.

Proposition 3.2.2. *Let H be a p-divisible group of height n and dimension d over a perfect field k. Let N be the associated isocrystal over k. Then $\mathscr{M}_{H,\infty}$ is isomorphic to the functor which inputs an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over $W(k)[1/p]$ and outputs the set of exact sequences*

$$0 \to \mathcal{O}^n_{X_S} \xrightarrow{s} \mathcal{E}_S(H) \to i_{\infty *}W \to 0, \tag{3.2.1}$$

where $i_\infty \colon \mathrm{Spec}\, R \to X_S$ is the inclusion, and W is a projective $\mathcal{O}_S$-module quotient of $N \otimes_{W(k)[1/p]} \mathcal{O}_S$ of rank d.

Proof. We briefly describe this isomorphism on the level of points over $S = \mathrm{Spa}(R, R^+)$. Suppose that we are given a point of $\mathscr{M}_{H,\infty}(S)$, corresponding to a p-divisible group G over R°, together with a quasi-isogeny $\iota \colon H \otimes_k R^\circ/p \to G \otimes_{R^\circ} R^\circ/p$ and an isomorphism $\alpha \colon \mathbf{Q}_p^n \to VG$ of sheaves of $\mathbf{Q}_p$-vector spaces on S. The logarithm map on G fits into an exact sequence of sheaves of $\mathbf{Z}_p$-modules on S,

$$0 \to G^{\mathrm{ad}}_\eta[p^\infty] \to G^{\mathrm{ad}}_\eta \to \mathrm{Lie}\, G[1/p] \to 0.$$

After taking projective limits along multiplication-by-p, this turns into an exact sequence of sheaves of $\mathbf{Q}_p$-vector spaces on S,

$$0 \to VG \to \tilde{G}^{\mathrm{ad}}_\eta \to \mathrm{Lie}\, G[1/p] \to 0.$$

The quasi-isogeny induces an isomorphism $\tilde{H}^{\mathrm{ad}}_\eta \times_{\mathrm{Spa}\, W(k)[1/p]} S \cong \tilde{G}^{\mathrm{ad}}_\eta$; composing this with the level structure gives an injective map $\mathbf{Q}_p^n \to \tilde{H}^{\mathrm{ad}}_\eta(S)$, whose cokernel W is isomorphic to the projective R-module $\mathrm{Lie}\, G$ of rank d. In light of Theorem 3.2.1, the map $\mathbf{Q}_p^n \to \tilde{H}^{\mathrm{ad}}_\eta(S)$ corresponds to an $\mathcal{O}_{X_S}$-linear map $s \colon \mathcal{O}^n_{X_S} \to \mathcal{E}_S(H)$, which fits into the exact sequence in (3.2.1). □

Similarly, we have a description of $\mathscr{M}_{\mathcal{D},\infty}$ in terms of modifications.

Proposition 3.2.3. *Let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum. Then $\mathscr{M}_{\mathcal{D},\infty}$ is isomorphic to the functor which inputs an affinoid perfectoid space S over $\breve{E}$ and outputs the set of exact sequences of $B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S}$-modules*

$$0 \to V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \xrightarrow{s} \mathcal{E}_S(H) \to i_{\infty *}W \to 0,$$

where W is a finite projective $\mathcal{O}_S$-module, which is locally isomorphic to $V_0 \otimes_{\mathbf{Q}_p} \mathcal{O}_S$ as a $B \otimes_{\mathbf{Q}_p} \mathcal{O}_S$-module (using notation from Definition 2.2.1).

3.3 The determinant morphism, and connected components

If we are given a rational EL datum $\mathcal{D}$, there is a *determinant morphism* $\det\colon \mathscr{M}_{\mathcal{D},\infty} \to \mathscr{M}_{\det \mathcal{D},\infty}$, which we review below. For an algebraically closed perfectoid field C containing $W(k)[1/p]$, the base change $\mathscr{M}_{\det \mathcal{D},\infty,C}$ is a locally profinite set of copies of $\operatorname{Spa} C$. For a point $\tau \in \mathscr{M}_{\det \mathcal{D},\infty}(C)$, let $\mathscr{M}^{\tau}_{\mathcal{D},\infty}$ be the fiber of $\mathscr{M}_{\mathcal{D},\infty} \to \mathscr{M}_{\det \mathcal{D},\infty}$ over τ. We will prove in Section 5 that each $\mathscr{M}^{\tau,\text{non-sp}}_{\mathcal{D},\infty}$ is cohomologically smooth if $\mathcal{D}$ is basic. This implies that $\pi_0(\mathscr{M}^{\tau,\text{non-sp}}_{\mathcal{D},\infty})$ is discrete, so that cohomogical smoothness of $\mathscr{M}^{\tau,\text{non-sp}}_{\mathcal{D},\infty}$ is inherited by each of its connected components. This is Theorem 1.0.1. In certain cases (for example Lubin-Tate space) it is known that $\mathscr{M}^{\tau}_{\mathcal{D},\infty}$ is already connected [Che14].

We first review the determinant morphism for the space $\mathscr{M}_{H,\infty}$, where H is a p-divisible group of height n and dimension d over a perfect field k of characteristic p. Let $\breve{E} = W(k)[1/p]$. For a perfectoid space S over $\breve{E}$, we have the vector bundle $\mathcal{E}_S(H)$ and its determinant $\det \mathcal{E}_S(H)$, a line bundle of degree d. (This does not correspond to a p-divisible group "$\det H$" unless $d \leqslant 1$.) We define $\mathscr{M}_{\det H,\infty}(S)$ to be the functor which inputs a perfectoid space $S = \operatorname{Spa}(R, R^+)$ over $\breve{E}$ and outputs the set of morphisms $s\colon \mathcal{O}_{X_S} \to \det \mathcal{E}_S(H)$, such that the cokernel of s is a projective $B^+_{\mathrm{dR}}(R)/(\xi)^d$-module of rank 1, where (ξ) is the kernel of $B^+_{\mathrm{dR}}(R) \to R$. Then for an algebraically closed perfectoid field $C/\breve{E}$, the set $\mathscr{M}_{\det H,\infty}(C)$ is a $\mathbf{Q}_p^\times$-torsor. The morphism $\det\colon \mathscr{M}_{H,\infty} \to \mathscr{M}_{\det H,\infty}$ is simply $s \mapsto \det s$.

For the general case, let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum. Let $F = Z(B)$ be the center of B. Then F is a semisimple commutative $\mathbf{Q}_p$-algebra, and V is free as an F-module. We put $\mathbf{G} = \operatorname{Aut}_B(V)$ (as an algebraic group), and then $\mathbf{G}^{\mathrm{ab}} = \mathbf{G}/\mathbf{G}^{\mathrm{der}} = \operatorname{Aut}_F(\det_F V) \cong \operatorname{Res}_{F/\mathbf{Q}_p} \mathbf{G}_m$. Let μ^{ab} be the composition of μ with $\mathbf{G} \to \mathbf{G}^{\mathrm{ab}}$. Let $\mathscr{M}_{\det \mathcal{D},\infty}$ be the functor which inputs a perfectoid space $S = \operatorname{Spa}(R, R^+)$ over $\breve{E}$ and outputs the set of F-linear morphisms $s\colon \det_F V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \to \det_F \mathcal{E}_S(H)$, such that the cokernel of s is of the form $i_{\infty *}W$, where W is a finite projective $\mathcal{O}_S$-module, locally isomorphic to $\det_F V_0 \otimes_{\mathbf{Q}_p} F$ as a $F \otimes_{\mathbf{Q}_p} \mathcal{O}_S$-module.

3.4 Basic Rapoport-Zink spaces

The main theorem of this article concerns basic Rapoport-Zink spaces, so we recall some facts about these here.

Let H be a p-divisible group over a perfect field k of characteristic p. The space $\mathcal{M}_{H,\infty}$ is said to be basic when the p-divisible group H (or rather, its Dieudonné module $M(H)$) is isoclinic. This is equivalent to saying that the natural map

$$\operatorname{End}^\circ H \otimes_{\mathbf{Q}_p} W(k)[1/p] \to \operatorname{End}_{W(k)[1/p]} M(H)[1/p]$$

is an isomorphism, where on the right the endomorphisms are not required to commute with Frobenius.

More generally we have a notion of basicness for a rational EL datum (B, H, V, μ), referring to the following equivalent conditions:

- The $\mathbf{G}$-isocrystal ($\mathbf{G} = \operatorname{Aut}_B V$) associated to H is basic in the sense of Kottwitz [Kot85].

- The natural map
$$\operatorname{End}^{\circ}_B(H) \otimes_{\mathbf{Q}_p} W(k)[1/p] \to \operatorname{End}_{B\otimes_{\mathbf{Q}_p} W(k)[1/p]} M(H)[1/p]$$
is an isomorphism.

- Considered as an algebraic group over $\mathbf{Q}_p$, the automorphism group $J = \operatorname{Aut}^{\circ}_B H$ is an inner form of $\mathbf{G}$.

- Let $D' = \operatorname{End}^{\circ}_B H$. For any algebraically closed perfectoid field C containing $W(k)$, the map
$$D' \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \to \mathscr{E}nd_{(B\otimes_{\mathbf{Q}_p}\mathcal{O}_{X_C})} \mathcal{E}_C(H)$$
is an isomorphism.

In brief, the duality theorem from [SW13] says the following. Given a basic EL datum $\mathcal{D}$, there is a dual datum $\check{\mathcal{D}}$, for which the roles of the groups $\mathbf{G}$ and J are reversed. There is a $\mathbf{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$-equivariant isomorphism $\mathscr{M}_{\mathcal{D},\infty} \cong \mathscr{M}_{\check{\mathcal{D}},\infty}$ which exchanges the roles of π_{GM} and π_{HT}.

3.5 The special locus

Let $\mathcal{D} = (B, V, H, \mu)$ be a basic rational EL datum relative to a perfect field k of characteristic p, with reflex field E. Let F be the center of B. Define F-algebras D and D' by
$$\begin{aligned} D &= \operatorname{End}_B V \\ D' &= \operatorname{End}_B H \end{aligned}$$
Finally, let $\mathbf{G} = \operatorname{Aut}_B V$ and $J = \operatorname{Aut}_B H$, considered as algebraic groups over $\mathbf{Q}_p$. Then $\mathbf{G}$ and J both contain $\operatorname{Res}_{F/\mathbf{Q}_p} \mathbf{G}_m$.

Let C be an algebraically closed perfectoid field containing $\breve{E}$, and let $x \in \mathscr{M}_{\mathcal{D},\infty}(C)$. Then x corresponds to a p-divisible group G over $\mathcal{O}_C$ with endomorphisms by B, and also it corresponds to a $B\otimes_{\mathbf{Q}_p}\mathcal{O}_{X_C}$-linear map $s\colon V\otimes_{\mathbf{Q}_p}\mathcal{O}_X \to \mathcal{E}_C(N)$ as in Proposition 3.2.3. Define $A_x = \operatorname{End}_B G$ (endomorphisms in the isogeny category). Then A_x is a semisimple F-algebra. In light of Proposition 3.2.3, an element of A_x is a pair (α, α'), where $\alpha \in \operatorname{End}_{B\otimes_{\mathbf{Q}_p}\mathcal{O}_{X_C}} V\otimes\mathcal{O}_{X_C} = \operatorname{End}_B V = D$ and $\alpha' \in \operatorname{End}_{B\otimes_{\mathbf{Q}_p}\mathcal{O}_{X_C}} \mathscr{E}_C(H) = D'$ (the last equality is due to basicness), such that $s\circ\alpha = \alpha'\circ s$. Thus:
$$A_x \cong \left\{(\alpha,\alpha') \in D\times D' \;\middle|\; s\circ\alpha = \alpha'\circ s\right\}.$$

Lemma 3.5.1. *The following are equivalent:*

1. *The F-algebra A_x strictly contains F.*
2. *The stabilizer of $\pi_{GM}(x) \in \mathcal{F}\ell_\mu(C)$ in $J(\mathbf{Q}_p)$ strictly contains $F^\times$.*
3. *The stabilizer of $\pi_{HT}(x) \in \mathcal{F}\ell'_\mu(C)$ in $\mathbf{G}(\mathbf{Q}_p)$ strictly contains $F^\times$.*

Proof. As in Proposition 3.2.3, let $s\colon V\otimes_{\mathbf{Q}_p}\mathcal{O}_{X_S} \xrightarrow{s} \mathcal{E}_S(H)$ be the modification corresponding to x.

Note that the condition (1) is equivalent to the existence of an invertible element $(\alpha,\alpha') \in A_x$ not contained in (the diagonally embedded) F. Also note that if one of α,α' lies in F, then so does the other, in which case they are equal.

Suppose $(\alpha,\alpha') \in A_x$ is invertible. The point $\pi_{GM}(x) \in \mathcal{F}\ell_\mu$ corresponds to the cokernel of the fiber of s at ∞. Since $\alpha'\circ s = s\circ\alpha$, the cokernels of $\alpha'\circ s$ and s are the same, which means exactly that $\alpha' \in J(\mathbf{Q}_p)$

stabilizes $\pi_{GM}(x)$. Thus (1) implies (2). Conversely, if there exists $\alpha' \in J(\mathbf{Q}_p)\backslash F^\times$ which stabilizes $\pi_{GM}(x)$, it means that the $B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}$-linear maps s and $\alpha' \circ s$ have the same cokernel, and therefore there exists $\alpha \in \operatorname{End}_{B\otimes_{\mathbf{Q}_p}\mathcal{O}_{X_C}} V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} = D$ such that $s \circ \alpha = \alpha' \circ s$, and then $(\alpha, \alpha') \in A_x\backslash F^\times$. This shows that (2) implies (1).

The equivalence between (1) and (3) is proved similarly. □

Definition 3.5.2. The *special locus* in $\mathscr{M}_{\mathcal{D},\infty}$ is the subset $\mathscr{M}^{\mathrm{sp}}_{\mathcal{D},\infty}$ defined by the condition $A_x \neq F$. The *non-special locus* $\mathscr{M}^{\text{non-sp}}_{\mathcal{D},\infty}$ is the complement of the special locus.

The special locus is built out of "smaller" Rapoport-Zink spaces, in the following sense. Let A be a semisimple F-algebra, equipped with two F-embeddings $A \to D$ and $A \to D'$, so that $A \otimes_F B$ acts on V and H. Also assume that a cocharacter in the conjugacy class μ factors through a cocharacter $\mu_0 \colon \mathbf{G}_m \to \operatorname{Aut}_{A\otimes_F B} V$. Let $\mathcal{D}_0 = (A \otimes_F B, V, H, \mu_0)$. Then there is an evident morphism $\mathscr{M}_{\mathcal{D}_0,\infty} \to \mathscr{M}_{\mathcal{D},\infty}$. The special locus $\mathscr{M}^{\mathrm{sp}}_{\mathcal{D},\infty}$ is the union of the images of all the $\mathscr{M}_{\mathcal{D}_0,\infty}$, as A ranges through all semisimple F-subalgebras of $D \times D'$ strictly containing F.

4 Cohomological smoothness

Let Perf be the category of perfectoid spaces in characteristic p, with its pro-étale topology [Sch17, Definition 8.1]. For a prime $\ell \neq p$, there is a notion of ℓ-cohomological smoothness [Sch17, Definition 23.8]. We only need the notion for morphisms $f \colon Y' \to Y$ between sheaves on Perf which are separated and representable in locally spatial diamonds. If such an f is ℓ-cohomologically smooth, and Λ is an ℓ-power torsion ring, then the relative dualizing complex $Rf^!\Lambda$ is an invertible object in $D_{\text{ét}}(Y', \Lambda)$ (thus, it is v-locally isomorphic to $\Lambda[n]$ for some $n \in \mathbf{Z}$), and the natural transformation $Rf^!\Lambda \otimes f^* \to Rf^!$ of functors $D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(Y', \Lambda)$ is an equivalence [Sch17, Proposition 23.12]. In particular, if f is projection onto a point, and $Rf^!\Lambda \cong \Lambda[n]$, one derives a statement of Poincaré duality for Y':

$$R\operatorname{Hom}(R\Gamma_c(Y', \Lambda), \Lambda) \cong R\Gamma(Y', \Lambda)[n].$$

We will say that f is cohomologically smooth if it is ℓ-cohomologically smooth for all $\ell \neq p$. As an example, if $f \colon Y' \to Y$ is a separated smooth morphism of rigid-analytic spaces over $\mathbf{Q}_p$, then the associated morphism of diamonds $f^\diamond \colon (Y')^\diamond \to Y^\diamond$ is cohomologically smooth [Sch17, Proposition 24.3]. There are other examples where f does not arise from a finite-type map of adic spaces. For instance, if $\tilde{B}_C = \operatorname{Spa} C \left\langle T^{1/p^\infty} \right\rangle$ is the perfectoid closed ball over an algebraically closed perfectoid field C, then $\tilde{B}_C$ is cohomologically smooth over C.

If Y is a perfectoid space over an algebraically closed perfectoid field C, it seems quite difficult to detect whether Y is cohomologically smooth over C. We will review in Section 4.2 a "Jacobian criterion" from [FS] which applies to certain kinds of Y. But first we give a classical analogue of this criterion in the context of schemes.

4.1 The Jacobian criterion: classical setting

Proposition 4.1.1. *Let X be a smooth projective curve over an algebraically closed field k. Let $Z \to X$ be a smooth morphism. Define $\mathscr{M}_Z$ to be the functor which inputs a k-scheme T and outputs the set of sections* *of $Z \to X$ over X_T, that is, the set of morphisms s making*

$$\begin{tikzcd} & Z \arrow[d] \\ X \times_k T \arrow[ur, "s"] \arrow[r] & X \end{tikzcd}$$

commute, subject to the condition that, fiberwise on T, the vector bundle $s^* \operatorname{Tan}_{Z/X}$ has vanishing H^1. Then $\mathscr{M}_Z \to \operatorname{Spec} k$ is formally smooth.

Here $\operatorname{Tan}_{Z/X}$ is the tangent bundle, equal to the $\mathcal{O}_Z$-linear dual of the sheaf of differentials $\Omega_{Z/X}$, which is locally free of finite rank. Let $\pi \colon X \times_k T \to T$ be the projection. For $t \in T$, let X_t be the fiber of π over t, and let $s_t \colon X_t \to Z$ be the restriction of s to X_t. By proper base change, the fiber of $R^1\pi_* s^* \operatorname{Tan}_{Z/X}$ at $t \in T$ is $H^1(X_t, s_t^* \operatorname{Tan}_{Z/X})$. The condition about the vanishing of H^1 in the proposition is equivalent to $H^1(X_t, s_t^* \operatorname{Tan}_{Z/X}) = 0$ for each $t \in T$. By Nakayama's lemma, this condition is equivalent to $R^1\pi_* s^* \operatorname{Tan}_{Z/X} = 0$.

Proof. Suppose we are given a commutative diagram

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d] & \mathscr{M}_Z \arrow[d] \\ T \arrow[r] & \operatorname{Spec} k, \end{tikzcd} \tag{4.1.1}$$

where $T_0 \to T$ is a first-order thickening of affine schemes; thus T_0 is the vanishing locus of a square-zero ideal sheaf $I \subset \mathcal{O}_T$. Note that I becomes an $\mathcal{O}_{T_0}$-module.

The morphism $T_0 \to \mathscr{M}_Z$ in (4.1.1) corresponds to a section of $Z \to X$ over T_0. Thus there is a solid diagram

$$\begin{tikzcd} X \times_k T_0 \arrow[r, "s_0"] \arrow[d] & Z \arrow[d] \\ X \times_k T \arrow[r] \arrow[ur, dotted, "s"] & X. \end{tikzcd} \tag{4.1.2}$$

We claim that there exists a dotted arrow making the diagram commute. Since $Z \to X$ is smooth, it is formally smooth, and therefore this arrow exists Zariski-locally on X. Let $\pi \colon X \times_k T \to T$ and $\pi_0 \colon X \times_k T_0 \to T_0$ be the projections. Then $X \times_k T_0$ is the vanishing locus of the ideal sheaf $\pi^* I \subset \mathcal{O}_{X \times_k T}$. Note that sheaves of sets on $X \times_k T$ are equivalent to sheaves of sets on $X \times_k T_0$; under this equivalence, $\pi^* I$ and $\pi_0^* I$ correspond. By [Sta14, Remark 36.9.6], the set of such morphisms form a (Zariski) sheaf of sets on $X \times_k T$, which when viewed as a sheaf on $X \times_k T_0$ is a torsor for

$$\mathscr{H}\!om_{\mathcal{O}_{X \times_k T_0}}(s_0^* \Omega_{Z/X}, \pi_0^* I) \cong s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I.$$

This torsor corresponds to class in

$$H^1(X \times_k T_0, s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I).$$

This H^1 is the limit of a spectral sequence with terms

$$H^p(T_0, R^q \pi_{0*}(s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I)).$$

But since T_0 is affine and $R^q\pi_{0*}(s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I)$ is quasi-coherent, the above terms vanish for all $p > 0$, and therefore

$$H^1(X \times_k T_0, s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I) \cong H^0(T_0, R^1\pi_{0*}(s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I)).$$

Since $s_0^* \operatorname{Tan}_{Z/X}$ is locally free, we have $s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I \cong s_0^* \operatorname{Tan}_{Z/X} \otimes^{\mathbf{L}} \pi_{0*} I$, and we may apply the projection formula [Sta14, Lemma 35.21.1] to obtain

$$R\pi_{0*}(s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I) \cong R\pi_{0*} s_0^* \operatorname{Tan}_{Z/X} \otimes^{\mathbf{L}} I.$$

Now we apply the hypothesis about vanishing of H^1, which implies that $R\pi_{0*} s_0^* \operatorname{Tan}_{Z/X}$ is quasi-isomorphic to the locally free sheaf $\pi_{0*} s_0^* \operatorname{Tan}_{Z/X}$ in degree 0. Therefore the complex displayed above has $H^1 = 0$.

Thus our torsor is trivial, and so a morphism $s\colon X \times_k T \to Z$ exists filling in (4.1.2). The final thing to check is that s corresponds to a morphism $T \to \mathscr{M}_Z$, i.e., that it satisfies the fiberwise $H^1 = 0$ condition. But this is automatic, since T_0 and T have the same schematic points. □

In the setup of Proposition 4.1.1, let $s\colon X \times_k \mathscr{M}_Z \to Z$ be the universal section. That is, the pullback of s along a morphism $T \to \mathscr{M}_Z$ is the section $X \times_k T \to Z$ to which this morphism corresponds. Let $\pi\colon X \times_k \mathscr{M}_Z \to \mathscr{M}_Z$ be the projection. By Proposition 4.1.1 $\mathscr{M}_Z \to \operatorname{Spec} k$ is formally smooth. There is an isomorphism

$$\pi_* s^* \operatorname{Tan}_{Z/X} \cong \operatorname{Tan}_{\mathscr{M}_Z/\operatorname{Spec} k}.$$

Indeed, the proof of Proposition 4.1.1 shows that $\pi_* s^* \operatorname{Tan}_{Z/X}$ has the same universal property with respect to first order deformations as $\operatorname{Tan}_{\mathscr{M}_Z/\operatorname{Spec} k}$.

The following example is of similar spirit as our main application of the perfectoid Jacobian criterion below.

Example 4.1.2. Let $X = \mathbf{P}^1$ over the algebraically closed field k. For $d \in \mathbf{Z}$, let $V_d = \underline{\operatorname{Spec}}_X \operatorname{Sym}_{\mathcal{O}_X}(\mathcal{O}(-d))$ be the geometric vector bundle over X whose global sections are $\Gamma(X, \mathcal{O}(d))$. Fix integers $n, d, \delta > 0$ and let P be a homogeneous polynomial over k of degree δ in n variables. Then P defines a morphism $P\colon \prod_{i=1}^n V_d \to V_{d\delta}$, by sending sections $(s_i)_{i=1}^n$ of V_d to the section $P(s_1, \dots, s_n)$ of $V_{d\delta}$. Fix a global section $f\colon X \to V_{d\delta}$ to the projection morphism and consider the pull-back of P along f:

$$\begin{tikzcd} Z \arrow[r, hook] & P^{-1}(f) \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[dr, "\mathrm{id}_X"] & \\ & \prod_{i=1}^n V_d \arrow[r, "P"] & V_{d\delta} \arrow[r] & X \end{tikzcd}$$

Moreover, let Z be the smooth locus of $P^{-1}(f)$ over X. It is an open subset. The derivatives $\frac{\partial P}{\partial x_i}$ of P are homogeneous polynomials of degree $\delta - 1$ in n variables, hence can be regarded as functions $\prod_{i=1}^n V_d \to V_{d(\delta-1)}$. A point $y \in P^{-1}(f)$ lies in Z if and only if $\frac{\partial P}{\partial x_i}(y)$, $i = 1, \dots, n$ are not all zero. We wish to apply Proposition 4.1.1 to Z/X. Let $\mathscr{M}'_Z$ denote the space of global sections of Z over X, that is for a k-scheme T, $\mathscr{M}'_Z(T)$ is the set of morphisms $s\colon X \times_k T \to Z$ as in the proposition (without any further conditions). A k-point $g \in \mathscr{M}'_Z(k)$ corresponds to a section $g\colon X \to \prod_{i=1}^n V_d$, satisfying $P \circ g = f$. In general, for a (geometric) vector bundle V on X with corresponding locally free $\mathcal{O}_X$-module $\mathscr{E}$, the pull-back of the tangent space $\operatorname{Tan}_{V/X}$ along a section $s\colon X \to V$ is canonically isomorphic to $\mathscr{E}$. Hence in our situation (using that $Z \subseteq P^{-1}(f)$ is open) the tangent space $g^* \operatorname{Tan}_{Z/X}$ can be computed from the short exact sequence,

$$0 \to g^* \operatorname{Tan}_{Z/X} \to \bigoplus_{i=1}^n \mathcal{O}(d) \xrightarrow{D_g P} \mathcal{O}(d\delta) \to 0,$$

where D_gP is the derivative of P at g. It is the $\mathcal{O}_X$-linear map given by $(t_i)_{i=1}^n \mapsto \sum_{i=1}^n \frac{\partial P}{\partial x_i}(g)t_i$ (note that $\frac{\partial P}{\partial x_i}(g)$ are global sections of $\mathcal{O}(d(\delta-1))$). Note that D_gP is surjective: by Nakayama, it suffices to check this fiberwise, where it is true by the condition defining Z.

The space $\mathscr{M}_Z$ is the subfunctor of $\mathscr{M}'_Z$ consisting of all g such that (fiberwise) $g^*\mathrm{Tan}_{Z/X} = \ker(D_gP)$ has vanishing H^1. Writing $\ker(D_gP) = \bigoplus_{i=1}^r \mathcal{O}(m_i)$ ($m_i \in \mathbf{Z}$), this is equivalent to $m_i \geqslant -1$. By the Proposition 4.1.1 we conclude that $\mathscr{M}_Z$ is formally smooth over k.

Consider now a numerical example. Let $n = 3$, $d = 1$ and $\delta = 4$ and let $g \in \mathscr{M}'_Z(k)$. Then $D_gP \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}(1)^{\oplus 3}, \mathcal{O}(4)) = \Gamma(X, \mathcal{O}(3)^{\oplus 3})$, a 12-dimensional k-vector space, and moreover, D_gP lies in the open subspace of surjective maps. We have the short exact sequence of $\mathcal{O}_X$-modules

$$0 \to g^*\mathrm{Tan}_{Z/X} \to \mathcal{O}(1)^{\oplus 3} \xrightarrow{D_gP} \mathcal{O}(4) \to 0 \tag{4.1.3}$$

This shows that $g^*\mathrm{Tan}_{Z/X}$ has rank 2 and degree -1. Moreover, being a subbundle of $\mathcal{O}(1)^{\oplus 3}$ it only can have slopes $\leqslant 1$. There are only two options, either $g^*\mathrm{Tan}_{Z/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}$ or $g^*\mathrm{Tan}_{Z/X} \cong \mathcal{O}(-2) \oplus \mathcal{O}(1)$. The point g lies in $\mathscr{M}_Z$ if and only if the first option occurs for g. Which option occurs can be seen from the long exact cohomology sequence associated to (4.1.3):

$$0 \to \Gamma(X, g^*\mathrm{Tan}_{Z/X}) \to \underbrace{\Gamma(X, \mathcal{O}(1))^{\oplus 3}}_{\text{6-dim'l}} \xrightarrow{\Gamma(D_gP)} \underbrace{\Gamma(X, \mathcal{O}(4))}_{\text{5-dim'l}} \to \mathrm{H}^1(X, g^*\mathrm{Tan}_{Z/X}) \to 0,$$

It is clear that $\Gamma(X, g^*\mathrm{Tan}_{Z/X})$ is 1-dimesional if and only if $g^*\mathrm{Tan}_{Z/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}$ and 2-dimensional otherwise. The first option is generic, i.e., $\mathscr{M}_Z$ is an open subscheme of $\mathscr{M}'_Z$.

4.2 The Jacobian criterion: perfectoid setting

We present here the perfectoid version of Proposition 4.1.1.

Theorem 4.2.1 (Fargues-Scholze [FS]). *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space in characteristic p. Let $Z \to X_S$ be a smooth morphism of schemes. Let $\mathscr{M}_Z^{>0}$ be the functor which inputs a perfectoid space $T \to S$ and outputs the set of sections of $Z \to X_S$ over T, that is, the set of morphisms s making*

$$\begin{tikzcd} & Z \arrow[d] \\ X_T \arrow[ur, "s"] \arrow[r] & X_S \end{tikzcd}$$

commute, subject to the condition that, fiberwise on T, all Harder-Narasimhan slopes of the vector bundle $s^*\,\mathrm{Tan}_{Z/X_S}$ are positive. Then $\mathscr{M}_Z^{>0} \to S$ is a cohomologically smooth morphism of locally spatial diamonds.

Example 4.2.2. Let $S = \eta = \mathrm{Spa}(C, \mathcal{O}_C)$, where C is an algebraically closed perfectoid field of characteristic 0, and let $Z = \mathbf{V}(\mathcal{E}_S(H)) \to X_S$ be the geometric vector bundle attached to $\mathcal{E}_S(H)$, where H is a p-divisible group over the residue field of C. Then $\mathscr{M}_Z = H^0(\mathcal{E}_S(H))$ is isomorphic to $\tilde{H}_\eta^{\mathrm{ad}}$ by Proposition 3.2.1. Let $s\colon X_{\mathscr{M}_Z} \to Z$ be the universal morphism; then $s^*\,\mathrm{Tan}_{Z/X_S}$ is the constant Banach-Colmez space associated to H (i.e., the pull-back of $\mathcal{E}_S(H)$ along $X_{\mathscr{M}_Z} \to X_S$). This has vanishing H^1 if and only if H has no étale part. This is true if and only if $\mathscr{M}_Z^{>0}$ is isomorphic to a perfectoid open ball. The perfectoid open ball is cohomologically smooth, in accord with Theorem 4.2.1. In contrast, if the étale quotient $H^{\text{ét}}$ has height $d > 0$, then $\pi_0(\tilde{H}_\eta^{\mathrm{ad}}) \cong \mathbf{Q}_p^d$ implies that $\tilde{H}_\eta^{\mathrm{ad}}$ is not cohomologically smooth.

In the setup of Theorem 4.2.1, suppose that $x = \mathrm{Spa}(C, \mathcal{O}_C) \to S$ is a geometric point, and that $x \to \mathscr{M}_Z^{>0}$ is an S-morphism, corresponding to a section $s\colon X_C \to Z$. Then $s^*\,\mathrm{Tan}_{Z/X_S}$ is a vector bundle on

X_C. In light of the discussion in the previous section, we are tempted to interpret $H^0(X_C, s^* \operatorname{Tan}_{Z/X_S})$ as the "tangent space of $\mathscr{M}_Z^{>0} \to S$ at x". At points x where $s^* \operatorname{Tan}_{Z/X_S}$ has only positive Harder-Narasimhan slopes, this tangent space is a perfectoid open ball.

5 Proof of the main theorem

5.1 Dilatations and modifications

As preparation for the proof of Theorem 1.0.1, we review the notion of a dilatation of a scheme at a locally closed subscheme [BLR90, §3.2].

Throughout this subsection, we fix some data. Let X be a curve, meaning that X is a scheme which is locally the spectrum of a Dedekind ring. Let $\infty \in X$ be a closed point with residue field C. Let $i_\infty \colon \operatorname{Spec} C \to X$ be the embedding, and let $\xi \in \mathcal{O}_{X,\infty}$ be a local uniformizer at ∞.

Proposition 5.1.1. *Let $V \to X$ be a morphism of finite type, and let $Y \subset V_\infty$ be a locally closed subscheme of the fiber of V at ∞.*

There exists a morphism of X-schemes $V' \to V$ which is universal for the following property: $V' \to X$ is flat at ∞, and $V'_\infty \to V_\infty$ factors through $Y \subset V_\infty$.

The X-scheme V' is the *dilatation* of V at Y. We review here its construction.

First suppose that $Y \subset V_\infty$ is closed. Let $\mathscr{I} \subset \mathcal{O}_V$ be the ideal sheaf which cuts out Y. Let $B \to V$ be the blow-up of V along Y. Then $\mathscr{I} \cdot \mathcal{O}_B$ is a locally principal ideal sheaf. The dilatation V' of V at Y is the open subscheme of B obtained by imposing the condition that the ideal $(\mathscr{I} \cdot \mathcal{O}_B)_x \subset \mathcal{O}_{B,x}$ is generated by ξ at all $x \in B$ lying over ∞.

We give here an explicit local description of the dilatation V'. Let $\operatorname{Spec} A$ be an affine neighborhood of ∞, such that $\xi \in A$, and let $\operatorname{Spec} R \subset V$ be an open subset lying over $\operatorname{Spec} A$. Let $I = (f_1, \dots, f_n)$ be the restriction of $\mathscr{I}$ to $\operatorname{Spec} R$, so that I cuts out $Y \cap \operatorname{Spec} A$. Then the restriction of $V' \to V$ to $\operatorname{Spec} R$ is $\operatorname{Spec} R'$, where

$$R' = R\left[\frac{f_1}{\xi}, \dots, \frac{f_n}{\xi}\right] / (\xi\text{-torsion}).$$

Now suppose $Y \subset V_\infty$ is only locally closed, so that Y is open in its closure $\overline{Y}$. Then the dilatation of V at Y is the dilatation of $V \backslash (\overline{Y} \backslash Y)$ at Y.

Note that a dilatation $V' \to V$ is an isomorphism away from ∞, and that it is affine.

Example 5.1.2. Let

$$0 \to \mathcal{E}' \to \mathcal{E} \to i_{\infty *} W \to 0$$

be an exact sequence of $\mathcal{O}_X$-modules, where $\mathcal{E}$ (and thus $\mathcal{E}'$) is locally free, and W is a C-vector space. (This is an elementary modification of the vector bundle $\mathcal{E}$.) Let $K = \ker(\mathcal{E}_\infty \to W)$.

Let $\mathbf{V}(\mathcal{E}) \to X$ be the geometric vector bundle corresponding to $\mathcal{E}$. Similarly, we have $\mathbf{V}(\mathcal{E}') \to X$, and an X-morphism $\mathbf{V}(\mathcal{E}') \to \mathbf{V}(\mathcal{E})$. Let $\mathbf{V}(K) \subset \mathbf{V}(\mathcal{E})_\infty$ be the affine space associated to $K \subset \mathcal{E}_\infty$. We claim that $\mathbf{V}(\mathcal{E}')$ is isomorphic to the dilatation $\mathbf{V}(\mathcal{E})'$ of $\mathbf{V}(\mathcal{E})$ at $\mathbf{V}(K)$. Indeed, by the universal property of dilatations, there is a morphism $\mathbf{V}(\mathcal{E}') \to \mathbf{V}(\mathcal{E})'$, which is an isomorphism away from ∞.

To see that $\mathbf{V}(\mathcal{E}') \to \mathbf{V}(\mathcal{E})'$ is an isomorphism, it suffices to work over $\mathcal{O}_{X,\infty}$. Over this base, we may give a basis $f_1, \dots, f_n$ of global sections of $\mathcal{E}$, with $f_1, \dots, f_k$ lifting a basis for $K \subset \mathcal{E}_\infty$. Then the localization of $\mathbf{V}(\mathcal{E})' \to \mathbf{V}(\mathcal{E})$ at ∞ is isomorphic to

$$\operatorname{Spec} \mathcal{O}_{X,\infty}\left[\frac{f_1}{\xi}, \dots, \frac{f_k}{\xi}, f_{k+1}, \cdots, f_n\right] \to \operatorname{Spec} \mathcal{O}_{X,\infty}[f_1, \dots, f_n].$$

This agrees with the localization of $\mathbf{V}(\mathcal{E}') \to \mathbf{V}(\mathcal{E})$ at ∞.

Lemma 5.1.3. *Let $V \to X$ be a smooth morphism, let $Y \subset V_\infty$ be a smooth locally closed subscheme, and let $\pi\colon V' \to V$ be the dilatation of V at Y. Then $V' \to X$ is smooth, and $\mathrm{Tan}_{V'/X}$ lies in an exact sequence of $\mathcal{O}_{V'}$-modules*

$$0 \to \mathrm{Tan}_{V'/X} \to \pi^* \mathrm{Tan}_{V/X} \to \pi^* j_* N_{Y/V_\infty} \to 0, \tag{5.1.1}$$

where N_{Y/V_∞} is the normal bundle of $Y \subset V_\infty$, and $j\colon Y \to V$ is the inclusion.

Finally, let $T \to X$ be a morphism which is flat at ∞, and let $s\colon T \to V$ be a morphism of X-schemes, such that s_∞ factors through Y. By the universal property of dilatations, s factors through a morphism $s'\colon T \to V'$. Then we have an exact sequence of $\mathcal{O}_V$-modules

$$0 \to (s')^* \mathrm{Tan}_{V'/X} \to s^* \mathrm{Tan}_{V/X} \to i_{T_\infty *} s_\infty^* N_{Y/V_\infty} \to 0. \tag{5.1.2}$$

Proof. One reduces to the case that Y is closed in V_∞. The smoothness of $V' \to X$ is [BLR90, §3.2, Proposition 3]. We turn to the exact sequence (5.1.1). The morphism $\mathrm{Tan}_{V'/X} \to \pi^* \mathrm{Tan}_{V/X}$ comes from functoriality of the tangent bundle. To construct the morphism $\pi^* \mathrm{Tan}_{V/X} \to \pi^* j_* N_{Y/V_\infty}$, we consider the diagram

$$\begin{tikzcd}
V'_\infty \arrow[r, "\pi'_\infty"] \arrow[dr, "\pi_\infty"'] \arrow[dd, "i_{V'}"'] & Y \arrow[d, "i_Y"] \arrow[dd, bend left, "j"] \\
& V_\infty \arrow[d, "i_V"] \\
V' \arrow[r, "\pi"'] & V
\end{tikzcd}$$

in which the outer rectangle is cartesian. For its part, the normal bundle N_{Y/V_∞} sits in an exact sequence of $\mathcal{O}_Y$-modules

$$0 \to \mathrm{Tan}_{Y/C} \to i_Y^* \mathrm{Tan}_{V_\infty/C} \to N_{Y/V_\infty} \to 0.$$

The composite

$$\begin{aligned}
i_{V'}^* \pi^* \mathrm{Tan}_{V/X} &= \pi_\infty^* i_V^* \mathrm{Tan}_{V/X} \\
&\cong \pi_\infty^* \mathrm{Tan}_{V_\infty/C} \\
&= (\pi'_\infty)^* i_Y^* \mathrm{Tan}_{V/C} \\
&\to (\pi'_\infty)^* N_{Y/V_\infty}
\end{aligned}$$

induces by adjunction a morphism

$$\pi^* \mathrm{Tan}_{V/X} \to i_{V'*}(\pi'_\infty)^* N_{Y/V_\infty} \cong \pi^* j_* N_{Y/V_\infty},$$

where the last step is justified because j is a closed immersion.

We check that (5.1.1) is exact using our explicit description of V'. The sequence is clearly exact away from the preimage of Y in V', since on the complement of this locus, the morphism π is an isomorphism, and $\pi^* j_* = 0$. Therefore we let $y \in Y$ and check exactness after localization at y. Let $\mathcal{I} \subset \mathcal{O}_V$ be the ideal sheaf which cuts out Y, and let $I \subset \mathcal{O}_{V,y}$ be the localization of $\mathcal{I}$ at y. Then $\mathcal{O}_{V_\infty,y} = \mathcal{O}_{V,y}/\xi$. Since $Y \subset V_\infty$ are both smooth at y, we can find a system of local coordinates $\overline{f}_1, \dots, \overline{f}_n \in \mathcal{O}_{V_\infty,y}$ (meaning that the differentials $d\overline{f}_i$ form a basis for $\Omega^1_{V_\infty/C,y}$), such that $\overline{f}_{k+1}, \dots, \overline{f}_n$ generate I/ξ. If $\partial/\partial\overline{f}_i$ are the dual basis, then the stalk of N_{Y/V_∞} at y is the free $\mathcal{O}_{Y,y}$-module with basis $\partial/\partial\overline{f}_{k+1}, \dots, \partial/\partial\overline{f}_n$.

Choose lifts $f_i \in \mathcal{O}_{V,y}$ of the $\overline{f}_i$. Then I is generated by $\xi, f_k, \dots, f_n$. The localization of $V' \to V$ over y is $\operatorname{Spec} \mathcal{O}_{V',y}$, where $\mathcal{O}_{V',y} = \mathcal{O}_{V,y}[g_{k+1}, \dots, g_n]/(\xi\text{-torsion})$, where $\xi g_i = f_i$ for $i = k+1, \dots, n$. Then the stalk of $\operatorname{Tan}_{V'/X}$ at y is the free $\mathcal{O}_{V',y}$-module with basis $\partial/\partial f_1, \dots, \partial/\partial f_k, \partial/\partial g_{k+1}, \dots, \partial/\partial g_n$, whereas the stalk of $\pi^* \operatorname{Tan}_{V/X}$ at y is the free $\mathcal{O}_{V',y}$-module with basis $\partial/\partial f_1, \dots, \partial/\partial f_n$. The quotient between these stalks is evidently the free module over $\mathcal{O}_{V',y}/\xi$ with basis $\partial/\partial f_{k+1}, \dots, \partial/\partial f_n$, and this agrees with the stalk of $\pi^* j_* N_{Y/V_\infty}$.

Given a morphism of X-schemes $s\colon T \to V$ as in the lemma, we apply $(s')^*$ to (5.1.1); this is exact because s' is flat. The term on the right is $s^* j_* N_{Y/V_\infty} \cong i_{T_\infty *} s_\infty^* N_{Y/V_\infty}$ (once again, this is valid because j is a closed immersion). □

5.2 The space $\mathscr{M}_{H,\infty}$ as global sections of a scheme over X_C

We will prove Theorem 1.0.1 for the Rapoport-Zink spaces of the form $\mathscr{M}_{H,\infty}$ before proceeding to the general case. Let H be a p-divisible group of height n and dimension d over a perfect field k. In this context, $\breve{E} = W(k)[1/p]$. Let $\mathcal{E} = \mathcal{E}_C(H)$. Throughout, we will be interpreting $\mathscr{M}_{H,\infty}$ as a functor on $\operatorname{Perf}_{\breve{E}}$ as in Proposition 3.2.2.

We have a determinant morphism $\det\colon \mathscr{M}_{H,\infty} \to \mathscr{M}_{\det H,\infty}$. Let $\tau \in \mathscr{M}_{\det H,\infty}(C)$ be a geometric point of $\mathscr{M}_{\det H,\infty}$. This point corresponds to a section τ of $\mathbf{V}(\det \mathcal{E}) \to X_C$, which we also call τ. Let $\mathscr{M}_{H,\infty}^\tau$ be the fiber of det over τ.

Our first order of business is to express $\mathscr{M}_{H,\infty}^\tau$ as the space of global sections of a smooth morphism $Z \to X_C$, defined as follows. We have the geometric vector bundle $\mathbf{V}(\mathcal{E}^n) \to X$, whose global sections parametrize morphisms $s\colon \mathcal{O}_{X_C}^n \to \mathcal{E}$. Let U_{n-d} be the locally closed subscheme of the fiber of $\mathbf{V}(\mathcal{E}^n)$ over ∞, which parametrizes all morphisms of rank $n-d$. We consider the dilatation $\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty = n-d} \to \mathbf{V}(\mathcal{E}^n)$ of $\mathbf{V}(\mathcal{E}^n)$ along U_{n-d}. For any flat X_C-scheme T, $\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty = n-d}(T)$ is the set of all $s\colon \mathcal{O}_T^n \to \mathcal{E}_T$ such that $\operatorname{cok}(s) \otimes C$ is projective $\mathcal{O}_T \otimes C$-module of rank d. Define Z as the Cartesian product:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty = n-d} \arrow[d, "\det"] \\ X_C \arrow[r, "\tau"'] & \mathbf{V}(\det \mathcal{E}). \end{tikzcd} \tag{5.2.1}$$

Lemma 5.2.1. *Let $\mathscr{M}_Z$ be the functor which inputs a perfectoid space T/C and outputs the set of sections of $Z \to X_C$ over X_T. Then $\mathscr{M}_Z$ is isomorphic to $\mathscr{M}_{H,\infty}^\tau$.*

Proof. Let $T = \operatorname{Spa}(R, R^+)$ be an affinoid perfectoid space over C. The morphism $X_T \to X_C$ is flat. (This can be checked locally: $B_{\mathrm{dR}}^+(R)$ is torsion-free over the discrete valuation ring $B_{\mathrm{dR}}^+(C)$, and so it is flat.) By the description in (5.2.1), an X_T-point of $\mathscr{M}_Z$ corresponds to a morphism $\sigma\colon \mathcal{O}_{X_T}^n \to \mathcal{E}_T(H)$ which has the properties:

(1) The cokernel of σ_∞ is a projective R-module quotient of $\mathcal{E}_T(H)_\infty$ of rank d.

(2) The determinant of σ equals τ.

On the other hand, by Proposition 3.2.2, $\mathscr{M}_{H,\infty}(T)$ is the set of morphisms $\sigma\colon \mathcal{O}_{X_T}^n \to \mathcal{E}_T(H)$ satisfying

(1′) The cokernel of σ is $i_{\infty *} W$, for a projective R-module quotient W of $\mathcal{E}_T(H)_\infty$ of rank d.

(2) The determinant of σ equals τ.

We claim the two sets of conditions are equivalent for a morphism $\sigma\colon \mathcal{O}_{X_T}^n \to \mathcal{E}_T(H)$. Clearly (1′) implies (1), so that (1′) and (2) together imply (1) and (2) together. Conversely, suppose (1) and (2) hold. Since τ represents a point of $\mathscr{M}_{\det H,\infty}$, it is an isomorphism outside of ∞, and therefore so is σ. This means that $\operatorname{cok}\sigma$ is supported at ∞. Thus $\operatorname{cok}\sigma$ is a $B_{\mathrm{dR}}^+(R)$-module. For degree reasons, the length of $(\operatorname{cok}\sigma)\otimes_{B_{\mathrm{dR}}^+(R)} B_{\mathrm{dR}}^+(C')$ has length d for every geometric point $\operatorname{Spa}(C',(C')^+) \to T$. Whereas condition (1) says that $(\operatorname{cok}\sigma)\otimes_{B_{\mathrm{dR}}^+(R)} R$ is a projective R-module of rank d. This shows that $(\operatorname{cok}\sigma)$ is already a projective R-module of rank d, which is condition (1′). □

Lemma 5.2.2. *The morphism $Z \to X_C$ is smooth.*

Proof. Let $\infty' \in X_C$ be a closed point, with residue field C'. It suffices to show that the stalk of Z at ∞' is smooth over $\operatorname{Spec} B_{\mathrm{dR}}^+(C')$.

If $\infty' \neq \infty$, then this stalk is isomorphic to the variety $(\mathbf{A}^{n^2})^{\det=\tau}$ consisting of $n \times n$ matrices with fixed determinant τ. Since τ is invertible in $B_{\mathrm{dR}}^+(C')$, this variety is smooth.

Now suppose $\infty' = \infty$. Let ξ be a generator for the kernel of $B_{\mathrm{dR}}^+(C) \to C$. Then the stalk of Z at ∞ is isomorphic to the flat $B_{\mathrm{dR}}^+(C)$-scheme Y, whose T-points for a flat $B_{\mathrm{dR}}^+(C)$-scheme T are $n \times n$ matrices with coefficients in $\Gamma(T, \mathcal{O}_T)$, which are rank $n-d$ modulo ξ, and which have fixed determinant τ (which must equal $u\xi^d$ for a unit $u \in B_{\mathrm{dR}}^+(C)$). Consider the open subset $Y_0 \subset Y$ consisting of matrices M where the first $(n-d)$ columns have rank $(n-d)$. Then the final d columns of M are congruent modulo ξ to a linear combination of the first $(n-d)$ columns. After row reduction operations only depending on those first $(n-d)$ columns, M becomes

$$\left(\begin{array}{c|c} I_{n-d} & P \\ \hline 0 & \xi Q \end{array}\right),$$

with $\det Q = w$ for a unit $w \in B_{\mathrm{dR}}^+(C)$ which only depends on the first $(n-d)$ columns of M. We therefore have a fibration $Y_0 \to \mathbf{A}^{n(n-d)}$, namely projection onto the first $(n-d)$ columns, whose fibers are $\mathbf{A}^{d(n-d)} \times (\mathbf{A}^{d^2})^{\det=w}$, which is smooth. Therefore Y_0 is smooth. The variety Y is covered by opens isomorphic to Y_0, and so it is smooth. □

We intend to apply Theorem 4.2.1 to the morphism $Z \to X$, and so we need some preparations regarding the relative tangent space of $\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty = n-d} \to X_C$.

5.3 A linear algebra lemma

Let $f\colon V' \to V$ be a rank r linear map between n-dimensional vector spaces over a field C. Thus there is an exact sequence

$$0 \to W' \to V' \xrightarrow{f} V \xrightarrow{q} W \to 0.$$

with $\dim W = \dim W' = n - r$.

Consider the minor map $\Lambda\colon \operatorname{Hom}(V', V) \to \operatorname{Hom}(\bigwedge^{r+1} V', \bigwedge^{r+1} V)$ given by $\sigma \mapsto \bigwedge^{r+1}\sigma$. This is a polynomial map, whose derivative at f is a linear map

$$D_f\Lambda\colon \operatorname{Hom}(V', V) \to \operatorname{Hom}\left(\bigwedge^{r+1} V', \bigwedge^{r+1} V\right).$$

Explicitly, this map is

$$D_f\Lambda(\sigma)(v_1 \wedge \cdots \wedge v_{r+1}) = \sum_{i=1}^{r+1} f(v_1) \wedge f(v_2) \wedge \cdots \wedge \sigma(v_i) \wedge \cdots \wedge f(v_{r+1}).$$

Lemma 5.3.1. *Let*

$$K = \ker\big(\mathrm{Hom}(V', V) \to \mathrm{Hom}(W', W)\big)$$

be the kernel of the map $\sigma \mapsto q \circ (\sigma|_{W'})$. *Then* $\ker D_f\Lambda = K$.

Proof. Suppose $\sigma \in K$. Since f has rank r, the exterior power $\bigwedge^{r+1} V'$ is spanned over C by elements of the form $v_1 \wedge \cdots \wedge v_{r+1}$, where $v_{r+1} \in \ker f = W'$. Since $f(v_{r+1}) = 0$, the sum in (5.3) reduces to

$$D_f\Lambda(\sigma)(v_1 \wedge \cdots \wedge v_{r+1}) = f(v_1) \wedge \cdots \wedge f(v_r) \wedge \sigma(v_{r+1}).$$

Since $\sigma \in K$ and $v_{r+1} \in W'$ we have $\sigma(v_{r+1}) \in \ker q = f(V')$, which means that $D_f\Lambda(\sigma)(v_1, \ldots, v_{r+1}) \in \bigwedge^{r+1} f(V') = 0$. Thus $\sigma \in \ker D_f\Lambda$.

Now suppose $\sigma \in \ker D_f\Lambda$. Let $w \in W'$. We wish to show that $\sigma(w) \in f(V')$. Let $v_1, \ldots, v_r \in V'$ be vectors for which $f(v_1), \cdots, f(v_r)$ is a basis for $f(V')$. Since $\sigma \in \ker D_f\Lambda$, we have $D_f\Lambda(\sigma)(v_1 \wedge \cdots \wedge v_r \wedge w) = 0$. On the other hand,

$$D_f\Lambda(\sigma)(v_1 \wedge \cdots \wedge v_r \wedge w) = f(v_1) \wedge \cdots \wedge f(v_r) \wedge \sigma(w),$$

because all other terms in the sum in (5.3) are 0, owing to $f(w) = 0$. Since the wedge product above is 0, and the $f(v_i)$ are a basis for $f(V')$, we must have $\sigma(w) \in f(V')$. Thus $\sigma \in K$. ☐

We interpret Lemma 5.3.1 as the calculation of a certain normal bundle. Let $Y = \mathbf{V}(\mathrm{Hom}(V', V))$ be the affine space over C representing morphisms $V' \to V$ over a C-scheme, and let $j\colon Y^{\mathrm{rk}=r} \to Y$ be the locally closed subscheme representing morphisms which are everywhere of rank r. Thus, $Y^{\mathrm{rk}=r}$ is an open subset of the fiber over 0 of (the geometric version of) the minor map Λ. It is well known that $Y^{\mathrm{rk}=r}/C$ is smooth of codimension $(n-r)^2$ in Y/C, and so the normal bundle $N_{Y^{\mathrm{rk}=r}/Y}$ is locally free of this rank.

We have a universal morphism of $\mathcal{O}_{Y^{\mathrm{rk}=r}}$-modules $\sigma\colon \mathcal{O}_{Y^{\mathrm{rk}=r}} \otimes_C V' \to \mathcal{O}_{Y^{\mathrm{rk}=r}} \otimes_C V$. Let $\mathcal{W}' = \ker\sigma$ and $\mathcal{W} = \mathrm{cok}\,\sigma$, so that $\mathcal{W}'$ and $\mathcal{W}$ are locally free $\mathcal{O}_{Y^{\mathrm{rk}=r}}$-modules of rank $n-r$. We also have the $\mathcal{O}_{Y^{\mathrm{rk}=r}}$-linear morphism $D\Lambda\colon \mathcal{O}_{Y^{\mathrm{rk}=r}} \otimes_C \mathrm{Hom}(V', V) \to \mathcal{O}_{Y^{\mathrm{rk}=r}} \otimes_C \mathrm{Hom}(\Lambda^{r+1}V', \Lambda^{r+1}V)$, whose kernel is precisely $\mathrm{Tan}_{Y^{\mathrm{rk}=r}/C}$. The geometric interpretation of Lemma 5.3.1 is a commutative diagram with short exact rows:

$$\begin{tikzcd}
\ker D\Lambda \arrow[r] \arrow[d, "\cong"'] & \mathcal{O}_{Y^{\mathrm{rk}=r}} \otimes_C \mathrm{Hom}(V', V) \arrow[r] \arrow[d, "\cong"'] & \mathscr{H}\!om(\mathcal{W}', \mathcal{W}) \arrow[d, "\cong"] \\
\mathrm{Tan}_{Y^{\mathrm{rk}=r}/C} \arrow[r] & j^*\,\mathrm{Tan}_{Y/C} \arrow[r] & N_{Y^{\mathrm{rk}=r}/Y}.
\end{tikzcd} \tag{5.3.1}$$

5.4 Moduli of morphisms of vector bundles with fixed rank at ∞

We return to the setup of §5.1. We have a curve X and a closed point $\infty \in X$, with inclusion map i_∞ and residue field C.

Let $\mathcal{E}$ and $\mathcal{E}'$ be rank n vector bundles over X, with fibers $V = \mathcal{E}_\infty$ and $V' = \mathcal{E}'_\infty$. We have the geometric vector bundle $\mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E})) \to X$. If $f\colon T \to X$ is a morphism, then T-points of $\mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E}))$ classify $\mathcal{O}_T$-linear maps $f^*\mathcal{E}' \to f^*\mathcal{E}$.

Let $\mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E}))^{\mathrm{rk}_\infty=r}$ be the dilatation of $\mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E}))$ at the locally closed subscheme $\mathbf{V}(\mathrm{Hom}(V', V))^{\mathrm{rk}=r}$ of the fiber $\mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E}))_\infty = \mathbf{V}(\mathrm{Hom}(V', V))$. This has the following property, for a flat morphism $f\colon T \to X$: the X-morphisms $s\colon T \to \mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E}))^{\mathrm{rk}_\infty=r}$ parametrize those $\mathcal{O}_T$-linear maps $\sigma\colon f^*\mathcal{E}' \to f^*\mathcal{E}$, for which the fiber $\sigma_\infty\colon f_\infty^* V' \to f_\infty^* V$ has rank r everywhere on T_∞.

Given a morphism s as above, corresponding to a morphism $\sigma\colon f^*\mathcal{E}' \to f^*\mathcal{E}$, we let $\mathcal{W}'$ and $\mathcal{W}$ denote the kernel and cokernel of σ_∞. Then $\mathcal{W}'$ and $\mathcal{W}$ are locally free $\mathcal{O}_{T_\infty}$-modules of rank r. Let $i_{T_\infty}\colon T_\infty \to T$ denote the pullback of i_∞ through f.

We intend to use Lemma 5.1.3 to compute $s^*\,\mathrm{Tan}_{\mathbf{V}(\mathscr{H}om(\mathcal{E}',\mathcal{E}))^{\mathrm{rk}_\infty=r}/X}$. The tangent bundle $\mathrm{Tan}_{\mathbf{V}(\mathscr{H}om(\mathcal{E}',\mathcal{E}))/X}$ is isomorphic to the pullback $f^*\,\mathscr{H}om(\mathcal{E}',\mathcal{E})$. Also, we have identified the normal bundle $N_{\mathbf{V}(\mathrm{Hom}(V',V))^{\mathrm{rk}=r}/\mathbf{V}(\mathrm{Hom}(V',V)}$ in (5.3.1). So when we apply the lemma to this situation, we obtain an exact sequence of $\mathcal{O}_T$-modules

$$0 \to s^*\mathrm{Tan}_{\mathbf{V}(\mathscr{H}om(\mathcal{E}',\mathcal{E}))^{\mathrm{rk}_\infty=r}/X} \to f^*\,\mathscr{H}om(\mathcal{E}',\mathcal{E}) \to i_{T_\infty *}\,\mathscr{H}om(\mathcal{W}',\mathcal{W}) \to 0, \tag{5.4.1}$$

where the third arrow is adjoint to the map

$$i_{T_\infty}^* f^*\,\mathscr{H}om(\mathcal{E}',\mathcal{E}) = \mathrm{Hom}(f_\infty^*V', f_\infty^*V) \to \mathscr{H}om(\mathcal{W}',\mathcal{W}),$$

which sends $\sigma \in \mathscr{H}om(f_\infty^*V', f_\infty^*V)$ to the composite

$$\mathcal{W}' \to f_\infty^*V' \xrightarrow{\sigma_\infty} f_\infty^*V \to \mathcal{W}.$$

The short exact sequence in (5.4.1) identifies the $\mathcal{O}_T$-module $s^*\mathrm{Tan}_{\mathbf{V}(\mathscr{H}om(\mathcal{E}',\mathcal{E}))^{\mathrm{rk}_\infty=r}/X}$ as a modification of $f^*\,\mathscr{H}om(\mathcal{E}',\mathcal{E})$ at the divisor T_∞. We can say a little more in the case that σ itself is a modification. Let us assume that σ fits into an exact sequence

$$0 \to f^*\mathcal{E}' \xrightarrow{\sigma} f^*\mathcal{E} \xrightarrow{\alpha} i_{T_\infty *}\mathcal{W} \to 0.$$

Dualizing gives another exact sequence

$$0 \to f^*(\mathcal{E}^\vee) \xrightarrow{\sigma^\vee} f^*(\mathcal{E}')^\vee \xrightarrow{\alpha'} i_{T_\infty *}(\mathcal{W}')^\vee \to 0.$$

Then

$$\begin{aligned} s^*\mathrm{Tan}_{\mathbf{V}(\mathscr{H}om(\mathcal{E}',\mathcal{E}))^{\mathrm{rk}_\infty=r}/X} &= \ker\left[f^*\,\mathscr{H}om(\mathcal{E}',\mathcal{E}) \to i_{T_\infty *}\,\mathscr{H}om(\mathcal{W}',\mathcal{W})\right] \\ &\cong \ker(\alpha \otimes \alpha') \end{aligned}$$

The kernel of $\alpha\otimes\alpha'$ can be computed in terms of $\ker\alpha = f^*\mathcal{E}'$ and $\ker\alpha' = f^*(\mathcal{E}^\vee)$, see Lemma 5.4.1 below. It sits in a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & f^*\,\mathscr{H}om(\mathcal{E},\mathcal{E}') \arrow[d] & & & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & f^*\,\mathscr{H}om(\mathcal{E},\mathcal{E}) \oplus f^*\,\mathscr{H}om(\mathcal{E}',\mathcal{E}') \arrow[r] & s^*\mathrm{Tan}_{\mathbf{V}(\mathscr{H}om(\mathcal{E}',\mathcal{E}))^{\mathrm{rk}_\infty=r}/X} \arrow[r] & 0. \\
 & \mathrm{Tor}_1(i_{\infty *}\mathcal{W}', i_{\infty *}\mathcal{W}) \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd} \tag{5.4.2}$$

Lemma 5.4.1. *Let $\mathcal{A}$ be an abelian $\otimes$-category. Let*

$$0 \to K \xrightarrow{i} A \xrightarrow{f} B \to 0$$

$$0 \to K' \xrightarrow{i'} A' \xrightarrow{f'} B' \to 0$$

be two exact sequences in $\mathcal{A}$, with A, A', K, K' projective. The homology of the complex

$$K \otimes K' \xrightarrow{(i\otimes 1_{K'}, 1_K \otimes i')} (A \otimes K') \oplus (K \otimes A) \xrightarrow{1_A \otimes i' - i \otimes 1_{A'}} A \otimes A'$$

is given by $H_2 = 0$, $H_1 \cong \mathrm{Tor}_1(B, B')$, and $H_0 \cong B \otimes B'$. Thus, $K'' = \ker(f \otimes f' \colon A \otimes A' \to B \otimes B')$ appears in a diagram

$$\begin{tikzcd} & 0 \arrow[d] & & & \\ & K \otimes K' \arrow[d] & & & \\ 0 \arrow[r] & L \arrow[r] \arrow[d] & (A \otimes K') \oplus (K \otimes A) \arrow[r] & K'' \arrow[r] & 0 \\ & \mathrm{Tor}_1(B, B') \arrow[d] & & & \\ & 0 & & & \end{tikzcd}$$

where both sequences are exact.

Proof. Let $C_\bullet$ be the complex $K \to A$, and let $C'_\bullet$ be the complex $K' \to A'$. Since $C'_\bullet$ is a projective resolution of B', we have a Tor spectral sequence [Sta14, Tag 061Z]

$$E^2_{i,j} \colon \mathrm{Tor}_j(H_i(C_\bullet), B') \implies H_{i+j}(C_\bullet \otimes C'_\bullet).$$

We have $E^2_{0,0} = B \otimes B'$ and $E^2_{0,1} = \mathrm{Tor}_1(B, B')$, and $E^2_{i,j} = 0$ for all other (i, j). Therefore $H_0(C_\bullet \otimes C'_\bullet) \cong B \otimes B'$ and $H_1(C_\bullet \otimes C'_\bullet) \cong \mathrm{Tor}_1(B, B')$, which is the lemma. $\square$

5.5 A tangent space calculation

We return to the setup of §5.2. Thus we have fixed a p-divisible group H over a perfect field k, and an algebraically closed perfectoid field C containing $W(k)[1/p]$. But now we specialize to the case that H is isoclinic. Therefore $D = \mathrm{End}\, H$ (up to isogeny) is a central simple $\mathbf{Q}_p$-algebra. Let $\mathcal{E} = \mathcal{E}_C(H)$; we have $\mathscr{H}\!om(\mathcal{E}, \mathcal{E}) \cong D \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}$.

Recall the scheme $Z \to X_C$, defined as a fiber product in (5.2.1). Let $s \colon X_C \to Z$ be a section. This corresponds to a morphism $\sigma \colon \mathcal{O}^n_{X_C} \to \mathcal{E}$. Let W' and W be the cokernel of σ_∞; these are C-vector spaces.

We are interested in the vector bundle $s^*\mathrm{Tan}_{Z/X_C}$. This is the kernel of the derivative of the determinant map:

$$s^*\mathrm{Tan}_{Z/X_C} = \ker\left(D_s \det \colon s^*\mathrm{Tan}_{\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty = n-d}/X_C} \to \det \mathcal{E}\right).$$

We apply (5.4.2) to give a description of $s^*\mathrm{Tan}_{\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty=n-d}/X_C}$. We get a diagram of $\mathcal{O}_{X_C}$-modules

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
& (\mathcal{E}^\vee)^n \arrow[d] & & & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & (M_n(\mathbf{Q}_p) \times D) \otimes \mathcal{O}_{X_C} \arrow[r] & s^*\mathrm{Tan}_{\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty=n-d}/X_C} \arrow[r] & 0. \\
& \mathrm{Tor}_1(i_{\infty*}W', i_{\infty*}W) \arrow[d] & & & \\
& 0 & & &
\end{tikzcd} \tag{5.5.1}$$

On the other hand, the horizontal exact sequence fits into a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] & (M_n(\mathbf{Q}_p) \times D) \otimes \mathcal{O}_{X_C} \arrow[r] \arrow[d, "\mathrm{tr}"'] & s^*\mathrm{Tan}_{\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty \leqslant n-d}/X_C} \arrow[r] \arrow[d, "D_s \det"] & 0 \\
& & \mathcal{O}_{X_C} \arrow[r, "\tau"'] & \det \mathcal{E} &
\end{tikzcd} \tag{5.5.2}$$

The arrow labeled tr is induced from the $\mathbf{Q}_p$-linear map $M_n(\mathbf{Q}_p) \times D \to \mathbf{Q}_p$ carrying (α', α) to $\mathrm{tr}(\alpha') - \mathrm{tr}(\alpha)$ (reduced trace on D). The commutativity of the lower right square boils down to the identity, valid for sections $s_1, \dots, s_n \in H^0(X_C, \mathcal{E})$ and $\alpha \in D$:

$$((\alpha s_1) \wedge s_2 \wedge \cdots \wedge s_n) + \cdots + (s_1 \wedge \cdots \wedge (\alpha s_n)) = (\mathrm{tr}\ \alpha)(s_1 \wedge \cdots \wedge s_n).$$

(There is a similar identity for $\alpha' \in M_n(\mathbf{Q}_p)$.) Because the arrow labeled τ is injective, we can combine (5.5.1) and (5.5.2) to arrive at a description of $s^*\,\mathrm{Tan}_{Z/X_C}$:

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
& (\mathcal{E}^\vee)^n \arrow[d] & & & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & (M_n(\mathbf{Q}_p) \times D)^{\mathrm{tr}=0} \otimes \mathcal{O}_X \arrow[r] & s^*\mathrm{Tan}_{Z/X_C} \arrow[r] & 0. \\
& \mathrm{Tor}_1(i_{\infty*}W', i_{\infty*}W) \arrow[d] & & & \\
& 0 & & &
\end{tikzcd} \tag{5.5.3}$$

We pass to duals to obtain

$$\begin{tikzcd}
& & & 0 \arrow[d] & \\
0 \arrow[r] & (s^*\mathrm{Tan}_{Z/X_C})^\vee \arrow[r] & ((M_n(\mathbf{Q}_p)\times D)/\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathcal{O}_{X_C} \arrow[r] \arrow[dr, dotted] & \mathcal{F}^\vee \arrow[r] \arrow[d] & 0 \\
& & & \mathcal{E}^n \arrow[d] & \\
& & & \mathrm{Tor}_1(i_{\infty *}((W')^\vee), i_{\infty *}W^\vee) \arrow[d] & \\
& & & 0 &
\end{tikzcd} \tag{5.5.4}$$

The dotted arrow is induced from the map $(M_n(\mathbf{Q}_p)\times D)\otimes_{\mathbf{Q}_p}\mathcal{O}_{X_C}\to\mathcal{E}^n$ sending $(\alpha',\alpha)\otimes 1$ to $\alpha\circ\sigma-\sigma\circ\alpha'$.

Theorem 5.5.1. *If s is a section to $Z\to X_C$ corresponding, under the isomorphism of Lemma 5.2.1, to a point $x\in\mathcal{M}^\tau_{H,\infty}(C)$, then the following are equivalent:*

1. *The vector bundle $s^*\mathrm{Tan}_{Z/X_C}$ has a Harder-Narasimhan slope which is $\leqslant 0$.*
2. *The point x lies in the special locus $\mathscr{M}^{\tau,\mathrm{sp}}_{H,\infty}$.*

Proof. Let $\sigma\colon\mathcal{O}^n_{X_C}\to\mathcal{E}$ denote the homomorphism corresponding to x. Condition (1) is true if and only if $H^0(X_C,s^*\mathrm{Tan}^\vee_{Z/X_C})\neq 0$. We now take H^0 of (5.5.4), noting that $H^0(X_C,\mathcal{F}^\vee)\to H^0(X_C,\mathcal{E}^n)$ is injective. We find that

$$\begin{aligned} H^0(X_C,s^*\mathrm{Tan}^\vee_{Z/X_C}) &\cong \left\{(\alpha',\alpha)\in M_n(\mathbf{Q}_p)\times D\;\middle|\;\alpha\circ\sigma=\sigma\circ\alpha'\right\}/\mathbf{Q}_p. \\ &= A_x/\mathbf{Q}_p. \end{aligned}$$

This is nonzero exactly when x lies in the special locus. ☐

Combining Theorem 5.5.1 with the criterion for cohomological smoothness in Theorem 4.2.1 proves Theorem 1.0.1 for the space $\mathscr{M}_{H,\infty}$.

Naturally we wonder whether it is possible to give a complete discription of $s^*\,\mathrm{Tan}_{Z/X_C}$, as this is the "tangent space" of $\mathscr{M}^\tau_{H,\infty}$ at the point x. Note that $s^*\,\mathrm{Tan}_{Z/X_C}$ can only have nonnegative slopes, since it is a quotient of a trivial bundle. Therefore Theorem 5.5.1 says that 0 appears as a slope of $s^*\,\mathrm{Tan}_{Z/X_C}$ if and only if s corresponds to a special point of $\mathscr{M}^\tau_{H,\infty}$.

Example 5.5.2. Consider the case that H has dimension 1 and height n, so that $\mathscr{M}_{H,\infty}$ is an infinite-level Lubin-Tate space. Suppose that $x\in\mathscr{M}_{H,\infty}(C)$ corresponds to a section $s\colon X_C\to Z$. Then $s^*\,\mathrm{Tan}_{Z/X_C}$ is a vector bundle of rank n^2-1 and degree $n-1$, with slopes lying in $[0,1/n]$; this already limits the possibilities for the slopes to a finite list.

If $n=2$ there are only two possibilities for the slopes appearing in $s^*\,\mathrm{Tan}_{Z/X_C}$: $\{1/3\}$ and $\{0,1/2\}$. These correspond exactly to the nonspecial and special loci, respectively.

If $n=3$, there are a priori five possibilities for the slopes appearing in $s^*\,\mathrm{Tan}_{Z/X_C}$: $\{1/4,1/4\}$, $\{1/3,1/5\}$, $\{1/3,1/3,0,0\}$, $\{2/7,0\}$, and $\{1/3,1/4,0\}$. But in fact the final two cases cannot occur: if 0 appears as a slope, then x lies in the special locus, so that $A_x\neq\mathbf{Q}_p$. But as A_x is isomorphic to a subalgebra of $\mathrm{End}^\circ H$, the division algebra of invariant $1/3$, it must be the case that $\dim_{\mathbf{Q}_p} A_x = 3$, which forces 0 to appear as a slope with multiplicity $\dim_{\mathbf{Q}_p} A_x/\mathbf{Q}_p = 2$. On the nonspecial locus, we suspect that the generic (semistable) case $\{1/4, 1/4\}$ always occurs, as otherwise there would be some unexpected stratification of $\mathscr{M}_{H,\infty}^{\circ,\text{non-sp}}$. But currently we do not know how to rule out the case $\{1/3, 1/5\}$.

5.6 The general case

Let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum over k, with reflex field E. Let F be the center of B. As in Section 3.5, let $D = \operatorname{End}_B V$ and $D' = \operatorname{End}_B H$, so that D and D' are both F-algebras.

Let C be a perfectoid field containing $\breve{E}$, and let $\tau \in \mathscr{M}_{\det \mathcal{D},\infty}(C)$. Let $\mathscr{M}_{\mathcal{D},\infty}^{\tau}$ be the fiber of the determinant map over τ. We will sketch the proof that $\mathscr{M}_{\mathcal{D},\infty}^{\tau} \to \operatorname{Spa} C$ is cohomologically smooth. It is along the same lines as the proof for $\mathscr{M}_{H,\infty}$, but with some extra linear algebra added.

The space $\mathscr{M}_{\mathcal{D},\infty}^{\tau}$ may be expressed as the space of global sections of a smooth morphism $Z \to X_C$, defined as follows. We have the geometric vector bundle $\mathbf{V}(\mathscr{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_X, \mathcal{E}_C(H)))$. In its fiber over ∞, we have the locally closed subscheme whose R-points for a C-algebra R are morphisms, whose cokernel is as a $B \otimes_{\mathbf{Q}_p} R$-module isomorphic to $V_0 \otimes_{\breve{E}} R$, where V_0 is the weight 0 subspace of $V \otimes_{\mathbf{Q}_p} \breve{E}$ determined by μ. We then have the dilatation $\mathbf{V}(\mathscr{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H)))^{\mu}$ of $\mathbf{V}(\mathscr{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_X, \mathcal{E}_C(H)))$ at this locally closed subscheme. Its points over $S = \operatorname{Spa}(R, R^+)$ parametrize B-linear morphisms $s\colon V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \to \mathcal{E}_S(H)$, such that (locally on S) the cokernel of the fiber s_∞ is isomorphic as a $(B \otimes_{\mathbf{Q}_p} R)$-module to $V_0 \otimes_{\breve{E}} R$. Finally, the morphism $Z \to X_C$ is defined by the cartesian diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & \mathbf{V}(\mathscr{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H)))^{\mu} \arrow[d, "\det"] \\
X_C \arrow[r, "\tau"'] & \mathbf{V}(\mathscr{H}om_F(\det_F V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \det_F \mathcal{E}_C(H))).
\end{tikzcd}$$

Let $x \in \mathscr{M}_{\mathcal{D},\infty}(C)$ correspond to a B-linear morphism $s\colon V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \to \mathcal{E}_C(H)$ and a section of $Z \to X_C$ which we also call s. Define $B \otimes_{\mathbf{Q}_p} C$-modules W' and W by

$$0 \to W' \to V \otimes_{\mathbf{Q}_p} C \stackrel{s_\infty}{\to} \mathcal{E}_C(H)_\infty \to W \to 0.$$

The analogue of (5.5.4) is a diagram which computes the dual of $s^* \operatorname{Tan}_{Z/X_C}$:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & (s^*\operatorname{Tan}_{Z/X_C})^\vee \arrow[r] & ((D' \times D)/F) \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \arrow[r] \arrow[dr, dotted] & \mathcal{F}^\vee \arrow[r] \arrow[d] & 0 \\
 & & & \mathscr{H}om(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H)) \arrow[d] & \\
 & & & \operatorname{Tor}_1^F(i_{\infty *}((W')^\vee, i_{\infty *} W^\vee) \arrow[d] & \\
 & & & 0 &
\end{tikzcd} \tag{5.6.1}$$

This time, the dotted arrow is induced from the map $(D' \times D) \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \to \mathscr{H}om(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H))$

sending $(\alpha', \alpha) \otimes 1$ to $\alpha \circ s - s \circ \alpha'$. Taking H^0 in (5.6.1) shows that $H^0(X_C, s^* \operatorname{Tan}^\vee_{Z/X_C}) = A_x/F$, and this is nonzero exactly when x lies in the special locus.

5.7 Proof of Corollary 1.0.2

We conclude with a discussion of the infinite-level modular curve $X(p^\infty)$. Recall from [Sch15] the following facts about the Hodge-Tate period map $\pi_{HT} \colon X(p^\infty) \to \mathbf{P}^1$. The ordinary locus in $X(p^\infty)$ is sent to $\mathbf{P}^1(\mathbf{Q}_p)$. The supersingular locus is isomorphic to finitely many copies of $\mathscr{M}_{H,\infty,C}$, where H is a connected p-divisible group of height 2 and dimension 1 over the residue field of C; the restriction of π_{HT} to this locus agrees with the π_{HT} we had already defined on each $\mathscr{M}_{H,\infty,C}$.

We claim that the following are equivalent for a C-point x of $X(p^\infty)^\circ$:

1. The point x corresponds to an elliptic curve $E/\mathcal{O}_C$, such that the p-divisible group $E[p^\infty]$ has $\operatorname{End} E[p^\infty] = \mathbf{Z}_p$.

2. The stabilizer of $\pi_{HT}(x)$ in $\mathrm{PGL}_2(\mathbf{Q}_p)$ is trivial.

3. There is a neighborhood of x in $X(p^\infty)^\circ$ which is cohomologically smooth over C.

First we discuss the equivalence of (1) and (2). If E is ordinary, then $E[p^\infty] \cong \mathbf{Q}_p/\mathbf{Z}_p \times \mu_{p^\infty}$ certainly has endomorphism ring larger than $\mathbf{Z}_p$, so that (1) is false. Meanwhile, the stabilizer of $\pi_{HT}(x)$ in $\mathrm{PGL}_2(\mathbf{Q}_p)$ is a Borel subgroup, so that (2) is false as well. The equivalence between (1) and (2) in the supersingular case is a special case of the equivalence discussed in Section 3.5.

Theorem 1.0.1 tells us that $\mathscr{M}^{\circ,\text{non-sp}}_{H,\infty}$ is cohomologically smooth, which implies that shows that (2) implies (3). We therefore are left with showing that if (2) is false for a point $x \in X(p^\infty)^\circ$, then no neighborhood of x is cohomologically smooth.

First suppose that x lies in the ordinary locus. This locus is fibered over $\mathbf{P}^1(\mathbf{Q}_p)$. Suppose U is a sufficiently small neighborhood of x. Then U is contained in the ordinary locus, and so $\pi_0(U)$ is nondiscrete. This implies that $H^0(U, \mathbf{F}_\ell)$ is infinite, and so U cannot be cohomologically smooth.

Now suppose that x lies in the supersingular locus, and that $\pi_{HT}(x)$ has nontrivial stabilizer in $\mathrm{PGL}_2(\mathbf{Q}_p)$. We can identify x with a point in $\mathscr{M}^{\circ,\text{sp}}_{H,\infty}(C)$. We intend to show that every neighborhood of x in $\mathscr{M}^\circ_{H,\infty}$ fails to be cohomologically smooth.

Not knowing a direct method, we appeal to the calculations in [Wei16], which constructed semistable formal models for each $\mathscr{M}^\circ_{H,m}$. The main result we need is Theorem 5.1.2, which uses the term "CM points" for what we have called special points. There exists a decreasing basis of neighborhoods $Z_{x,0} \supset Z_{x,1} \supset \cdots$ of x in $\mathscr{M}^\circ_{H,\infty}$. For each affinoid $Z = \operatorname{Spa}(R, R^+)$, let $\overline{Z} = \operatorname{Spec} R^+ \otimes_{\mathcal{O}_C} \kappa$, where κ is the residue field of C. For each $m \geqslant 0$, there exists a nonconstant morphism $\overline{Z}_{x,m} \to C_{x,m}$, where $C_{x,m}$ is an explicit nonsingular affine curve over κ. This morphism is equivariant for the action of the stabilizer of $Z_{x,m}$ in $\mathrm{SL}_2(\mathbf{Q}_p)$. For infinitely many m, the completion $C^{\text{cl}}_{x,m}$ of $C_{x,m}$ is a projective curve with positive genus.

Let $U \subset \mathscr{M}^\circ_{H,\infty}$ be an affinoid neighborhood of x. Then there exists $N \geqslant 0$ such that $Z_{x,m} \subset U$ for all $m \geqslant N$. Let $K \subset \mathrm{SL}_2(\mathbf{Q}_p)$ be a compact open subgroup which stabilizes U, so that U/K is an affinoid subset of the rigid-analytic curve $\mathscr{M}^\circ_{H,\infty}/K$. For each $m \geqslant N$, let $K_m \subset K$ be the stabilizer of $Z_{x,m}$, so that K_m acts on $C_{x,m}$.

There exists an integral model of U/K whose special fiber contains as a component the completion of each $\overline{Z}_{x,m}/K_m$ which has positive genus. Since there is a nonconstant morphism $\overline{Z}_{x,m}/K_m \to C_{x,m}/K_m$, we must have

$$\dim_{\mathbf{F}_\ell} H^1(U/K, \mathbf{F}_\ell) \geqslant \sum_{m \geqslant N} \dim_{\mathbf{F}_\ell} H^1(C^{\text{cl}}_{x,m}/K_m, \mathbf{F}_\ell).$$

Now we take a limit as K shrinks. Since $U \sim \varprojlim U/K$, we have $H^1(U, \mathbf{F}_\ell) \cong \varinjlim H^1(U/K, \mathbf{F}_\ell)$. Also, for each m, the action of K_m on $C_{x,m}$ is trivial for all sufficiently small K. Therefore

$$\dim_{\mathbf{F}_\ell} H^1(U, \mathbf{F}_\ell) \geqslant \sum_{m \geqslant N} \dim_{\mathbf{F}_\ell} H^1(C_{x,m}^{\mathrm{cl}}, \mathbf{F}_\ell) = \infty.$$

This shows that U is not cohomologically smooth.

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