VECTOR BUNDLES ON $\mathbf{P}^1_{\mathbb{Z}}$ WITH THE GENERIC FIBER $\mathcal{O} \oplus \mathcal{O}(1)$ AND SIMPLE JUMPS

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Introduction

In this paper we study vector bundles on the arithmetic surface $\mathbf{P}^1_A$, where A is a Dedekind domain.

The problem of classification of vector bundles on complex projective spaces is quite difficult (see [1]). Relatively little is known in the arithmetic setting. Namely, Hanna showed that every bundle admits a filtration with linear bundles as quotients in the case when A is a Euclidean domain (see [2] or Theorem 1.1.4). An algorithm for constructing such a filtration was obtained by Smirnov and the author in [3].

Smirnov classified vector bundles of rank two with trivial generic fiber and simple jumps in [4] and proved that every such bundle on $\mathbf{P}^1_{\mathbb{Z}}$ has $\mathcal{O}(-2)$ as a subbundle (see [5]).

The purpose of the current article is to classify vector bundles generically isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$ and having simple jumps: that is, either $E_y \cong \mathcal{O} \oplus \mathcal{O}(1)$ or $E_y \cong \mathcal{O}(-1) \oplus \mathcal{O}(2)$ for every closed point $y \in \operatorname{Spec} A$. In the case where A is a PID, we get a complete classification; this is the content of Theorem 2.2.1 and Propositions 2.3.2, 2.3.3. To every such bundle E we attached an important invariant $\Delta(E)$, its discriminant. For example, in the case $A = \mathbb{Z}$ the classification implies that up to isomorphism there are only finitely many vector bundles of given discriminant Δ (see Example 2.3.5).

Notice that $\mathcal{O}(-1)$ is a subbundle of the pullback of E to $\mathbf{P}^1_{\mathbb{Z}} \times \mathbb{Z}/n\mathbb{Z}$ for every non-zero integer n, so the interesting question to ask is whether $\mathcal{O}(-1) \subset E$ globally. It turns out that this is not always the case; the answer is given by Theorem 2.4.3 and involves in fact cubic nonresidues modulo $\Delta(E)$.

In Section 3, we restrict ourselves to the case $A = \mathbb{Z}$. Let Δ be a non-zero integer, $\Delta \neq \pm 1$. Let C be an integral binary cubic form $pv_0^3 + 3qv_0^2v_1 + 3rv_0v_1^2 + sv_1^3$ of discriminant D, and suppose that C satisfies the following condition: C has a triple root modulo d if and only if d divides Δ, where $d \neq 0, \pm 1$. To every such cubic C we associate a vector bundle of discriminant Δ; see Theorem 3.4.2.

It is natural to ask whether any isomorphism class of vector bundles with generic fiber $\mathcal{O} \oplus \mathcal{O}(1)$ and simple jumps can be obtained as a bundle associated to a binary cubic form (this is equivalent to proving that every such bundle has $\mathcal{O}(-2)$ as a subbundle). If it is not the case, then can we describe the obstructions?

1. Preliminaries

We shall study vector bundles over $\mathbf{P}^1_A$ for a Dedekind domain A, in particular for $A = \mathbb{Z}$. It is instructive and more natural to state a few results in greater generality. Thus let A be a Noetherian commutative ring.

We write usually $\mathcal{O}$ and $\mathcal{O}(d)$ instead of $\mathcal{O}_X$ and $\mathcal{O}_X(d)$ for a suitable scheme X, especially if X can be easily specified. As usual, let

$$\mathbf{P}^1_A = \operatorname{Proj} A[t_0, t_1], \quad \deg t_0 = \deg t_1 = 1.$$

In addition, $\mathcal{O}(U_0) = A[x]$, $\mathcal{O}(U_1) = A[x^{-1}]$, and $\mathcal{O}(U_{01}) = A[x, x^{-1}]$, where $x = t_1/t_0$, U_i denotes the complement to the zero locus of t_i, and $U_{01} = U_0 \cap U_1$.

1.1. We shall start with a brief review of vector bundles on $\mathbf{P}^1_A$.

Theorem 1.1.1 (Grothendieck, [1])**.** *Let F be a field. Any vector bundle on $\mathbf{P}^1_F$ is isomorphic to a sum of line bundles with uniquely defined summands.*

Line bundles can be described as follows:

Theorem 1.1.2 ([6])**.** *Any line bundle on $\mathbf{P}^n_A$ is isomorphic to a bundle of the form $p^*L \otimes \mathcal{O}(d)$, where L is a line bundle on* $\operatorname{Spec} A$ *and $p : \mathbf{P}^n_A \to \operatorname{Spec} A$ is a structure morphism.*

In particular, any line bundle on $\mathbf{P}^1_F$ is isomorphic to $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$.

Note that the situation for $A = \mathbb{Z}$ is more complicated. More generally, for certain Dedekind domains A, there exist indecomposable rank 2 vector bundles on $\mathbf{P}^1_A$ (see [4], [9]). It is an open question whether any vector bundle on $\mathbf{P}^1_A$ for a Dedekind ring A admits a filtration with linear bundles as quotients. Let us cite a few known results in this direction.

Theorem 1.1.3 (Hanna, [2])**.** *Let A be a PID, and let E be any vector bundle on $\mathbf{P}^1_A$. Then E has a filtration*

$$0 \subseteq E_0 \subseteq E_1 \subseteq \ldots \subseteq E_t = E$$

such that E_i/E_{i-1} is a line bundle when $i < t$ and E_t/E_{t-1} has rank at most two.

The question is answered completely for any Euclidean ring.

Theorem 1.1.4 (Hanna, [2])**.** *Let A be a Euclidean domain, and let F be any vector bundle on $\mathbf{P}^1_A$. Then E has a filtration*

$$0 = E_0 \subseteq E_1 \subseteq \ldots \subseteq E_n = E$$

such that E_i/E_{i-1} is a line bundle ($1 \leq i \leq n = \operatorname{rk} E$).

In particular, every bundle on $\mathbf{P}^1_{\mathbb{Z}}$ admits a filtration with linear bundles as quotients.

1.2. Cohomology and base change ([7]). We recall the classical theorem and its corollaries.

Let $f : X \to Y$ be a proper morphism of Noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. By X_y we denote the fiber of f over $y \in Y$, $\mathcal{F}_y$ denotes the fiber $\mathcal{F}|_{X_y}$ and said to be a fiber of $\mathcal{F}$ over the point $y \in Y$.

Then $y \mapsto \chi(\mathcal{F}_y)$ is a locally constant function on Y, and the function $y \mapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ is upper semicontinuous. Moreover, if Y is reduced and connected, then

1. The function $y \mapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ is constant on Y if and only if $\mathcal{E} = R^p f_* \mathcal{F}$ is a locally free $\mathcal{O}_Y$-module, and for every $y \in Y$ the natural morphism $\mathcal{E} \otimes_{\mathcal{O}_Y} k(y) \to H^p(X_y, \mathcal{F}_y)$ is an isomorphism.
2. If the preceding equivalent conditions are satisfied, then for every $y \in Y$ the natural map $R^{p-1} f_* \mathcal{F} \otimes_{\mathcal{O}_Y} k(y) \to H^{p-1}(X_y, \mathcal{F}_y)$ is an isomorphism.

1.3. Beilinson spectral sequence. Some of the methods for constructing vector bundles on $\mathbf{P}^n_{\mathbb{C}}$ described in [1] can be applied to construct bundles on $\mathbf{P}^1_A$. Namely, Beilinson spectral sequences are very useful tools in the study of vector bundles in the arithmetic setting.

Theorem 1.3.1 (Beilinson). *Let F be a vector bundle on $\mathbf{P}^1_A$, and let $\pi : \mathbf{P}^1_A \to \operatorname{Spec} A$ be a structure morphism. There is a spectral sequence E^{pq} with E_1-term $E_1^{pq} = R\pi^q_*(F(p)) \otimes \Omega^{-p}(p)$ which converges to*

$$F^i = \begin{cases} F & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases},$$

that is, $E^{pq}_\infty = 0$ for $p + q \neq 0$.

In particular, the E_1-term is concentrated in the second quadrant. Moreover, its nontrivial part is concentrated in the first two rows:

$$H^1(F(-1)) \otimes \mathcal{O}(-1) \xrightarrow{d^1} H^1(F) \otimes \mathcal{O}$$

$$H^0(F(-1)) \otimes \mathcal{O}(-1) \xrightarrow{d^1} H^0(F) \otimes \mathcal{O}.$$

1.4. Gluing. Let A be a PID, and let $\sigma \in \mathrm{GL}_n(A[x, x^{-1}])$. To σ one associates a vector bundle on $\mathbf{P}^1_A$ as follows: $E\big|_{U_0} = \mathcal{O}e_1 + \cdots + \mathcal{O}e_r$, $E\big|_{U_1} = \mathcal{O}f_1 + \cdots + \mathcal{O}f_r$, and

$$[e_1, \ldots, e_r]\sigma = [f_1, \ldots, f_r] \tag{1}$$

over U_{01}, so that $f_j = \sum_{i=1}^{r} \sigma_{i,j} e_i$.

By the Quillen–Suslin Theorem, every finitely generated projective A-module is free, thus any vector bundle of rank r on $\mathbf{P}^1_A$ can be obtained in that way. In this case, the isomorphism class of such a bundle is an element of the double quotient

$$\mathrm{Vect}_r(\mathbf{P}^1) = \mathrm{GL}_r(A[x]) \diagdown \mathrm{GL}_r(A[x, x^{-1}]) \diagup \mathrm{GL}_r(A[x^{-1}]). \tag{2}$$

1.5. Jumps. Given a vector bundle E on $\mathbf{P}^1_A$, we will say that E has a jump over $y \in \operatorname{Spec} A$, or simply E_y is a jump of E, if the fiber E_y of E over the point y is not isomorphic to the generic fiber of E (that is, the fiber over the generic point of $\operatorname{Spec} A$). In this case, y is called a jump point.

The set of jump points is obviously finite.

2. Classification

Let A be a Dedekind domain. We shall study vector bundles of rank 2 on $\mathbf{P}^1_A$ with fixed structure of fibers over the closed points of $\operatorname{Spec} A$. Namely, we consider vector bundles generically isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$ such that their set of jump points is non-empty and all their jumps over the closed points of $\operatorname{Spec} A$ are isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(2)$. In this case, we say that the jumps are simple.

2.1. Non-degenerate maps and vector bundles.

2.1.1. Let E be a vector bundle as above. Consider the bundle $F = E(1)$. We obtain that $H^1(F) = H^1(F(-1)) = 0$, $H^0(F) \simeq A^5$, and $H^0(F(-1)) \simeq A^3$ as a consequence of the proper base change theorem (see 1.2). Thus the E_1-term of the Beilinson spectral sequence is of the form (see 1.3)

$$E_1^{-1,0} \xrightarrow{d_1^{-1,0}} E_1^{0,0}, \tag{3}$$

where $E_1^{-1,0} = H^0(F(-1)) \otimes \mathcal{O}(-1) \simeq \mathcal{O}^3(-1)$, and $E_1^{0,0} = H^0(F) \otimes \mathcal{O} \simeq \mathcal{O}^5$.

Since the spectral sequence with E_1-term (3) degenerates at the second page, it follows from Theorem 1.3.1 that

$$E_\infty^{0,0} = E_2^{0,0} = \operatorname{Coker}\left(d_1^{-1,0}\right) = F. \tag{4}$$

This implies that we have an exact sequence

$$0 \to \mathcal{O}^3(-2) \to \mathcal{O}^5(-1) \to E \to 0.$$

Given a morphism $\varphi \in \operatorname{Hom}(\mathcal{O}^3(-2), \mathcal{O}^5(-1))$, we will say that φ is nondegenerate if $\operatorname{Coker}(\varphi)$ is a locally free sheaf of rank 2. We note that this is equivalent to the assertion that φ is locally split. Consequently, bundles with the generic fiber $\mathcal{O} \oplus \mathcal{O}(1)$ and simple jumps are classified by nondegenerate morphisms.

2.1.2. Let e_1, e_2, e_3 and $f_1, \dots, f_5$ be the standard bases of $\mathcal{O}^3$ and $\mathcal{O}^5$, respectively. The choice of bases fixes an identification

$$\operatorname{Hom}(\mathcal{O}^3(-2), \mathcal{O}^5(-1)) \cong M_{5,3}(\operatorname{Hom}(\mathcal{O}, \mathcal{O}(1))).$$

We identify $\operatorname{Hom}(\mathcal{O}, \mathcal{O}(1)) = At_0 + At_1$, where t_0, t_1 is the basis of $H^0(\mathcal{O}(1))$. Thus, for every $\varphi \in \operatorname{Hom}(\mathcal{O}^3(-2), \mathcal{O}^5(-1))$, we have

$$\varphi = t_0\varphi_0 + t_1\varphi_1, \text{ where } \varphi_0,\ \varphi_1 \in M_{5,3}(A).$$

Let φ, φ' be nondegenerate arrows. We write $\varphi \sim \varphi'$ if $\varphi' = \theta\varphi\lambda$, where $\theta \in \mathrm{GL}_5(A)$ and $\lambda \in \mathrm{GL}_3(A)$. It is clear that $\varphi \sim \varphi'$ implies $\operatorname{Coker}(\varphi) \simeq \operatorname{Coker}(\varphi')$.

2.1.3. Let φ be a nondegenerate arrow. Restricting φ to the point $t_0 = 0$, we obtain

$$\varphi_1 \sim \varphi_1^{(1)} = \begin{pmatrix} I_3 \\ 0_{2,3} \end{pmatrix} \in M_{5,3}(A),$$

where I_3 is the 3×3 identity matrix and $0_{m,n}$ is the zero matrix in $\in M_{m,n}(A)$.

The stabilizer of $\varphi_1^{(1)}$ in $\mathrm{GL}_5(A) \times \mathrm{GL}_3(A)$ consists of the pairs (θ, α^{-1}) with $\theta = \begin{pmatrix} \alpha & \beta \\ 0_{2,2} & \delta \end{pmatrix}$, $\delta \in \mathrm{GL}_2(A)$, and $\beta \in M_{3,2}(A)$. This follows from a straightforward computation.

The theory of elementary divisors implies that $\varphi_0 \sim \varphi_0^{(1)}$, where

$$\varphi_0^{(1)} = \begin{pmatrix} M \\ N \end{pmatrix}, \quad M \in M_{3,3}(A), \quad N = \begin{pmatrix} 0 & \nu\nu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}. \tag{5}$$

Since φ is nondegenerate, it follows that $\nu_1 \in A^*$. Indeed, otherwise we could find a prime π dividing ν_1. Then $\varphi^{(1)}$ has only one 3-minor which can be nontrivial modulo π, but this minor has roots on $\mathbf{P}^1_{\overline{k}}$, where $\overline{k}$ denotes the algebraic closure of $k = A/\pi$, which contradicts the fact that the pullback of E along the morphism $\mathbf{P}^1_{\overline{k}} \to \mathbf{P}^1_A$ is a vector bundle of rank 2 on $\mathbf{P}^1_{\overline{k}}$.

Without loss of generality, we can assume that

$$\varphi = \varphi^{(1)} = t_1 \begin{pmatrix} I_3 \\ 0_{2,3} \end{pmatrix} + t_0 \begin{pmatrix} M(\varepsilon) \\ N(\nu) \end{pmatrix}, \tag{6}$$

where

$$N(\nu) = \begin{pmatrix} 0 & \nu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } M(\varepsilon) = \begin{pmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} & 0 \\ \varepsilon_{2,1} & \varepsilon_{2,2} & 0 \\ \varepsilon_{3,1} & \varepsilon_{3,2} & 0 \end{pmatrix}. \tag{7}$$

Proposition 2.1.1. *Let A be a field. Suppose that φ is nondegenerate, and $E = \operatorname{Coker}(\varphi)$. Then*

$$E \cong \begin{cases} \mathcal{O} + \mathcal{O}(1) & \text{for } \nu \neq 0; \\ \mathcal{O}(-1) + \mathcal{O}(2) & \text{otherwise.} \end{cases}$$

Proof. The exactness of (2.1.1) shows that $\operatorname{Det} E \simeq \mathcal{O}_X(1)$, and $E \simeq \mathcal{O}_X(-d) + \mathcal{O}_X(d+1)$ for some $d \geq 0$. Consider the long exact sequence of cohomology associated to (2.1.1):

$$0 \longrightarrow H^0(X, E) \longrightarrow H^1(X, \mathcal{O}_X^3(-2)) \simeq H^1(X, \mathcal{O}_X(-2))^3 \simeq A^3 \longrightarrow 0.$$

It follows immediately that $h^0(X, E) = 3$ and $d \leq 1$. To distinguish between the cases $d = 0$ and $d = 1$, we use the long exact sequence associated to (2.1.1) twisted by $\mathcal{O}(-1)$. Namely, we have

$$0 \to H^0(E(-1)) \longrightarrow H^1(\mathcal{O}_X^3(-3)) \xrightarrow{H^1(\varphi(-1))} H^1(\mathcal{O}_X^5(-2)) \longrightarrow H^1(E(-1)) \to 0. \tag{8}$$

When computing the middle arrow $H^1(\varphi(-1))$, it is more convenient to work with the adjoint arrow

$$H^0([\varphi(-1)]^\vee \otimes K_X) : H^0(\mathcal{O}_X)^5 \to H^0(\mathcal{O}_X(1))^3,$$

where K_X is the canonical bundle. We note that $H^1(\varphi)^\vee = H^0(\varphi^\vee)$. Consequently, the arrow $H^0([\varphi(-1)]^\vee \otimes K_X)$ is given by multiplication by

$$\varphi^* \in M_{5,3}(\operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X(1))),$$

the adjoint of φ. With respect to the bases $f_1^*, \dots, f_5^*$, and $t_0e_1, \dots, t_0e_3, t_1e_1, \dots, t_1e_3$ the matrix of φ^* has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & 0 & 0 \\ \varepsilon_{1,2} & \varepsilon_{2,2} & \varepsilon_{3,2} & \nu & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We finally conclude that $H^0(X, E(-1)) = A \oplus A/\nu$, and $H^1(X, E(-1)) = A/\nu$, thus $d = 1$ if and only if $\nu = 1$, as desired. □

Corollary 2.1.2. *Let A be a domain, and $\nu \neq 0$. Then $E = \operatorname{Coker}(\varphi)$ is generically isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$, all the jumps have the form $\mathcal{O}(-1) \oplus \mathcal{O}(2)$, and lie exactly over the divisors of ν. In particular, the ideal generated by ν depends only on the isomorphism class of E. In this situation, we say that (ν) is a discriminant ideal of E and write $\Delta(E) = (\nu)$.*

We will generally abuse notation by simply saying that ν is a discriminant of E and writing $\Delta(E) = \nu$.

Further, let us assume that A is a PID. This assumption allows us to describe explicitly the orbits of the action of $\mathrm{GL}_n(A)$ on A^n. As usual, given $a, b \in A$, we write $(a, b) = c$ if $Aa + Ab = (c)$.

2.1.4. The arrow φ is nondegenerate if and only if its pullbacks to U_0 and U_1 are nondegenerate. We first treat the case of the restriction to the open set U_0:

$$\varphi\,|_{U_0} = \varphi_0 + x\varphi_1 = \begin{pmatrix} \varepsilon_{1,1} + x & \varepsilon_{1,2} & 0 \\ \varepsilon_{2,1} & \varepsilon_{2,2} + x & 0 \\ \varepsilon_{3,1} & \varepsilon_{3,2} & x \\ 0 & \nu & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

In this case, we obtain an obvious necessary condition for φ to be nondegenerate:

$$\gcd(\varepsilon_{2,1}, \varepsilon_{3,1}) = 1, \tag{10}$$

since the restriction of $\varphi\,|_{U_0}$ to the point $x = -\,\varepsilon_{1,1}$ must be of rank 3. Let $\overline{\varepsilon_{2,1}}$, $\overline{\varepsilon_{3,1}} \in A$ be such that

$$\varepsilon_{2,1}\,\overline{\varepsilon_{2,1}} + \varepsilon_{3,1}\,\overline{\varepsilon_{3,1}} = 1. \tag{11}$$

2.1.5. A straightforward computation shows that the set of pairs $(\rho', \alpha') \in \mathrm{GL}_5(A) \times \mathrm{GL}_3(A)$, where

$$\rho' = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \beta_{1,1} & \beta_{1,2} \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \beta_{2,1} & \beta_{2,2} \\ 0 & \alpha_{3,2} & \alpha_{3,3} & \beta_{3,1} & \beta_{3,2} \\ 0 & 0 & 0 & \alpha_{2,2} & \alpha_{2,3}\nu \\ 0 & 0 & 0 & \alpha_{3,2}/\nu & \alpha_{3,3} \end{pmatrix}, \quad \alpha' = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ 0 & \delta_{1,1} & \delta_{1,2} \\ 0 & \delta_{2,1} & \delta_{2,2} \end{pmatrix}^{-1}$$

contains the stabilizer of the set of matrices of the form (6). We also note that $\alpha_{3,2} \equiv 0 \pmod{\nu}$, thus

$$\begin{pmatrix} \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} \in \tilde{\Gamma}_0(\nu); \tag{12}$$

here $\tilde{\Gamma}_0(\nu)$ denotes the group $\{\gamma = (\gamma_{i,j}) \in \mathrm{GL}_2(A) : \ \gamma_{2,1} \equiv 0 \pmod{\nu}\}$.

Now set $\alpha_{1,2} = -\alpha_{1,1}\overline{\varepsilon_{2,1}}\,\varepsilon_{1,1}$, $\alpha_{1,3} = -\alpha_{1,1}\overline{\varepsilon_{2,1}}\,\varepsilon_{1,1}$, where $\overline{\varepsilon_{2,1}}$ and $\overline{\varepsilon_{3,1}}$ were defined in (11). According to the arbitrariness of the $\beta_{i,j}$, it follows that, for any nondegenerate arrow φ, there exists an equivalent arrow φ' of the form (6) such that $\varepsilon_{1,1} = 0$; moreover, for $1 \le k \le 3$, the coefficients $\varepsilon_{k,2}$ are defined modulo ν.

Using the above argument, that is, restricting $\varphi\,|_{U_0}$ to the point $x = -\,\varepsilon_{2,2}$, we obtain that at least one of the coefficients $\varepsilon_{2,1}$, $\varepsilon_{3,1}$ is prime to ν. It will therefore suffice to treat two cases: namely, $(\varepsilon_{3,1}, \nu) = 1$ and $(\varepsilon_{3,1}, \nu) \neq 1$.

2.1.6. *The case* $(\varepsilon_{3,1}, \nu) = 1$. Since $\varepsilon_{3,1}$ is prime to ν and $\varepsilon_{2,1}$ (see (10)), it follows that there exist $\tau, \omega \in A$ such that $\varepsilon_{2,1}\,\nu\tau + \varepsilon_{3,1}\,\omega = 1$. Then set

$$\begin{pmatrix} \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} = \begin{pmatrix} \varepsilon_{3,1} & -\,\varepsilon_{2,1} \\ \nu\tau & \omega \end{pmatrix} \in \tilde{\Gamma}_0(\nu). \tag{13}$$

In this case, we have

$$\varphi \sim t_0 \begin{pmatrix} 0 & \varepsilon_{1,2} & 0 \\ 0 & \varepsilon_{2,2} & 0 \\ 1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 1 \end{pmatrix} + t_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

2.1.7. *The case* $(\varepsilon_{3,1}, \nu) \neq 1$. A simple computation shows that

$$\varphi \sim t_0 \begin{pmatrix} 0 & \varepsilon_{1,2} & 0 \\ 1 & \varepsilon_{2,2} & 0 \\ 0 & \varepsilon_{3,2} & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 1 \end{pmatrix} + t_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

2.1.8. We note that any nondegenerate φ is equivalent either to (14) or (15) in the sense of 2.1.2; thus we are reduced to proving nondegeneracy conditions for arrows of such form.

Proposition 2.1.3. *Let φ be as in (14). Then φ is nondegenerate if and only if $\varepsilon_{2,2} = 0$ and $(\varepsilon_{1,2}, \nu) = 1$.*

Proof. We must show that the restrictions $\varphi\big|_{U_0}$ and $\varphi\big|_{U_1}$ are nondegenerate. We first consider the restriction

$$\varphi\big|_{U_1} = y\varphi_0 + \varphi_1 = \begin{pmatrix} 1 & \varepsilon_{1,2}\, y & 0 \\ 0 & \varepsilon_{2,2}\, y + 1 & 0 \\ y & 0 & 1 \\ 0 & \nu y & 0 \\ 0 & 0 & y \end{pmatrix}.$$

It is easy to see that $\varphi\big|_{U_1}$ is nondegenerate if and only if the map

$$A[y] \to A[y]^3 : 1 \mapsto (\varepsilon_{2,2}\, y + 1,\ \nu y,\ \varepsilon_{1,2}\, y^3)$$

is injective and its cokernel is a projective module. We now observe that the latter is equivalent to the assertion that the inclusion $A[y] \to A[y]^3$ is a split morphism; thus the map $\varphi\big|_{U_1}$ is nondegenerate if and only if the row $(\varepsilon_{2,2}\, y+1,\ \nu y,\ \varepsilon_{1,2}\, y^3)$ is unimodular.

If $(\varepsilon_{2,2}\, y + 1,\ \nu y,\ \varepsilon_{1,2}\, y^3)$ is unimodular, then its restriction to the point $y = \nu$ given by

$$(\varepsilon_{2,2}\, \nu + 1,\ \nu^2, \varepsilon_{1,2}\, \nu^3),$$

is also unimodular; consequently there exist a triple $(a, b, c) \in A^3$ such that

$$a(\varepsilon_{2,2}\, \nu + 1) + b\nu^2 + c\, \varepsilon_{1,2}\, \nu^3 = 1.$$

It follows immediately that $a = 1$ and $\varepsilon_{2,2} \equiv 0 \pmod{\nu}$; consequently, according to the remark in the end of 2.1.5, we obtain an equality $\varepsilon_{2,2} = 0$.

Next, we consider the restriction of φ to the open set U_0:

$$\varphi\big|_{U_0} = \varphi_0 + x\varphi_1 = \begin{pmatrix} x & \varepsilon_{1,2} & 0 \\ 0 & \varepsilon_{2,2} + x & 0 \\ 1 & 0 & x \\ 0 & \nu & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By repeating the above argument, we deduce that $\varphi\big|_{U_0}$ is nondegenerate if and only if the row

$$(\varepsilon_{1,2}, \varepsilon_{2,2} + x, \nu) \tag{16}$$

is unimodular. Restricting to the point $x = 0$ (and using the equality $\varepsilon_{2,2} = 0$), we get the desired condition $(\varepsilon_{1,2}, \nu) = 1$.

We now complete the proof by showing that every φ satisfying the conditions of Proposition 2.1.3 is nondegenerate. Let φ be a map of the form (14) such that $\varepsilon_{2,2} = 0$ and $(\varepsilon_{1,2}, \nu) = 1$.

Since the restriction $\varphi\big|_{U_1}$ is obviously nondegenerate, it suffices to prove that $\varphi\big|_{U_0}$ is nondegenerate. As we observed above, the latter is equivalent to the unimodularity of the row (16). Define ζ and ξ by $\varepsilon_{1,2}\,\zeta + \nu\xi = 1$. It is easy to check that

$$(\zeta + x\zeta) \cdot \varepsilon_{1,2} - 1 \cdot x + (\xi + x\xi) \cdot \nu = 1,$$

so the row $(\varepsilon_{1,2}, x, \nu)$ is unimodular. □

Proposition 2.1.4. *Let φ be as in (15). Then φ is nondegenerate if and only if $\varepsilon_{2,2} = 0$, $\varepsilon_{1,2} = 0$, and $(\nu, \varepsilon_{3,2}) = 1$.*

Proof. The proof is similar to that of Proposition 2.1.3 but easier.

We first consider the restriction of φ to U_1 given by

$$\varphi\big|_{U_1} = y\varphi_0 + \varphi_1 = \begin{pmatrix} 1 & \varepsilon_{1,2}\,y & 0 \\ y & \varepsilon_{2,2}\,y + 1 & 0 \\ 0 & \varepsilon_{3,2}\,y & 1 \\ 0 & \nu y & 0 \\ 0 & 0 & y \end{pmatrix}.$$

Suppose that $\varphi\big|_{U_1}$ is nondegenerate. It then follows easily that $\varepsilon_{2,2} = 0$, $\varepsilon_{1,2} = 0$, and the row $(\nu, \varepsilon_{3,2})$ is unimodular; conversely, if φ satisfies the conditions of the proposition, then its restriction to the open set U_1 is obviously nondegenerate; consequently, it will suffice to prove the non-degeneracy of $\varphi\big|_{U_0}$. This can be done by checking the unimodularity of the row $(-x^2, \varepsilon_{3,2}, \nu)$. Since $(\nu, \varepsilon_{3,2}) = 1$ (see 2.1.4), there exist $\zeta, \xi \in A$ such that $\nu\zeta + \varepsilon_{3,2}\,\xi = 1$. Then we have

$$-x^2 \cdot 1 + \varepsilon_{3,2} \cdot (\xi + x^2\xi) + \nu \cdot (\zeta + x^2\zeta) = 1,$$

which completes the proof. □

2.2. A classification theorem. First, we need to introduce a bit of notation.

Let $\varepsilon \in A$. We will denote the matrices $\begin{pmatrix} 0 & \varepsilon & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \varepsilon & 0 \end{pmatrix}$ by $M_1(\varepsilon)$ and $M_2(\varepsilon)$, respectively.

For each $\nu \in A$, we let $N(\nu)$ denote the matrix $\begin{pmatrix} 0 & \nu & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and we define $\widetilde{N}(\nu) = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}$.

Let $(\nu, \varepsilon) \in A^2$ be a pair such that $\nu \notin A^* \cup \{0\}$ and $(\varepsilon,\, \nu) = 1$. For any such pair, we let $V_1(\nu, \varepsilon)$ denote the bundle $\operatorname{Coker}(\varphi)$, where φ is given by the formula

$$\varphi = t_0 \begin{pmatrix} M_1(\varepsilon) \\ N(\nu) \end{pmatrix} + t_1 \begin{pmatrix} I_3 \\ 0_{2,3} \end{pmatrix}. \tag{17}$$

If the pair $(\nu, \varepsilon) \in A^2$ is as above, we let $V_2(\nu, \varepsilon)$ denote the bundle $\operatorname{Coker}(\varphi)$, where φ is the matrix

$$\varphi = t_0 \begin{pmatrix} M_2(\varepsilon) \\ N(\nu) \end{pmatrix} + t_1 \begin{pmatrix} I_3 \\ 0_{2,3} \end{pmatrix}. \tag{18}$$

Theorem 2.2.1. *Let A be a PID, and let E be a vector bundle on $\mathbf{P}^1_A$ such that E is generically isomorphic to $\mathcal{O}\oplus\mathcal{O}(1)$ and $E_y \simeq \mathcal{O}(-1)\oplus\mathcal{O}(2)$ for every closed jump point $y \in \operatorname{Spec} A$. Suppose moreover that the set of jump points is nonempty. Then E is isomorphic either to $V_1(\nu, \varepsilon)$ or to $V_2(\nu, \varepsilon)$ for some pair (ν, ε) such that $\nu \in A \setminus \{0\} \cup A^*$ and ε is prime to ν.*

Proof. Suppose that $\nu \notin A^* \cup \{0\}$, $\varepsilon \in A$, and $(\nu, \varepsilon) = 1$. Then Propositions 2.1.3 and 2.1.4 immediately imply that that $V_1(\nu, \varepsilon)$ and $V_2(\nu, \varepsilon)$ define vector bundles with the generic fiber $\mathcal{O}\oplus\mathcal{O}(1)$ and simple jumps.

It follows by construction that every such bundle can be obtained as a bundle $\operatorname{Coker}(\varphi)$, where φ is either as in (17) or as in (18). □

Remark 2.2.2. Suppose that $\nu, \nu' \notin A^* \cup \{0\}$. It follows from Theorem 2.2.1 that, for every unimodular pair (ν, ε) and every unimodular pair (ν', ε'), $V_1(\nu, \varepsilon) \not\simeq V_2(\nu', \varepsilon')$.

2.3. Morphisms between the bundles. We shall describe morphisms between the bundles in question. Let F and G be vector bundles with the generic fiber $\mathcal{O}\oplus\mathcal{O}(1)$ and simple jumps. We can use the functoriality of Beilinson spectral sequence (see 2.1.1) and canonical isomorphisms

$$H^0(F(1)) \simeq A^5, \quad H^0(F) \simeq A^5, \quad H^0(G(1)) \simeq A^5, \text{ and } H^0(G) \simeq A^3 \tag{19}$$

to reduce to the problem of describing commutative diagrams of the following form

$$\begin{array}{ccccccc}
\mathcal{O}^3(-2) & \xrightarrow{\varphi} & \mathcal{O}^5(-1) & \longrightarrow & F & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle\theta} & & \big\downarrow{\scriptstyle\lambda} & & & & \\
\mathcal{O}^3(-2) & \xrightarrow{\psi} & \mathcal{O}^5(-1) & \longrightarrow & G & \longrightarrow & 0,
\end{array}$$

where φ and ψ are the arrows defining F and G, respectively. We have the following commutativity equation:

$$\lambda\varphi = \psi\theta, \tag{20}$$

where $\lambda \in \mathrm{GL}_5(A)$ and $\theta \in \mathrm{GL}_3(A)$.

According to Theorem 2.2.1, we may choose these arrows either as in (17) or as in (18).

Proposition 2.3.1. *Let $F \simeq V_i(\nu, \varepsilon)$ and $G \simeq V_j(\mu, \zeta)$, where $(\nu, \varepsilon) = 1$, $(\mu, \zeta) = 1$, $\nu, \mu \notin A^* \cup \{0\}$, and $1 \le i, j \le 2$. The functor H^0 and the canonical isomorphisms in (19) identify $\operatorname{Hom}_{\mathcal{O}}(F, G)$ with the set of those $\theta = (\theta_{k,l}) \in M_{3,3}(A)$ satisfying the following conditions:*

(1) $\theta_{2,1} = \theta_{3,1} = 0$.

(2) $\widetilde{N}(\mu)\tilde{\theta}\widetilde{N}(\nu)^{-1} \in M_{2,2}(A)$, *where* $\tilde{\theta} = \begin{pmatrix} \theta_{2,2} & \theta_{2,3} \\ \theta_{3,2} & \theta_{3,3} \end{pmatrix}$.

(3) *The matrix* $M^\theta = ((M_j(\zeta)\theta - \theta M_i(\varepsilon))_{l,k})$, *where* $1 \le l \le 3$ *and* $2 \le k \le 3$, *satisfies*

$$M^\theta \widetilde{N}(\nu)^{-1} \in M_{3,2}(A),\ 2 \le k \le 3.$$

(4) *The first column of the matrix* $M_j(\zeta)\theta - \theta M_i(\varepsilon)$ *is zero.*

Proof. Suppose that there exist a pair of morphisms (λ, θ) such that (20) holds. Then set $\lambda = (\lambda_{i,j})$, where $\lambda_{1,1} \in M_{3,3}(A)$, $\lambda_{1,2} \in M_{2,3}(A)$, $\lambda_{2,1} \in M_{3,2}(A)$, and $\lambda_{2,2} \in M_{2,2}(A)$.

Restricting the commutativity equation to the point $t_0 = 0$, we obtain the following equalities

$$\lambda_{1,1} = \theta \quad \text{and } \lambda_{2,1} = 0_{3,2}.$$

For $t_1 = 0$, the restriction of (20) holds if and only if

$$\theta M_i(\varepsilon) + \lambda_{1,2}N(\nu) = M_j(\zeta)\theta, \quad \text{and } \lambda_{2,2}N(\nu) = N(\mu)\theta. \tag{21}$$

Since $\lambda_{1,2}$ and $\lambda_{2,2}$ are defined by θ, we are reduced to proving the equivalence of the condition for these matrices to be integral (that is, defined over A) and the conditions in the statement. The result now follows from a simple explicit computation (see the definition of $N(\nu)$ in the beginning of 2.2). □

To complete the classification, it suffices to describe isomorphic bundles of the form $V_i(\nu, \varepsilon)$, $i = 1, 2$.

Proposition 2.3.2 (Isomorphic bundles of the form V_1)**.** *Let* (ν, ε) *and* (μ, ζ) *be as in the statement of Theorem 2.2.1. Then* $V_1(\nu, \varepsilon) \simeq V_1(\mu, \zeta)$ *if and only if* $(\mu) = (\nu)$ *and there exists* $\eta \in A^*$ *satisfying*

$$\zeta \equiv \eta\, \varepsilon \pmod{\nu}. \tag{22}$$

Proof. Suppose that F and G are isomorphic. Then it follows from Corollary 2.1.2 that $(\nu) = (\mu)$. We may identify the set of isomorphisms between F and G with the set of those $\theta \in \mathrm{GL}_3(A)$ which satisfy the conditions (1) through (4) of Proposition 2.3.1. Namely, applying (1) and (4), we see that

$$\begin{pmatrix} -\theta_{1,3} \\ -\theta_{2,3} \\ \theta_{1,1} - \theta_{3,3} \end{pmatrix} = 0_3;$$

thus $\theta_{1,3} = \theta_{2,3} = 0$, $\theta_{1,1} = \theta_{3,3}$, and

$$\theta = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} & 0 \\ 0 & \theta_{2,2} & 0 \\ 0 & \theta_{3,2} & \theta_{1,1}. \end{pmatrix}.$$

Now we observe that $\theta_{1,1}$ and $\theta_{2,2}$ are units in A. The condition (3) is equivalent to the assertion that $\theta_{1,2} \equiv 0 \pmod{\nu}$, and $\theta_{1,1}$, $\theta_{2,2}$ satisfy $\varepsilon\, \theta_{2,2} \equiv \zeta\theta_{1,1} \pmod{\nu}$; this proves that (22) is a necessary condition.

Suppose we are given bundles $V_1(\nu, \varepsilon)$ and $V_1(\mu, \zeta)$, and suppose that $\varepsilon\, \eta = \zeta$ for some $\eta \in A^*$. To prove that these bundles are isomorphic, it suffices to find $\theta = (\theta_{i,j}) \in \mathrm{GL}_3(A)$ which satisfies the integrality conditions of Proposition 2.3.1.

It is now easy to check that θ given by the matrix $\begin{pmatrix}\eta & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \eta\end{pmatrix}$ has the desired properties. □

Proposition 2.3.3 (Isomorphic bundles of the form V_2)**.** *Let (ν,ε) and (μ,ζ) be as in the statement of Theorem 2.2.1. Then $V_1(\nu,\varepsilon)\simeq V_1(\mu,\zeta)$ if and only if $(\mu)=(\nu)$ and there exists $\eta\in A^*$ satisfying*

$$\zeta\equiv\eta\,\varepsilon \pmod{\nu}. \tag{23}$$

Proof. Suppose that $V_1(\nu,\varepsilon)\simeq V_1(\mu,\zeta)$. Arguing as in the proof of Proposition 2.3.2, we deduce that $(\mu)=(\nu)$, that $\theta_{1,2}=\theta_{3,2}=0$ and $\theta_{1,1}=\theta_{2,2}$, and that the units $\theta_{1,1}$ and $\theta_{3,3}$ satisfy the congruence $\zeta\theta_{1,1}\equiv\theta_{3,3}\,\varepsilon \pmod{\nu}$; thus the congruence (23) holds for $\eta=\theta_{1,1}^{-1}\theta_{3,3}$. The conditions (2) and (3) of Proposition 2.3.1 imply that $\theta_{1,3}\equiv\theta_{2,3}\equiv 0 \pmod{\nu}$.

It is straightforward to check that the condition (23) is sufficient. For example, we may set

$$\theta=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \eta\end{pmatrix}. \tag{24}$$

This $\theta\in\mathrm{GL}_3(A)$ induces the desired isomorphism. □

Remark 2.3.4. Let K be the field of fractions of A. Then $E_K\simeq\mathcal{O}\oplus\mathcal{O}(1)$ and $\mathrm{Aut}(E)$ is isomorphic to a subgroup of $\mathrm{Aut}(E_K)\simeq(K^*)^2\times K^2$. It follows from the proofs of the above propositions that $\mathrm{Aut}(E)\simeq(A^*)^2\times(\nu A)^2$.

Example 2.3.5. Let $A=\mathbb{Z}$. It follows easily from Proposition 2.3.2 and Proposition 2.3.3 that there are only finitely many bundles of a given discriminant Δ (up to isomorphism): namely, the number of non-isomorphic bundles of discriminant Δ is given by $\varphi(\Delta)$, where φ is Euler's totient function.

2.4. Gluing matrices. In this section, we phrase the classification in terms of gluing matrices (see 1.4). Let ν and ε be as in Theorem 2.2.1, and let α, $\beta\in A$ be such that $\nu\alpha-\varepsilon\,\beta=-1$. Two results below follow from an explicit computation; we refer the reader to [4] for details.

Proposition 2.4.1. *Given a vector bundle $E=V_1(\nu,\varepsilon)$, then there exist frames e_1,e_2 and f_1,f_2 of E over U_0 and U_1, respectively, such that over the open set U_{01} we have*

$$(e_1,e_2)\begin{pmatrix}\varepsilon\,x^{-1} & \nu x\\ \alpha & \beta x^2\end{pmatrix}=(f_1,f_2).$$

Proposition 2.4.2. *Given a vector bundle $E=V_1(\nu,\varepsilon)$, then there exist frames e_1,e_2 and f_1,f_2 of E over U_0 and U_1, respectively, such that over the open set U_{01} we have*

$$(e_1,e_2)\begin{pmatrix}\varepsilon\,x^{-1} & \nu\\ \alpha x & \beta x^2\end{pmatrix}=(f_1,f_2).$$

This description has several geometrical consequences. Combining Propositions 2.4.1 and 2.4.2 with Propositions 2.3.2 and 2.3.3, we see that if $\varepsilon\in A^*$ then, for every $\nu\notin A^*\cup\{0\}$ and $i=1,2$, the bundles $V_i(\nu,\varepsilon)$ have $\mathcal{O}(-1)$ as a subbundle (indeed, since the corresponding gluing matrices can be chosen to be upper triangular).

Moreover, given a vector bundle E with the generic fiber $\mathcal{O}\oplus\mathcal{O}(1)$ and simple jumps, it gives rise to an exact sequence of vector bundles on $\mathbf{P}^1_{A_\pi}$, where π is a prime divisor of $\Delta(E)$ and A_π is a localization of A at π (or its completion); namely

$$0\to\mathcal{O}(-1)\to E_\pi\to\mathcal{O}(2)\to 0,$$

where E_π is the pullback of E along the morphism $\mathbf{P}^1_{A_\pi}\to\mathbf{P}^1_A$. The sequence is split modulo π^k if and only if $k\le v_\pi(\Delta(E))$.

A question arises: whether every bundle from our classification has $\mathcal{O}(-1)$ as a subbundle globally?

In the case $A=\mathbb{Z}$, the answer is negative and has an arithmetic nature.

Theorem 2.4.3. *Let $\nu\neq 0$ be a noninvertible integer, and let ε be relatively prime to ν. If ε is not a perfect cube modulo ν, then $\mathcal{O}(-1)\not\subseteq V_i(\nu,\varepsilon)$ for $i=1$, 2.*

Proof. Let $V=V_i(\nu,\varepsilon)\otimes\mathcal{O}(1)$.

Since V has no jumps outside ν, we can choose an isomorphism $f_{gen}: V\simeq\mathcal{O}(1)\oplus\mathcal{O}(2)$ over $\mathbb{Z}[\nu^{-1}]$. Moreover, we may assume that $H^0(f_{gen})(s)=g_1+g_2$, where $g_1\in H^0(\mathcal{O}(1))$ and $g_2\in H^0(\mathcal{O}(2))$. On the other hand, there exists an isomorphism $f_{sp}: V\simeq\mathcal{O}\oplus\mathcal{O}(3)$ over $\mathbb{Z}/(\nu)$ that can be chosen so that $H^0(f_{sp})(s)=h_1+h_2$ with $h_1\in H^0(\mathcal{O})$ and $h_2\in H^0(\mathcal{O}(3))$.

Suppose that $\mathcal{O}\subseteq V$, that is there exists a section $s\in H^0(V)$, which has no zeroes on $\mathbf{P}^1_{\mathbb{Z}}$. This is equivalent to the assertion that $H^0(f_{gen})(s)$ has no zeroes on $\mathbf{P}^1_{\mathbb{Z}}\times\mathbb{Z}[\nu^{-1}]$ and $H^0(f_{sp})(s)$ has no zeroes on $\mathbf{P}^1_{\mathbb{Z}}\times\mathbb{Z}/(\nu)$; in addition, the existence of common zeroes of corresponding polynomials could be verified using their resultants. Thus, s is nowhere zero if and only if $Res(g_1,g_2)\in\mathbb{Z}[\nu^{-1}]^*$ and $Res(h_1,h_2)\in(\mathbb{Z}/(\nu))^*$. Since $h_1\in H^0(\mathcal{O})\simeq\mathbb{Z}/(\nu)$, the latter condition is satisfied if and only if $h_1\in(\mathbb{Z}/(\nu))^*$.

Then we fix isomorphisms f_{gen} and f_{sp} defined by the following identities (see Section 1.4)

$$\begin{pmatrix}-\beta x^{1+j} & \nu\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varepsilon\, x^{-1} & \nu x^{1-j}\\ \alpha x^j & \beta x^2\end{pmatrix}\begin{pmatrix}-1 & 0\\ \varepsilon\,\nu^{-1}x^{-2+j} & \nu^{-1}\end{pmatrix}=\begin{pmatrix}x^j & 0\\ 0 & x^{1-j}\end{pmatrix},$$

and

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\varepsilon\, x^{-1} & \nu x^{1-j}\\ \alpha x^j & \beta x^2\end{pmatrix}\begin{pmatrix}\varepsilon^{-1} & 0\\ -\alpha(\beta\,\varepsilon)^{-1}x^{-2+j} & \beta^{-1}\end{pmatrix}\equiv\begin{pmatrix}x^{-1} & 0\\ 0 & x^2\end{pmatrix}\pmod{\nu}$$

where $j=0$, 1; we note that ε and β are invertible in $\mathbb{Z}/(\nu)$, since $\varepsilon\,\beta-\alpha\nu=1$.

From here, the proof is straightforward. By Lemma 2.4.4 below, it follows that $h_1=u_0 \pmod{\nu}$, so that

$$u_0\in(\mathbb{Z}/(\nu))^*. \tag{25}$$

In the case $V(-1)=V_1(\nu,\varepsilon)$, applying Lemma 2.4.4 and the isomorphism f_{gen}, we obtain $g_1=\nu v_0t_0+(\nu v_1-\beta u_1)t_1$ and $g_2=u_0t_0^2+\nu u_1t_0t_1+\nu u_2t_1^2$. If we restrict these sections to U_0, we get the following Diophantine equation

$$Res(g_1\big|_{U_0},g_2\big|_{U_0})=u_0(\nu v_1-\beta u_0)^2-\nu^2v_0(u_1(\nu v_1-\beta u_0)-\nu v_0u_2)=\pm\nu^l, \tag{26}$$

where $l\in\mathbb{Z}$, but (25) implies that $l=0$. To finish the proof of the theorem in this case it remains to reduce (26) modulo ν. Indeed, we have

$$-\beta^2u_0^3\equiv\pm1\pmod{\nu},$$

and $\beta\equiv\varepsilon^{-1} \pmod{\nu}$.

When $V(-1) = V_2(\nu,\varepsilon)$, the argument is quite analogous to what we did above. In this case, we obtain

$$Res(g_1\big|_{U_0}, g_2\big|_{U_0}) = -u_0(\nu^2 u_2 v_1 - u_0(\nu v_2 - \beta u_0)) + \nu^3 v_0 u_2 = \pm 1. \tag{27}$$

To prove the theorem, we again consider the equation (27) reduced modulo ν. □

Lemma 2.4.4. *Let $V = V_i(\nu,\varepsilon)\otimes\mathcal{O}(1)$, where ν and ε are integers as in Theorem 2.2.1, and let α, $\beta\in\mathbb{Z}$ be such that $\varepsilon\beta-\nu\alpha = 1$. Then*

$$H^0(V) = \begin{cases} \{(u_0t_0^2 + \nu u_1 t_0 t_1 + \nu u_2 t_1^2,\ v_0 t_0^3 + v_1 t_0^2 t_1^3 + \beta u_1 t_0 t_1^2 + \beta u_2 t_1^3)\}, & \text{if } i = 1,\\ \{(u_0 t_0 + \nu u_1 t_1, v_0 t_0^3 + v_1 t_0^2 t_1 + v_2 t_0 t_1^2 + \beta u_1 t_1^3)\}, & \text{if } i = 2,\end{cases}$$

where $u_k, v_k\in\mathbb{Z}$, $k = 0,1,2$.

Proof. Using Propositions 2.4.1 and 2.4.2 we can easily compute $H^0(V(1))$; indeed, every global section can be written in the form $s = (s_1e_1 + s_2e_2)t_0$, where e_1, e_2 is a frame of V over U_0, and s_1, $s_2\in\mathbb{Z}[x]$ satisfy certain compatibility conditions. Namely, if $i = 1$, then Proposition 2.4.1 provides the following condition $-\alpha s_1 y^2 + \varepsilon s_2 y^3, \beta s_1 - \nu s_2 y\in\mathbb{Z}[y]$, where $y = x^{-1}$. Applying Proposition 2.4.2, we obtain $-\alpha s_1 y + \varepsilon s_2 y^3$, $\beta s_1 - \nu s_2 y^2\in\mathbb{Z}[y]$ in the case $i = 2$. The lemma now follows immediately. □

Remark 2.4.5. It follows from the proof of Theorem 2.4.3 that the existence of integer solutions of (26) with $l = 0$, and (27) is equivalent to the assertion that $\mathcal{O}(-1)\subset V_i(\nu,\varepsilon)$. Using a computer program we checked this for pairs $(\nu,\varepsilon)\in\mathbb{Z}^2$, such that $|\nu|\le 30$ and ε being a perfect square modulo ν. In these cases, the integer solutions exist, which leads us to the natural question, if the latter condition on ε is sufficient for a bundle $V_i(\nu,\varepsilon)$ to have $\mathcal{O}(-1)$ as a subbundle.

3. Binary cubic forms and subbundles

There is another approach to obtaining vector bundles with generic fiber $\mathcal{O}\oplus\mathcal{O}(1)$ and simple jumps; namely, those which have $\mathcal{O}(-2)$ as a subbundle. Moreover, given such a bundle E, there is a filtration

$$0\to\mathcal{O}(-2)\to E\to\mathcal{O}(3)\to 0. \tag{28}$$

We first note that every such bundle defines an element in $\mathrm{Ext}^1(\mathcal{O}(3),\mathcal{O}(-2))\simeq H^0(\mathcal{O}(3))^\vee$: that is, a binary cubic form. We are interested in obtaining a characterization of such cubic forms.

Throughout this section, we will focus our attention on the case $A = \mathbb{Z}$.

3.1. Serre duality. Let V be a free $\mathbb{Z}$-module of rank 2. Consider the scheme $\mathbb{P}(V)\simeq\mathbf{P}^1_{\mathbb{Z}}\simeq\mathrm{Proj}(\mathbb{Z}[t_0,t_1])$, where t_0, t_1 is a basis of $V^\vee$ (dual to some basis v_0, v_1 of V). As usual, $H^0(\mathbb{P}(V),\mathcal{O}(i)) = \Gamma(\mathbb{P}(V),\mathcal{O}(i))\simeq Sym^i(V^\vee)$.

According to Serre duality for projective spaces, there is a natural isomorphism $H^1(\mathbb{P}(V),\mathcal{O}(-2-i))\simeq(\widetilde{Sym}^i(V^\vee))^\vee$. An explicit identification can be described as follows:

$$t_0^{l_0}t_1^{l_1}\mapsto\sum_{(m_1,\dots m_i)} v_{m_1}\otimes\ldots\otimes v_{m_i}, \tag{29}$$

where $l_j < 0$, $l_0 + l_1 = -2-i$ and the sum is taken over all sequences $(m_1,\dots,m_i)$ that contain $-1-l_j$ terms equal to j.

3.2. Bundles associated to cubic forms. Let $(p, q, r, s) \in A^4$. To the section $f(t_0, t_1) = pt_0^{-4}t_1^{-1} + qt_0^{-3}t_1^{-2} + rt_0^{-2}t_1^{-3} + st_0^{-1}t_1^{-4} \in H^1(\mathcal{O}(-5))$, we associate an isomorphism class of the bundle E_f defined by the gluing matrix

$$\sigma_f = \begin{pmatrix} x^{-2} & \tilde{f} \\ 0 & x^3 \end{pmatrix}, \tag{30}$$

where $\tilde{f}$ denotes the Laurent polynomial $px^{-1} + q + rx + sx^2$; thus we obtain a map from $\mathrm{Ext}^1(\mathcal{O}(-3), \mathcal{O}(2))$ to the set of the isomorphism classes of vector bundles admitting a filtration of the form (28). It follows immediately that $\mathrm{Det}(E_f) \simeq \mathcal{O}(1)$; consequently the fibers of E_f over the closed points of $\mathrm{Spec}(A)$ are isomorphic to $\mathcal{O}(-l) \oplus \mathcal{O}(l + 1)$, where $0 \leq l \leq 2$. We can compute cohomology of $E_f(-1)$ to distinguish between the cases $l = 0$, $l = 1$, and $l = 2$.

For every $d \in \mathbb{Z}$ such that $d \neq 0, \pm 1$, we set $X_d = \mathbf{P}^1_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}/(d)$.

Proposition 3.2.1. *Let E_f be as above, and let d be an integer such that $d \neq 0, \pm 1$. If F is the pullback of $E_f(-1)$ along the morphism $X_d \to \mathbf{P}^1_{\mathbb{Z}}$, then*

$$h^0(X_d, F) = 3 - \mathrm{rk} \begin{pmatrix} p & q & r \\ q & r & s \end{pmatrix}. \tag{31}$$

Proof. To prove this, fix frames g_1, g_2 and h_1, h_2 of F over the open sets U_0 and U_1, respectively, such that

$$(g_1, g_2)\sigma_f = (h_1, h_2) \text{ over } U_{01}. \tag{32}$$

Let s be an element of $H^0(X_d, F)$. Then we can write $s = s_1 g_1 + s_2 g_2$ for some polynomials s_1, $s_2 \in \mathcal{O}_{X_d}(U_0) = (\mathbb{Z}/(d))[x]$. Applying (32), we see that

$$s = (s_1 x^3 - s_2 \tilde{f})h_1 + s_2 x^{-2} h_2 = (s_1 y^{-3} - s_2 \tilde{f})h_1 + s_2 y^2 h_2,$$

where $y = x^{-1}$.

Thus s is a global section if and only if the polynomials s_1 and s_2 satisfy the following conditions:

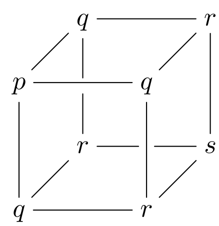
(1) $s_1 y^{-3} - s_2 \tilde{f} \in \mathcal{O}_{X_d}(U_1) = (\mathbb{Z}/(d))[y]$.
(2) $s_2 y^2 \in \mathcal{O}_{X_d}(U_1) = (\mathbb{Z}/(d))[y]$.

It follows immediately from the second condition that $\deg(s_2) \leq 2$. Consequently, we deduce from the first condition that $\deg(s_1) \leq 1$; moreover, coefficients of s_1 are uniquely determined by this property and coefficients of s_2. Let $s_2 = u_0 + u_1 x + u_2 x^2$. Then the first condition is reduced to the following system of linear equations:

$$\begin{cases} su_0 + ru_1 + qu_2 = 0 \\ ru_0 + qu_1 + pu_2 = 0. \end{cases} \tag{33}$$

This completes the proof of the proposition. □

3.3. Applying the identification (29) to f, we obtain an element of $(Sym^3V^\vee)^\vee$ which can be represented by the following Bhargava cube



and the binary cubic form $C(v_0, v_1) = pv_0^3 + 3qv_0^2v_1 + 3rv_0v_1^3 + sv_1^3$ associated to it; see [12]. The form C will play an important role in the construction below. Let H be the Hessian of C, its covariant defined by the formula

$$H(v_0, v_1) = (q^2 - pr)v_0^2 + (ps - qr)v_0v_1 + (r^2 - qs)v_1^2. \tag{34}$$

We note that a binary cubic form C has vanishing Hessian, $H \equiv 0$, if and only if C has a single triple root.

3.4. Combining Proposition 3.2.1 with the above observations, we deduce the following results:

Proposition 3.4.1. *Let F be as in the statement of Proposition 3.2.1. Then*

$$F \simeq \begin{cases} \mathcal{O}(-1)\oplus\mathcal{O} & \text{if } \overline{H} \not\equiv 0. \\ \mathcal{O}(-2)\oplus\mathcal{O}(1) & \text{if } \overline{H} \equiv 0 \text{ and } \overline{C} \not\equiv 0. \\ \mathcal{O}(-3)\oplus\mathcal{O}(2) & \text{if } \overline{C}, \overline{H} \equiv 0. \end{cases} \tag{35}$$

Where $\overline{C}, \overline{H} \in (\mathbb{Z}/(d))[x]$ denote the reductions of C and H modulo d, respectively.

Theorem 3.4.2. *Let E_f be a vector bundle on $\mathbf{P}^1_{\mathbb{Z}}$ associated to $f \in \operatorname{Ext}^1(\mathcal{O}(3), \mathcal{O}(-2))$ in Section 3.2, and let $C = pv_0^3 + 3qv_0^2v_1 + 3rv_0v_1^2 + sv_1^3$ be a binary cubic form with the Hessian H defined in Section 3.3. Then E_f is generically isomorphic to $\mathcal{O}\oplus\mathcal{O}(1)$ and its jumps have the form $\mathcal{O}(-1)\oplus\mathcal{O}(2)$ if and only if the following conditions hold:*

1. $\gcd(p, q, r, s) = 1$.
2. *There is an integer $d \neq 0, \pm 1$ such that the reduction of H modulo d is identically zero in $(\mathbb{Z}/(d))[x]$.*

Moreover, d satisfies the condition (2) if and only if $d \notin \mathbb{Z}^*$ is a non-zero divisor of $\Delta(E_f)$.

Proof. This is clear from the characterization of special fibers of E_f in Proposition 3.4.1. The last assertion follows from the definition of Δ (see 2.1.2). □

Suppose that $\gcd(p, q, r, s) = 1$. Thus given a binary cubic form $C = pv_0^3 + 3qv_0^2v_1 + 3rv_0v_1^3 + sv_1^3$ of discriminant D, its reductions modulo divisors of D have double or triple roots. This data determines a vector bundle E of discriminant $\Delta(E)$ such that $D = \Delta(E)\tilde{D}$ and the reduction of C modulo d has a triple root whenever d divides $\Delta(E)$. We also note that the conditions of Theorem 3.4.2 are invariant under the GL_2-equivalence of binary forms. This raises the following question: can we interpret the statement of Theorem 2.2.1 in terms of the classification of cubic forms (or equivalently, in terms of orders in cubic fields)?

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