A VOLUME ESTIMATE FOR PIECEWISE SMOOTH METRICS ON SIMPLICIAL COMPLEXES

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Abstract. We prove L. Green's volume estimate in the case of piecewise smooth Riemannian metrics on simplicial complexes.

A famous conjecture of W. Blaschke states that a Auf Wiedersehensfläche is a round sphere. One of the crucial elements in L. Green's celebrated proof of this conjecture is a lower estimate for the volume of a compact Riemannian manifold in terms of conjugate radius and mean scalar curvature, see [Gr] or [Be]. In this paper, we prove a generalization of this estimate for piecewise smooth Riemannian metrics on simplicial complexes. We hope that the estimate is useful in investigations concerning the rigidity of piecewise smooth Riemannian metrics.

Let X be a finite simplicial complex of dimension n. We say that X is *closed* if the following two conditions hold:

(C1) every simplex of X is contained in an n-simplex;
(C2) every $(n-1)$-simplex of X is adjacent to at least two n-simplices.

We always assume that X is closed. The n-simplices of X will be called *chambers*, the $(n-1)$-simplices *panels*. The m-skeleton of X is denoted X^m.

A *piecewise smooth Riemannian metric* g on X is a family of smooth Riemannian metrics g_A on the simplices A of X such that $g_A|B = g_B$ for any simplices A and B with $B \subset A$. For such a metric on X, the volume element of a simplex A of X is denoted μ_A. If $\dim A \geq 2$, the sectional, Ricci respectively scalar curvature of g_A is denoted K_A, Ric_A respectively scal_A.

From now on, let X be a closed simplicial complex with a piecewise smooth Riemannian metric g. Let P be a panel in X and C a chamber adjacent to P. For a point x in the interior of P, denote by $N_P^C(x)$ the unit normal vector to P at x pointing inside C. If S_P^C denotes the second fundamental form of $P \subset C$ with respect to N_P^C, then the *mean curvature* h_P^C is given by

$$h_P^C = \frac{1}{n-1} \operatorname{trace} S_P^C.$$

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If γ is a geodesic and C is a chamber, then the part of γ inside C is a geodesic with respect to g_C. Whenever γ passes transversally through the interior of a panel P, $x = \gamma(t_0) \in P$, coming in through a chamber C and going out through a chamber C', then the incoming direction $\dot\gamma(t_0 - 0)$ and outgoing direction $\dot\gamma(t_0 + 0)$ of γ coincide with respect to the natural identification $T_x C = T_x C'$.

We say that the metric g has *conjugate radius* $\operatorname{conj}(g) \geq a$ if there is no conjugate point along geodesics of length $< a$ which miss X^{n-2} and intersect X^{n-1} transversally.

Theorem. *Let X be a closed simplicial complex with a piecewise smooth Riemannian metric g. If $\operatorname{conj}(g) \geq \pi$, then*

$$\operatorname{vol}(X) := \sum_C \operatorname{vol} C \geq \frac{1}{n(n-1)} \sum_C \int_C \operatorname{scal}_C \, d\mu_C + \frac{2}{n} \sum_{P|C} \int_P h_P^C d\mu_P,$$

where the summation is over all chambers C respectively all pairs $P|C$ of a panel P adjacent to a chamber C. Equality holds if and only if

(1) *for each chamber C, the metric g_C has constant sectional curvature $K_C = 1$;*
(2) *for each panel P adjacent to exactly two chambers C and C', the metrics of C and C' fit together smoothly at P;*
(3) *each panel P adjacent to more than two chambers is totally geodesic.*

The proof of the theorem is along the lines of Green's proof, except that we have to consider possible branchings of geodesics along panels. Moreover, the panels contribute a term in the estimate since in general, the metrics of different chambers adjacent to a given panel do not fit together smoothly.

In dimension $n = 2$, the scalar curvature is twice the Gauß curvature, and under the assumptions as in the theorem we obtain

$$\operatorname{area}(X) \geq \sum_f \int_f K_f da_f + \sum_{e|f} \int_e \kappa_e^f ds_e,$$

where e and f denote edges and faces of X respectively, a_f and s_e area of face f and arc length of edge e and κ_e^f the geodesic curvature of e with respect to f. By the Gauß-Bonnet formula we have

$$\sum_f \int_f K_f da_f + \sum_{e|f} \int_e \kappa_e^f ds_e = 2\pi\chi(X) - \sum_v \big(2\pi - \chi(v)\pi - \alpha(v)\big),$$

where the sum on the RHS is over the vertices v of X, $\chi(v)$ is the Euler characteristic of the link X_v of X at v and $\alpha(v)$ the length of X_v. Here X_v is endowed with the length metric d_v induced from angle measurement in the faces of X adjacent to v.

Corollary. *Let X be a closed 2-dimensional simplicial complex with a piecewise smooth Riemannian metric g. If* $\operatorname{conj}(g) \geq \pi$*, then*

$$\operatorname{area}(X) \geq 2\pi\chi(X) + \sum_v \big(\chi(v)\pi + \alpha(v) - 2\pi\big)$$

with equality if and only if g is piecewise spherical.

The surface F_m of genus $m \geq 1$, where the $4m$-gon in the usual construction of F_m is subdivided into $4m$ equilateral spherical triangles with interior angle $\alpha > \pi/3$, is an example where equality holds.

Before we start with the proof of the theorem we need some preparations. To this end, let X be a closed n-dimensional simplicial complex with a piecewise smooth Riemannian metric g. If X happens to be a smooth Riemannian manifold, then the geodesic flow (φ^t) of X acts on the unit sphere bundle SX of X. Recall that (φ^t) leaves the Liouville measure on SX invariant. This is crucial in Green's proof of his volume estimate. A corresponding property will be crucial in our proof of the above theorem.

Now in general, geodesics branch on the spaces we consider, and it does not make sense to speak of *the geodesic* tangent to a given vector or direction. Therefore we do not consider the unit sphere bundle of X. Instead, we consider the space UX of unit speed geodesics. The geodesic flow, again denoted (φ^t), operates on UX by reparameterization,

$$\varphi^t(\gamma)(s) = \gamma(t+s), \quad s, t \in \mathbb{R}.$$

If X happens to be a smooth manifold, then the isomorphism $UX \cong SX$, $\gamma \mapsto \dot\gamma(0)$, conjugates the geodesic flow on UX and the standard one on SX.

A *geodesic chain* is a finite sequence $\gamma_0, \dots, \gamma_k$ of geodesic segments in X such that there are times $s_0, \dots, s_{k-1}$ and $t_1, \dots, t_k$ and a constant $\varepsilon > 0$ with the property that

$$\gamma_{i-1}(s_{i-1} + \tau) = \gamma_i(t_i + \tau), \quad 0 \leq \tau < \varepsilon.$$

We say that a geodesic segment γ is *admissible* if for any chain $\gamma_0, \dots, \gamma_k$ as above with $\gamma_0 = \gamma$, the geodesic γ_k misses X^{n-2} and intersects X^{n-1} transversally.

There is a natural generalization of the standard Liouville measure to a measure $\mathcal{L}$ on UX, also called the *Liouville measure*, see [BB]. We recall some of its properties.

(L1) $\mathcal{L}$ is concentrated on $U^*X = \{\gamma \in UX \mid \gamma \text{ is admissible}\}$.

(L2) $\mathcal{L}$ is invariant under the geodesic flow.

Any chamber C is a Riemannian manifold, and its unit tangent bundle SC carries the standard Liouville measure $\mathcal{L}_C$.

(L3) For every chamber C of X and measurable subset $A \subset SC$, the $\mathcal{L}$-measure of the set of $\gamma \in UX$ with $\dot\gamma(0) \in A$ is equal to $\mathcal{L}_C(A)$.

Let P be a panel of X and x be a point in the interior of P. Denote by $\nu = \nu(P) \geq 2$ the number of chambers adjacent to P. For a chamber C adjacent to P, denote by H_xC the hemisphere of unit tangent vectors of C at x pointing inside C and by λ_x the Lebesgue measure on H_xC. For $v \in H_xC$, let $\theta(v)$ be the angle between v and the normal $N_P^C(x)$. Now let $C' \neq C$ be two chambers adjacent to P. Let I be an interval and consider, for $\gamma \in U^*X$, the times

$$t_1 < \ldots < t_k, \quad t_i = t_i(\gamma), \quad 1 \le i \le k = k(\gamma) \ge 0,$$

in the interior of I at which γ enters P through C' and exits through C, $1 \le i \le k$.

(L4) For any bounded measurable function $f : I \times \cup_{x \in P} H_xC \to \mathbb{R}$

$$\int \sum_{i=1}^{k(\gamma)} f(t_i, \dot\gamma(t_i)) d\mathcal{L}(\gamma) = \frac{1}{\nu - 1} \int_I \int_P \int_{H_xC} f(t,v) \cos\theta(v) d\lambda_x(v) d\mu_P(x) dt.$$

Proof of theorem. Let $\gamma : [0,\pi] \to X$ be an admissible geodesic. Choose a subdivision $t_0 = 0 < t_1 < \ldots < t_k = \pi$ of $[0,\pi]$ such that $\gamma([t_{i-1}, t_i])$ is contained in a chamber C_i, $1 \le i \le k$, and $\gamma(t_i)$ is contained in a panel P_i, $1 \le i \le k-1$. Let N_i^- be the interior normal to C_i at $\gamma(t_i)$ and N_i^+ be the interior normal to C_{i+1} at $\gamma(t_i)$, $1 \le i \le k-1$. We choose the natural identification of $T_{\gamma(t_i)}C_i$ with $T_{\gamma(t_i)}C_{i+1}$ which is the identity on the tangent space to the common panel P_i of C_i and C_{i+1} and sets $N_i^- = -N_i^+$. Then we have $\dot\gamma(t_i - 0) = \dot\gamma(t_i + 0)$.

The vector space $\mathcal{V}$ of all piecewise smooth vector fields V along γ with $V(0) = V(\pi) = 0$ and $V(t_i) \in T_{\gamma(t_i)}P_i$ corresponds to piecewise smooth variations $c_s : [0,\pi] \to X$ of $\gamma = c_0$, $-\varepsilon < s < \varepsilon$, leaving the endpoints fixed and such that $c_s([t_{i-1}, t_i]) \subset C_i$, $1 \le i \le k$. The second variation of the arc length of such variations gives the index from I on $\mathcal{V}$; the formula for I is

$$I(U,V) = \int_0^\pi \big(\langle U_\perp', V_\perp' \rangle - \langle R(U_\perp, \dot\gamma)\dot\gamma, V_\perp \rangle \big) dt - \sum_{i=1}^{k-1} \cos\theta_i \big(S_i^-(U,V) + S_i^+(U,V) \big).$$

Here $U_\perp$ and $V_\perp$ are the components of U and V perpendicular to γ, θ_i is the angle of $\dot\gamma(t_i)$ with N_i^+ and S_i^- respectively S_i^+ are the second fundamental forms of $P_i \subset C_i$ with respect to N_i^- and of $P_i \subset C_{i+1}$ with respect to N_i^+. By the assumption on the conjugate radius we have $I \ge 0$.

We say that a vector field W along γ is *parallel* if $W|[t_{i-1}, t_i]$ is parallel, $1 \le i \le k$. For any vector $w \in T_{\gamma(0)}C_1$ there is a unique parallel vector field W along γ with $W(0) = w$. Now for W parallel along γ with $\|W\| = 1$, choose $U \in \mathcal{V}$ with

$$U_\perp(t) = \sin t \, W(t), \quad 0 \le t \le \pi.$$

For $1 \le i \le k-1$, denote by θ_i the angle between $v = \dot\gamma(t_i)$ and $N = N_i^+$. Furthermore, denote by η_i the angle between $W(t_i)$ and the plane spanned by v and N. We do not define η_i if $v = N$ — the set of geodesics in U^*X which hit X^{n-1} perpendicularly at some time is of Liouville measure 0. The condition $U(t_i) \in T_{\gamma(t_i)}P$ implies

$$\|U(t_i)\|^2 = \sin^2 t \big(\sin^2 \eta_i + \frac{\cos^2 \eta_i}{\cos^2 \theta_i} \big).$$

Hence $I(U,U) \geq 0$ gives

$$
\begin{aligned}
(1) \qquad \int_0^\pi \cos^2 t\, dt \geq \int_0^\pi \sin^2 t\, K(\dot\gamma(t) \wedge W(t))\, dt \\
+ \sum_{i=1}^{k-1} \big(\kappa_i^-(U(t_i)) + \kappa_i^+(U(t_i))\big) \sin^2 t \,\big(\cos\theta_i \sin^2\eta_i + \frac{\cos^2\eta_i}{\cos\theta_i}\big),
\end{aligned}
$$

where for a non-zero tangent vector u to P_i at $\gamma(t_i)$,

$$\kappa_i^\pm(u) = \langle S_i^\pm(u/\|u\|,\, u/\|u\|), N_i^\pm\rangle$$

is the curvature of the direction $u/\|u\|$ with respect to $N_i^\pm$.

Now we integrate this inequality over all unit vectors w perpendicular to $\dot\gamma(0)$, and the resulting inequality we integrate over the space U^*X of all admissible geodesics. By (L3) the above integrations turn the LHS of inequality (1) into

$$(2) \qquad \frac{\pi}{2} \operatorname{vol}(S^{n-2}) \sum_C \operatorname{vol}(SC) = \frac{\pi}{2} \operatorname{vol}(S^{n-2}) \operatorname{vol}(S^{n-1}) \operatorname{vol}(X).$$

Since $\mathcal{L}$ is invariant under the geodesic flow, the first term on the RHS in (1) becomes (see [Be, p.146])

$$(3) \quad \frac{\pi}{2} \frac{\operatorname{vol}(S^{n-2})}{n-1} \sum_C \int_{SC} \operatorname{Ric}_C \, d\mathcal{L}_C = \frac{\pi}{2} \frac{1}{n(n-1)} \operatorname{vol}(S^{n-2}) \operatorname{vol}(S^{n-1}) \sum_C \int_C \operatorname{scal}_C \, d\mu_C.$$

To treat the second term on the RHS in (1), we recall that $U(t_i)$ is obtained from $U_\perp(t_i)$ by projection along $v = \dot\gamma(t_i)$ onto $T_{\gamma(t_i)}P$. Since $\kappa_i^\pm$ only depends on the direction, not on the norm, we can as well use the projection $u = p(v,w)$ of $w = W(t_i)$ along v to $T_{\gamma(t_i)}P_i$ to evaluate $\kappa_i^\pm$ or, respectively, $f(v,w) = p(v,w)/\|p(v,w)\|$. The first integration (over all unit vectors perpendicular to $\dot\gamma(0)$) then leads to

$$(4) \quad \sum_{i=1}^{k-1} \int_{w \perp \dot\gamma(t_i)} \big(\kappa_i^-(f(\dot\gamma(t_i),w)) + \kappa_i^+(f(\dot\gamma(t_i),w))\big) \sin^2 t_i \big(\cos\theta_i \sin^2\eta_i + \frac{\cos^2\eta_i}{\cos\theta_i}\big)\, dw,$$

where dw is the volume element on the $(n-2)$-sphere of unit vectors perpendicular to $v = \dot\gamma(t_i)$. Now we use Property (L4) of the Liouville measure above. We count the contributions of each triple $C|P|C'$ of chambers $C \neq C'$ adjacent to a panel P separately. We get that the integral of (4) over U^*X is equal to

$$
\begin{aligned}
\sum_{C|P|C'} \frac{1}{\nu(P)-1} \int_0^\pi \int_P \int_{H_x C} \int_{\substack{w \perp v \\ w \in S_x C}} \big(\kappa_P^{C'}(f(v,w)) + \kappa_P^C(f(v,w))\big) \cdot \\
\sin^2 t \big(\cos^2\theta \sin^2\eta + \cos^2\eta\big)\, dw\, d\lambda_x(v)\, d\mu_P(x)\, dt.
\end{aligned}
$$

It is easy to see that this expression is equal to

$$\frac{\pi}{2}\sum_{P|C}\int_P\int_{S_xC}\int_{\substack{w\perp v\\ w\in S_xC}} \kappa_P^C(f(v,w))(\cos^2\theta\sin^2\eta+\cos^2\eta)\,dwd\lambda_x(v)d\mu_P(x), \tag{5}$$

where θ is the angle between v and N_P^C and η the angle between w and the plane spanned by v and N_P^C. To evaluate (5), we first compute the push forward $f_*\nu$ of the measure $\nu=(\cos^2\theta\sin^2\eta+\cos^2\eta)\,dwd\lambda_x(v)$. We note that f is equivariant with respect to the group of rotations of T_xC fixing $N_P^C(x)$. Hence $f_*\nu$ is a constant multiple of the standard Lebesgue measure λ_x^P of the unit $(n-2)$-sphere S_xP in T_xP. Since for $v\in S_xC$ fixed,

$$\int_{\substack{w\perp v\\ w\in S_xC}} \sin^2\eta\,dw=\frac{n-2}{n-1}\operatorname{vol}(S^{n-2}) \quad\text{and}\quad \int_{\substack{w\perp v\\ w\in S_xC}} \cos^2\eta\,dw=\frac{1}{n-1}\operatorname{vol}(S^{n-2}),$$

the total mass of ν is

$$\begin{aligned}\operatorname{vol}(S^{n-2})&\int_{S_xC}\Big(\frac{n-2}{n-1}\cos^2\theta+\frac{1}{n-1}\Big)\,d\lambda_x(v)\\ &=\operatorname{vol}(S^{n-2})\operatorname{vol}(S^{n-1})\Big(\frac{n-2}{n(n-1)}+\frac{1}{n-1}\Big)=\frac{2}{n}\operatorname{vol}(S^{n-2})\operatorname{vol}(S^{n-1}).\end{aligned}$$

Therefore, $f_*\nu=\dfrac{2}{n}\operatorname{vol}(S^{n-1})\lambda_x^P$ and hence (5) is equal to

$$\frac{\pi}{2}\frac{2}{n}\operatorname{vol}(S^{n-1})\sum_{P|C}\int_P\int_{S_xP}\kappa_P^C(u)\,d\lambda_x^P(u)d\mu_P=\frac{\pi}{2}\frac{2}{n}\operatorname{vol}(S^{n-2})\operatorname{vol}(S^{n-1})\sum_{P|C}\int_P h_P^C d\mu_P.$$

Combined with (2) and (3), this gives the inequality in the theorem.

As for the equality discussion, recall that $I\geq 0$ for each admissible geodesic $\gamma:[0,\pi]\to X$. Hence for a vector field U along γ as in the proof, $I(U,U)=0$ implies that U belongs to the null space of I. This in turn implies that $U_\perp|[t_{i-1},t_i]$ is a Jacobi field, $1\leq i\leq k$, and that $S_i^-(U,U)+S_i^+(U,U)=0$, $1\leq i\leq k-1$. The equality discussion in the estimate of the theorem follows easily. □

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