The Global Torelli Theorem: classical, derived, twisted.

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These notes survey work on various aspects and generalizations of the Global Torelli Theorem for K3 surfaces done over the last ten years. The classical Global Torelli Theorem was proved a long time ago (see [**10, 35, 48, 53**]), but the interest in similar questions has been revived by the new approach to K3 surfaces suggested by mirror symmetry.

Kontsevich proposed to view mirror symmetry as an equivalence between the bounded derived category of coherent sheaves on a Calabi–Yau manifold and the derived Fukaya category of its mirror dual. For an algebraic geometer the bounded derived category of coherent sheaves on a variety is a familiar object, but to view it as an interesting invariant of the variety rather than a technical tool to deal with cohomology is rather surprising. Due to results of Mukai, Orlov, and Polishchuk, the bounded derived category of coherent sheaves on an abelian variety is completely understood, i.e. we know when two abelian varieties yield equivalent derived categories and what the group of autoequivalences looks like.

Independently of their importance in mirror symmetry, K3 surfaces form the next most simple class of Calabi–Yau manifolds and one would next like to study them from a derived category point of view, too. This has been started already by Mukai in [**39**], but the break-through has been obtained by Orlov in [**44**]. This was still not the end. For many reasons (mirror symmetry considerations, existence of non-fine moduli spaces, geometric interpretation of conformal field theories, etc.) one wants to incorporate B-fields or, closely related, Brauer classes in the picture. These notes will mostly concentrate on aspects that are related to generalizations of the Global Torelli Theorem in this direction. The following list of topics gives an idea of what will be discussed:

- Hitchin's generalized Calabi–Yau structures.
- The period of a twisted K3 surface.
- Brauer group and B-fields.
- Derived categories of twisted sheaves.

This survey contains essentially no proofs. I have tried to emphasize ideas and refer to the literature for details. Some of the results can very naturally be viewed in terms of moduli spaces of generalized K3 surfaces and the action of the mirror symmetry group, but I have decided to neglect these aspects almost

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completely. Although mirror symmetry has shaped our way of thinking about derived categories, the symplectic side of the theory will not be touched upon.

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1. The classical Torelli

A poetic explanation of the name 'K3' was given in André Weil's remarks on his only article on the subject (which was in fact a report for a U.S. Air Force grant). He writes:

... il s'agit des variétés kählériennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira, et de la belle montagne K2 au Cachemire.

The official definition goes as follows:

Definition 1.1. A connected compact complex surface X is a *K3 surface* if its canonical bundle is trivial, i.e. $\omega_X \cong \mathcal{O}$, and $H^1(X, \mathcal{O}) = 0$.

A trivializing section of ω_X, i.e. a non-trivial holomorphic two-form, will usually be denoted $\sigma \in H^0(X, \omega_X)$. It is unique up to scaling.

Examples 1.2. i) The *Fermat quartic* is a concrete example of a (projective) K3 surface. It is given as the hypersurface in $\mathbb{P}^3$ described by the equation $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$. The adjunction formula shows that the canonical bundle is trivial and standard vanishing results for the cohomology of line bundles on the projective space prove the required vanishing.

ii) *Kummer surfaces* form another distinguished type of K3 surfaces. If T is an abelian surface or a complex torus of dimension two, then the associated Kummer surface $\mathrm{Kum}(T)$ is the minimal resolution of the quotient $T/\pm$ by the standard involution $x \longmapsto -x$ (which has sixteen fixed-points). In particular, $\mathrm{Kum}(T)$ contains sixteen (-2)-curves, i.e. smooth irreducible rational curves C_i which by adjunction satisfy $(C_i.C_i) = -2$. Note that $\mathrm{Kum}(T)$ is projective if and only if T is projective.

In the following we briefly recall a few standard facts from the theory of K3 surfaces, for further details see [**1, 3**]:

1. Any K3 surface is diffeomorphic to the Fermat quartic. This is shown by a simple deformation argument. As it turns out, quartic K3 surfaces as well as Kummer surfaces are arbitrarily close to any K3 surface. In particular, K3 surfaces are simply-connected and the second cohomology $H^2(X, \mathbb{Z})$ is therefore torsion free.

2. The intersection pairing endows the middle cohomology $H^2(X, \mathbb{Z})$ with the structure of a unimodular lattice of rank 22 which is abstractly isomorphic to

$$-E_8 \oplus -E_8 \oplus U \oplus U \oplus U.$$

Here, E_8 is the unique unimodular, positive-definite, even lattice of rank eight and U is the hyperbolic plane generated by two isotropic vectors e_1, e_2 with $(e_1.e_2) = 1$.

3. Any K3 surface is Kähler. This is a deep theorem; a complete proof of it was given by Siu in [**53**]. The analogous statement for higher dimensional (simply-connected) holomorphic symplectic manifolds does not hold. Usually a Kähler form will be denoted ω and its Kähler class by $[\omega] \in H^2(X, \mathbb{R})$. Although it will not be explicitly mentioned anywhere in the text, Yau's result on the existence of Ricci-flat metrics plays a central rôle in the theory. One way to phrase it is to say that any Kähler class can be uniquely represented by a Kähler form ω with $\omega^2 = \lambda(\sigma\bar{\sigma})$ for some positive scalar factor λ.

4. The weight-two Hodge structure

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

on $H^2(X, \mathbb{Z})$ is of great importance in the study of K3 surfaces. By definition $H^{2,0}(X) \cong H^0(X, \omega_X) \cong \mathbb{C}\sigma$ is of dimension one. Moreover, $H^{0,2}(X)$ is complex conjugate to $H^{2,0}(X)$ and $H^{1,1}(X)$ is orthogonal (with respect to the intersection pairing) to $H^{2,0}(X) \oplus H^{0,2}(X)$. Thus, the weight-two Hodge structure on the intersection lattice $H^2(X, \mathbb{Z})$ of a K3 surface is determined by the line $\mathbb{C}\sigma \subset H^2(X, \mathbb{C})$.

The lattice $H^2(X, \mathbb{Z})$ together with the natural Hodge structure of weight two is called the *Hodge lattice* of X. A *Hodge isometry* between two lattices endowed with additional Hodge structures is by definition a lattice isomorphism that respects both structures, the quadratic form of the lattices and the Hodge structures.

In our geometric situation any integral $(1,1)$-class $\delta \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ with $(\delta.\delta) = -2$ induces a Hodge isometry of the Hodge lattice $H^2(X, \mathbb{Z})$ given as the reflection s_δ in the hyperplane $\delta^\perp$. More explicitly,

$$s_\delta : \gamma \longmapsto \gamma + (\gamma.\delta)\delta.$$

5. The Kähler cone is the open cone

$$\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R}) := H^2(X, \mathbb{R}) \cap H^{1,1}(X)$$

formed by all Kähler classes $[\omega]$. As $([\omega].[\omega]) = \int_X \omega^2 > 0$, it is contained in one connected component $\mathcal{C}_X$ of the positive cone of all classes $\gamma \in H^{1,1}(X, \mathbb{R})$ with $(\gamma.\gamma) > 0$. (Since the intersection pairing on $H^{1,1}(X, \mathbb{R})$ has signature $(1, 19)$, the only other connected component is $-\mathcal{C}_X$.)

Conversely, a class $\gamma \in \mathcal{C}_X$ is a Kähler class if and only if $(\gamma.[C]) > 0$ for all (-2)-curves $C \subset X$. For a higher-dimensional analogue see [**24**].

The hyperplanes $\delta^\perp$ orthogonal to integral (-2)-classes $\delta \in H^{1,1}(X)$ cut $\mathcal{C}_X$ in possibly infinitely many chambers. If $\gamma \in \mathcal{C}_X$ is a class in the interior of one such chamber then there exist (-2)-curves $C_1, \ldots, C_n \subset X$ such that $s_{[C_1]}(\ldots s_{[C_n]}(\gamma)\ldots)$ is a Kähler class. The Kähler cone $\mathcal{K}_X$ forms one chamber.

All these results are interwoven with the culmination of the theory:

Theorem 1.3. **(Classical Global Torelli)** *Two K3 surfaces X and X' are isomorphic if and only if there exists a Hodge isometry*

$$H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z}).$$

The theorem has a long and complicated history. It has been proved in various degrees of generality by: Piateckii-Shapiro and Shafarevich, Burns and Rapoport, Peters and Looijenga, and Siu.

Remark 1.4. i) Although the Global Torelli Theorem asserts that there is an isomorphism $f : X \cong X'$ whenever there exists a Hodge isometry $g : H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$, a given Hodge isometry g might not be realized as a Hodge isometry of the form f_* for any isomorphism f.

ii) One important example of a Hodge isometry g that cannot be realized as f_* is provided by the reflection $s_{[C]}$ associated to a (-2)-curve $C \subset X$. Indeed, if $[\omega]$ is a Kähler class, then $f_*[\omega]$ is also a Kähler class. Hence, $(f_*[\omega].[C]) = \int_C f_*\omega > 0$. On the other hand, $(s_{[C]}[\omega].[C]) = -([\omega].[C]) < 0$.

It can be shown that any given Hodge isometry $g : H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ can be composed with finitely many reflections $s_{[C_1]},\ldots,s_{[C_n]}$ associated to (-2)-curves $C_i \subset X$, such that either the Hodge isometry $g \circ \left(s_{[C_1]} \circ \ldots \circ s_{[C_n]}\right)$ or its negative maps a Kähler class on X to a Kähler class on X'. This new Hodge isometry can then be lifted to a unique isomorphism due to the following remark.

iii) The full Global Torelli Theorem proves the following assertion: If $g : H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ is a Hodge isometry that sends at least one Kähler class on X to a Kähler class on X', then g is induced by a unique(!) isomorphism $f : X \cong X'$.

As has been mentioned before, any two K3 surfaces are diffeomorphic. So, instead of viewing K3 surfaces as different complex surfaces, they might be viewed as complex structures on one differentiable manifold of (real) dimension four. (It is known that any complex structure on M does define the structure of a K3 surface on M, see [**16**].) To be more specific, we denote by M the differentiable manifold underlying the Fermat quartic in i), Example 1.2. (The complex structure of the Fermat quartic induces a natural orientation of M, which will we fix throughout.)

Note, that in order to view a K3 surface X as a complex structure on M, one first needs to fix a diffeomorphism of the differentiable manifold underlying X with our fixed manifold M. The choice of the diffeomorphism is by no means unique, as the diffeomorphism group of M is far from being trivial. Due to Borcea [**6**] and Donaldson [**15**] one knows that the cohomology representation of the diffeomorphism group yields a surjection

$$\mathrm{Diff}(M) \twoheadrightarrow \mathrm{O}_+(H^2(M,\mathbb{Z})). \tag{1.1}$$

Here $\mathrm{O}_+(H^2(M,\mathbb{Z}))$ is the group of all lattice isomorphisms preserving the orientation of the positive directions (see the explanations in Remark 2.6).

Suppose now that a K3 surface X is described by a complex structure I on M. We write $X = (M, I)$. The holomorphic two-form σ (unique up to scaling) can be viewed as a complex two-form $\sigma \in \mathcal{A}^2_{\mathbb{C}}(M)$ on M. This two-form satisfies:

$$\text{i) } \sigma \text{ is closed, i.e. } d\sigma = 0, \text{ ii) } \sigma \wedge \sigma \equiv 0, \text{ and iii) } \sigma \wedge \bar{\sigma} > 0.$$

The last condition means that $\sigma\wedge\bar\sigma$ is a positive multiple of the fixed orientation at every point of M. The two-form σ is also called the *holomorphic volume form* or the *Calabi–Yau form* of $X = (M, I)$.

An easy observation, presumably due to Andreotti, shows that the converse also holds. Indeed, any complex two-form $\sigma \in \mathcal{A}^2_{\mathbb{C}}(M)$ satisfying i)-iii) is induced by a complex structure in the above sense. More explicitly, one defines $T^{0,1}M$ as the kernel of $\sigma : T_{\mathbb{C}}M \longrightarrow T^*_{\mathbb{C}}M$ and $T^{1,0}M$ as its complex conjugate. Conditions ii) and iii) ensure that this results in a decomposition of $T_{\mathbb{C}}M$ which defines an almost complex structure. This almost complex structure is integrable due to i).

The *period* of a K3 surface $X = (M, I)$ is the point $[\sigma] \in \mathbb{P}(H^2(M,\mathbb{C}))$, where σ is the holomorphic two-form that determines I. The Local Torelli Theorem, stated in [**59**] and most likely due to Andreotti, roughly says that any small change of the complex structure I on M up to diffeomorphisms of M is reflected by a change of the period, i.e. of the line $\mathbb{C}[\sigma] \subset H^2(M,\mathbb{C})$. This has led Weil to conjecture a global version of this result and, in fact, he gives two versions of it (see [**59**]):

Conjecture 1.5. i) Suppose $\mathbb{C}[\sigma] = \mathbb{C}[\sigma'] \subset H^2(M,\mathbb{C})$. Then there exists a diffeomorphism $f \in \mathrm{Diff}(M)$ isotopic to the identity, i.e. f is contained in the identity component $\mathrm{Diff}_o(M)$ of the full diffeomorphism group $\mathrm{Diff}(M)$, such that $\sigma = f^*\sigma'$. [1]

ii) If $[\sigma] = g([\sigma'])$ for a lattice isomorphism $g \in \mathrm{O}(H^2(M,\mathbb{Z}))$, then there exists a diffeomorphism $f \in \mathrm{Diff}(M)$ such that $\sigma = f^*\sigma'$.

Remark 1.6. The second version has been established by the classical Global Torelli Theorem as stated in the more algebro-geometric form in Theorem 1.3, but i) is still open. So, the Global Torelli Theorem for K3 surfaces has not been fully proven yet! In order to deduce i) from ii) one would have to prove that $\mathrm{Diff}_o(M)$ coincides with the kernel $\mathrm{Diff}_*(M)$ of $\mathrm{Diff}(M) \longrightarrow \mathrm{O}(H^2(M,\mathbb{Z}))$.

Remark 1.7. We conclude this section with the surjectivity of the period map. The Global Torelli Theorem is equivalent to the assertion that the period map

$$\mathcal{P} : \{I\}/\mathrm{Diff}_*(M) \longrightarrow Q, \quad X = (M,I) \longmapsto [\sigma]$$

is generically injective. Here, $Q \subset \mathbb{P}(H^2(M,\mathbb{C}))$ is the period domain

$$Q = \{x \in \mathbb{P}(H^2(M,\mathbb{C})) \mid (x.x) = 0,\ (x.\bar{x}) > 0\}.$$

Using the Global Torelli Theorem it has been proved (see [**1, 34, 52, 57**]) that any x in the period domain is the period of some K3 surface $X = (M,I)$. In other words, the period map $\mathcal{P}$ is surjective. Although a Global Torelli Theorem does not hold in higher dimensions, the surjectivity of the period map could nevertheless be established in broader generality, see [**23**].

2. Generalized K3 surfaces

In 2002 Hitchin [**20**] introduced generalized complex and generalized Calabi–Yau structures. Generalized Calabi–Yau structures on K3 surfaces were in detail investigated in [**25**].

If we think of a K3 surface as given by a Calabi–Yau form σ on the differentiable manifold M, then the following definition is very natural

Definition 2.1. A *generalized Calabi–Yau form* on M is an even complex form $\varphi = \varphi_0 + \varphi_2 + \varphi_4 \in \mathcal{A}^{2*}_{\mathbb{C}}(M) = \mathcal{A}^0_{\mathbb{C}}(M) \oplus \mathcal{A}^2_{\mathbb{C}}(M) \oplus \mathcal{A}^4_{\mathbb{C}}(M)$ satisfying:

$$\text{i) } d\varphi = 0,\ \text{ ii) } \langle\varphi,\varphi\rangle := \varphi_2 \wedge \varphi_2 - 2\varphi_0\varphi_4 \equiv 0,\ \text{ and iii) } \langle\varphi,\bar{\varphi}\rangle > 0.$$

The inequality in iii) means that the real four-form $\langle\varphi,\bar{\varphi}\rangle$ is in any point of M a positive multiple of the fixed volume form.

Example 2.2. The example that will interest us most is provided by B-field shifts of σ. For a given ordinary Calabi–Yau form σ and any real two-form B the form $\exp(B)\sigma = \sigma + B \wedge \sigma$ is a generalized Calabi–Yau form. It is different from σ only if $B^{0,2} \neq 0$.

[1] The expert reader of course knows that this has no chance to be true as stated. One either has to restrict to general complex structures σ and σ' or to add the assumption that at least one Kähler class with respect to σ is also a Kähler class with respect to σ'.

The quadratic form $\langle\ ,\ \rangle$ defined in ii) carries over to cohomology and yields the Mukai lattice:

Definition 2.3. Let X be a K3 surface. Then the *Mukai lattice* $\widetilde{H}(X,\mathbb{Z})$ of X is the integral cohomology $H^*(X,\mathbb{Z})$ endowed with the quadratic form

$$\langle\varphi,\psi\rangle := -\varphi_0\wedge\psi_4 + \varphi_2\wedge\psi_2 - \varphi_4\wedge\psi_0.$$

Of course, the complex structure did not matter for the definition of the Mukai lattice, so if the K3 surface X is viewed as a complex structure on M, then $\widetilde{H}(X,\mathbb{Z}) \cong \widetilde{H}(M,\mathbb{Z})$. Later we shall often mean by $\widetilde{H}(X,\mathbb{Z})$ the Mukai lattice of X together with its natural weight-two Hodge structure, which will be defined shortly.

Since the odd cohomology of a K3 surface is trivial, one has

$$\widetilde{H}(M,\mathbb{Z}) = H^2(M,\mathbb{Z}) \oplus - \left(H^0(M,\mathbb{Z})\oplus H^4(M,\mathbb{Z})\right).$$

So, as an abstract lattice it can be described by (use $-U\cong U$):

$$\widetilde{H}(M,\mathbb{Z}) \cong 4U \oplus 2(-E_8).$$

In analogy to the classical situation, the *period of a generalized Calabi–Yau structure* φ on M is its cohomology class $[\varphi]$ or rather the line spanned by it viewed as a point in $\mathbb{P}(\widetilde{H}(M,\mathbb{C}))$.

Moreover, the period of φ can be used to introduce a Hodge structure of weight two on $\widetilde{H}(M,\mathbb{Z})$, which shall be denoted $\widetilde{H}(M,\varphi,\mathbb{Z})$. One defines

$$\widetilde{H}^{2,0}(M,\varphi) := \mathbb{C}[\varphi] \subset \widetilde{H}(M,\mathbb{C})$$

and then $\widetilde{H}^{0,2}(M,\varphi)$ is necessarily spanned by $[\bar\varphi]$. By definition $\widetilde{H}^{1,1}(M,\varphi)$ is the orthogonal (with respect to the Mukai pairing) complement of $\widetilde{H}^{2,0}(M,\varphi)\oplus\widetilde{H}^{0,2}(M,\varphi)$.

Examples 2.4. i) In the case of a classical Calabi–Yau form $\varphi=\sigma$ defining a K3 surface X one recovers Mukai's original definition of the weight two Hodge structure $\widetilde{H}(X,\mathbb{Z})$ on the Mukai lattice, whose $(2,0)$-part is spanned by σ and whose $(1,1)$ part is $H^{1,1}(X)\oplus(H^0\oplus H^4)(X)$.

ii) For the B-field twist of an ordinary Calabi–Yau structure we introduce the Hodge structure

$$\widetilde{H}(X,B,\mathbb{Z}) := \widetilde{H}(X,\varphi:=\exp(B)\sigma,\mathbb{Z}).$$

Remark 2.5. Similar to the classical Global Torelli Theorem and eventually by reducing to it, one proves a 'generalized' Global Torelli Thoerem (see [**25**]). This can be phrased in terms of the period map as follows. Consider

$$\widetilde{\mathcal{P}} : \{\mathbb{C}\varphi \mid \varphi = \text{generalized CY form}\}/\mathrm{Diff}_*(M) \longrightarrow \widetilde{Q},$$

where $\widetilde{Q}\subset\mathbb{P}(\widetilde{H}(M,\mathbb{C}))$ is the period domain $\{x \mid \langle x,x\rangle=0,\ \langle x,\bar x\rangle>0\}$ and $\widetilde{\mathcal{P}}(\mathbb{C}\varphi)=\mathbb{C}[\varphi]$. Then $\widetilde{\mathcal{P}}$ restricted to the subset of those φ satisfying a generalized Kähler condition (see [**25**, Sect.3]) is generically injective. In analogy to Remark 1.7 one proves also that $\widetilde{\mathcal{P}}$ is surjective.

The most fascinating aspect of Hitchin's notion of generalized Calabi–Yau structures is the occurence of classical Calabi–Yau forms σ as well as of symplectic generalized Calabi–Yau forms $\exp(i\omega)$ (with ω a symplectic form) in the same moduli space. This allows one to pass from the symplectic to the complex world in a continuous fashion.

Note that the analogue of Siu's result has not been proved. For the time being we do not know whether any generalized Calabi–Yau structure is Kähler.

REMARK 2.6. The Mukai lattice $\widetilde{\Gamma} := \widetilde{H}(M, \mathbb{Z})$ has four positive directions and twenty negative ones. If $\widetilde{\Gamma} \otimes \mathbb{R} \cong V_1 \oplus W_1 \cong V_2 \oplus W_2$ are two orthogonal decompositions of the real vector space $\widetilde{\Gamma} \otimes \mathbb{R}$ such that V_1, V_2 are positive-definite and W_1, W_2 are negative-definite. Then orthogonal projection yields an isomorphism $V_1 \cong V_2$. This allows us to compare orientations of the four-dimensional real vector spaces V_1 and V_2. By definition, an *orientation of the positive directions* (or simply an orientation) of the Mukai lattice is given by an orientation of a positive-definite four-space $V \subset \widetilde{\Gamma} \otimes \mathbb{R}$ and two such orientations given by orientations of $V_1, V_2 \subset \widetilde{\Gamma} \otimes \mathbb{R}$ are equal if they correspond to each other under $V_1 \cong V_2$.

If X is a K3 surface, then $\widetilde{H}(X, \mathbb{Z})$ is naturally endowed with an orientation (of the four positive directions). Indeed, if $[\omega]$ is a Kähler class, then $\langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1 - \omega^2/2, \omega \rangle$ is a positive four-dimensional subspace of $\widetilde{H}(X, \mathbb{R})$ and the chosen basis determines an orientation. Any other choice of $[\omega]$ or σ yields the same orientation in the above sense. Similarly, $\widetilde{H}(X, B, \mathbb{Z})$ can be endowed with the orientation obtained as the image of the previous one under $\exp(B)$.

3. Twisted K3 surfaces

Although most of the things we shall explain or recall hold true for a fairly general class of complex manifolds, we will restrict to K3 surfaces. So, as before X will denote a K3 surface.

An *Azumaya algebra* on X is an associative, $\mathcal{O}$-algebra $\mathcal{A}$ such that locally (in the analytic topology) it is isomorphic to a matrix algebra $\mathrm{M}_r(\mathcal{O})$ for some r. In particular, $\mathcal{A}$ is locally free of constant rank r^2.

Two Azumaya algebras are isomorphic if they are isomorphic as $\mathcal{O}$-algebras. By the Skolem-Noether theorem $\mathrm{Aut}(\mathrm{M}_r(\mathbb{C})) \cong \mathrm{PGL}_r(\mathbb{C})$. Hence, isomorphism classes of Azumaya algebras of rank r^2 are parametrized by the set $H^1(X, \mathrm{PGL}_r)$.

To any vector bundle E of rank r one associates the 'trivial' Azumaya algebra $\mathcal{A} = \mathcal{E}nd(E)$ of rank r^2. This gives rise to the following notion of equivalence between Azumaya algebras: Two Azumaya algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are called *equivalent* if there exist vector bundles E_1 and E_2 such that $\mathcal{A}_1 \otimes \mathcal{E}nd(E_1)$ and $\mathcal{A}_2 \otimes \mathcal{E}nd(E_2)$ are isomorphic Azumaya algebras.

DEFINITION 3.1. The *Brauer group* $\mathrm{Br}(X)$ is the set of isomorphism class of Azumaya algebras modulo the above equivalence relation.

The group structure of $\mathrm{Br}(X)$ is given by the tensor product of Azumaya algebras.

A cohomological approach to Brauer groups is provided by the long exact cohomology sequence of

$$1 \longrightarrow \mathcal{O}^* \longrightarrow \mathrm{GL}_r \longrightarrow \mathrm{PGL}_r \longrightarrow 1,$$

which yields natural maps $\delta_r : H^1(X, \mathrm{PGL}_r) \longrightarrow H^2(X, \mathcal{O}^*)$ and eventually an injection

$$\delta : \mathrm{Br}(X) \hookrightarrow H^2(X, \mathcal{O}_X^*).$$

Using the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mu_r \arrow[r] \arrow[d, hook] & \mathrm{SL}_r \arrow[r] \arrow[d, hook] & \mathrm{PGL}_r \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & \mathcal{O}^* \arrow[r] & \mathrm{GL}_r \arrow[r] & \mathrm{PGL}_r \arrow[r] & 1 \end{tikzcd}$$

one finds that the image of δ_r is contained in the r-torsion part of $H^2(X,\mathcal{O}^*)$. Hence $\mathrm{Br}(X) \subset H^2(X,\mathcal{O}^*)$ is contained in the subgroup $H^2(X,\mathcal{O}^*)_{\mathrm{tor}}$ of torsion classes.

THEOREM 3.2. *Let X be a K3 surface. Then* $\mathrm{Br}(X) = H^2(X,\mathcal{O}^*)_{\mathrm{tor}}$.

This result goes back to Grothendieck for projective K3 surfaces (see [**18**]) and was proved in [**28**] for arbitrary K3 surfaces.

REMARK 3.3. If X is projective or, more generally, any scheme, one defines analogously the algebraic Brauer group and compares it with the étale cohomology $H^2(X,\mathbb{G}_m)$, which is sometimes called the cohomological Brauer group $\mathrm{Br}'(X)$. The latter contains only torsion classes and Grothendieck asked whether the natural injection $\mathrm{Br}(X) \longrightarrow H^2(X,\mathbb{G}_m)$ is bijective. An affirmative answer to this question has recently been published by de Jong in [**13**] for the case of quasi-projective schemes.

DEFINITION 3.4. A *twisted K3 surface* (X,α) consists of a K3 surface X together with a class $\alpha \in H^2(X,\mathcal{O}^*)$. We say that $(X,\alpha) \cong (X',\alpha')$ if there exists an isomorphism $f : X \cong X'$ with $f^*\alpha' = \alpha$.

Geometrically passing from ordinary K3 surfaces to twisted K3 surfaces means that we pass from ordinary (coherent, quasi-coherent) sheaves to twisted (coherent, quasi-coherent) sheaves. This notion in its various incarnations will be explained next.

1. Twisted sheaves. Suppose we represent a class $\alpha \in H^2(X,\mathcal{O}^*)$ by a Čech 2-cocycle $\{\alpha_{ijk} \in \Gamma(U_{ijk},\mathcal{O}_X^*)\}$ with respect to an open analytic covering $X = \bigcup U_i$. (Here and in the sequel we write U_{ij} and U_{ijk} for the intersections $U_i \cap U_j$ and $U_i \cap U_j \cap U_k$, respectively.)

An $\{\alpha_{ijk}\}$*-twisted (coherent) sheaf* E consists of pairs $(\{E_i\},\{\varphi_{ij}\})$ such that the E_i are (coherent) sheaves on U_i and $\varphi_{ij} : E_i|_{U_{ij}} \to E_j|_{U_{ij}}$ are isomorphisms satisfying the following conditions:

$$\text{i) } \varphi_{ii} = \mathrm{id}, \text{ ii) } \varphi_{ji} = \varphi_{ij}^{-1}, \text{ and iii) } \varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}.$$

Morphisms between $\{\alpha_{ijk}\}$-twisted sheaves are defined in the obvious way and one verifies that kernel and cokernel exist. Thus, we can speak about the abelian category of $\{\alpha_{ijk}\}$-twisted sheaves which we shall denote

$$\mathbf{Coh}(X,\{\alpha_{ijk}\}).$$

If $\{\alpha'_{ijk}\}$ is another Čech 2-cocycle based on the same open covering and representing the same class α, then there exist $\{\lambda_{ij} \in \mathcal{O}^*(U_{ij})\}$ such that $\alpha'_{ijk}\alpha_{ijk}^{-1} = \lambda_{ij} \cdot \lambda_{jk} \cdot \lambda_{ki}$. The $\{\alpha_{ijk}\}$-twisted sheaves are in bijection with the $\{\alpha'_{ijk}\}$-twisted sheaves via

$$(\{E_i\},\{\varphi_{ij}\}) \longmapsto (\{E_i\},\{\varphi_{ij} \cdot \lambda_{ij}\}).$$

In particular, this yields an equivalence of abelian categories

$$\mathbf{Coh}(X, \{\alpha_{ijk}\}) \cong \mathbf{Coh}(X, \{\alpha'_{ijk}\}), \tag{3.1}$$

which is non-canonical as it depends on the choice of $\{\lambda_{ij}\}$.

E.g. if $\{\lambda_{ij}\}$ satisfies $\lambda_{ij} \cdot \lambda_{jk} \cdot \lambda_{ki} = 1$ and thus defining a line bundle $\mathcal{L}$ on X, then the induced equivalence $\mathbf{Coh}(X, \{\alpha_{ijk}\}) \cong \mathbf{Coh}(X, \{\alpha_{ijk}\})$ in (3.1) is given by the tensor product

$$E \longmapsto E \otimes \mathcal{L}.$$

Similarly, if one passes to a finer open covering, then twisted sheaves are in a natural bijection. This allows us to speak of α-twisted sheaves without mentioning an explicit Čech representative of the cohomology class $\alpha \in H^2(X, \mathcal{O}^*)$. More precisely, all the abelian categories of twisted sheaves with respect to some Čech cocycle representing a fixed class α are equivalent, though not naturally. By abuse of notation, we call the equivalence class of these categories $\mathbf{Coh}(X, \alpha)$.

2. $\mathcal{A}$-modules. Fix $\alpha \in \mathrm{Br}(X)$ and pick a locally free coherent α-twisted sheaf G (which always exists, see below). Then $\mathcal{A}_G := G \otimes G^*$ is an Azumaya algebra whose Brauer class is α. If E is any α-twisted sheaf (with respect to the same choice of the cycle representing α), then $E \otimes G^*$ is an untwisted sheaf. Moreover, $E \otimes G^*$ has the structure of an $\mathcal{A}_G$-module.

The map $E \longmapsto E \otimes G^*$ then defines a bijective correspondence between α-twisted sheaves and $\mathcal{A}_G$-modules.

If we let $\mathbf{Coh}(X, \mathcal{A}_G)$ be the abelian category of coherent $\mathcal{A}$-modules, then this map yields an equivalence

$$\mathbf{Coh}(X, \mathcal{A}_G) \cong \mathbf{Coh}(X, \alpha).$$

3. Sheaves on gerbes. To an Azumaya algebra $\mathcal{A}$ as well as to a cocycle $\{\alpha_{ijk}\}$ representing a class $\alpha \in H^2(X, \mathcal{O}^*)$ one can associate a $\mathbb{G}_m$-gerbe over X, which are called $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}_{\{\alpha_{ijk}\}}$, respectively.

The gerbe $\mathcal{M}_{\mathcal{A}}$ associates to $T \longrightarrow X$ the category $\mathcal{M}_{\mathcal{A}}(T)$ whose objects are pairs (E, η) with E a locally free coherent sheaf on T and $\eta : \mathcal{E}nd(E) \cong \mathcal{A}_T$ an isomorphism of $\mathcal{O}_T$-algebras (see [**36**]). A morphism $(E, \eta) \longrightarrow (E', \eta')$ is given by an isomorphism $E \cong E'$ that commutes with the $\mathcal{A}_T$-actions induced by η and η'. It is easy to see that the group of automorphisms of an object (E, η) is $\mathcal{O}^*(T)$.

The gerbe $\mathcal{M}_{\{\alpha_{ijk}\}}$ associates to $T \longrightarrow X$ the category $\mathcal{M}_{\{\alpha_{ijk}\}}(T)$ whose objects are collections $\{\mathcal{L}_i, \varphi_{ij}\}$ with $\mathcal{L}_i \in \mathrm{Pic}(T_{U_i})$ and $\varphi_{ij} : \mathcal{L}_i|_{T_{U_{ij}}} \cong \mathcal{L}_j|_{T_{U_{ij}}}$ satisfying $\varphi_{ij} \cdot \varphi_{jk} \cdot \varphi_{ki} = \alpha_{ijk}$ (see [**33**]). A morphism $\{\mathcal{L}_i, \varphi_{ij}\} \longrightarrow \{\mathcal{L}'_i, \varphi'_{ij}\}$ is given by isomorphisms $\mathcal{L}_i \cong \mathcal{L}'_i$ compatible with φ_{ij} and φ'_{ij}.

Any sheaf $\mathcal{F}$ on a $\mathbb{G}_m$-gerbe $\mathcal{M} \longrightarrow X$ comes with a natural $\mathbb{G}_m$-action and thus decomposes as $\mathcal{F} = \bigoplus \mathcal{F}^m$, where the $\mathbb{G}_m$-action on $\mathcal{F}^m$ is given by the character $\lambda \longmapsto \lambda^m$. If $\mathcal{F} = \mathcal{F}^m$, then $\mathcal{F}$ is called of weight m.

There are natural bijections (of sets of isomorphism classes)

$$\{\mathcal{A} - \text{modules}\} \longleftrightarrow \{\text{sheaves on } \mathcal{M}_{\mathcal{A}} \text{ of weight one}\}$$

$$\{\{\alpha_{ijk}\} - \text{twisted sheaves}\} \longleftrightarrow \{\text{sheaves on } \mathcal{M}_{\{\alpha_{ijk}\}} \text{ of weight one}\},$$

which hold for (quasi)-coherent sheaves. This yields equivalences

$$\mathbf{Coh}(X,\mathcal{A})\cong\mathbf{Coh}(\mathcal{M}_{\mathcal{A}})_1 \text{ and } \mathbf{Coh}(X,\{\alpha_{ijk}\})\cong\mathbf{Coh}(\mathcal{M}_{\{\alpha_{ijk}\}})_1.$$

Here, the abelian categories on the gerbes are the categories of coherent sheaves of weight one. One can also show that $\mathbf{Coh}(X,\{\alpha^\ell_{ijk}\})\cong\mathbf{Coh}(\mathcal{M}_{\{\alpha_{ijk}\}})_\ell$. [**13, 14, 33**] for more details.

Isomorphism classes of $\mathbb{G}_m$-gerbes are in bijection with classes in $H^2(X,\mathcal{O}_X^*)$ (see [**36**]) and the isomorphism class of $\mathcal{M}_{\mathcal{A}}$ corresponds indeed to $\delta[\mathcal{A}]$. Similarly, the above construction of $\mathcal{M}_{\{\alpha_{ijk}\}}$ ensures that its isomorphism class corresponds to the class α.

In order to construct a concrete gerbe in the isomorphism class determined by α we had to choose a specific Azumaya algebra $\mathcal{A}$ or a cocycle $\{\alpha_{ijk}\}$ representing α. In the first case we have to assume $\alpha\in\mathrm{Br}(X)$.

One can construct directly an isomorphism $\mathcal{M}_{\mathcal{A}}\cong\mathcal{M}_{\{\alpha_{ijk}\}}$ for an appropriate cocycle α_{ijk} representing $\alpha=\delta[\mathcal{A}]$. This goes as follows: Choose a covering $X=\bigcup U_i$, isomorphisms $\eta_i:\mathcal{E}nd(E_i)\cong\mathcal{A}_{U_i}$, where the E_i are (locally) free, and isomorphisms $\xi_{ij}:E_i|_{U_{ij}}\cong E_j|_{U_{ij}}$ compatible with the η_i. Then $\xi_{ij}\cdot\xi_{jk}\cdot\xi_{ki}$ is given by multiplication with a scalar function α_{ijk} whose cohomology class represents $\delta[\mathcal{A}]$. If $(E,\eta)\in\mathcal{M}_{\mathcal{A}}(T)$, then $E_i\cong E_{U_i}\otimes\mathcal{L}_i$ for certain line bundles $\mathcal{L}_i$ on T_{U_i} and one may choose isomorphisms φ_{ij} between them (over U_{ij}) such that ξ_{ij} is $\mathrm{id}\otimes\varphi_{ij}$. Associating to (E,η) the collection $\{\mathcal{L}_i,\varphi_{ij}\}$ defines an isomorphism of gerbes $\mathcal{M}_{\mathcal{A}}\longrightarrow\mathcal{M}_{\{\alpha_{ijk}\}}$.

4. Sheaves on the Brauer–Severi variety. The following has been explained in [**60**]. Suppose $E=(\{E_i\},\{\varphi_{ij}\})$ is a locally free $\{\alpha_{ijk}\}$-twisted sheaf. The projective bundles $\mathbb{P}(E_i)\longrightarrow U_i$ glue to the Brauer–Severi variety $\pi:\mathbb{P}(E)\longrightarrow X$ and the relative tautological line bundles $\mathcal{O}_{\mathbb{P}(E_i)}(1)$ glue to a $\{\pi^*\alpha_{ijk}^{-1}\}$-twisted line bundle $\mathcal{O}_\pi(1)$ on $\mathbb{P}(E)$. If $F=(\{F_i\},\{\psi_{ij}\})$ is any $\{\alpha_{ijk}\}$-twisted sheaf, then $\pi^*F\otimes\mathcal{O}_\pi(1)$ is a true sheaf in a natural way. This yields an equivalence of $\mathbf{Coh}(X,\{\alpha_{ijk}\})$ with the full subcategory $\mathbf{Coh}(\mathbb{P}(E)/X)$ of $\mathbf{Coh}(\mathbb{P}(E))$ of all coherent sheaves F' on $\mathbb{P}(E)$ for which the natural morphism $\pi^*\pi_*(F'\otimes(\pi^*E\otimes\mathcal{O}_\pi(1))^*)\longrightarrow F'\otimes(\pi^*E\otimes\mathcal{O}_\pi(1))^*$ is an isomorphism:

$$\mathbf{Coh}(X,\{\alpha_{ijk}\})\cong\mathbf{Coh}(\mathbb{P}(E)/X).$$

Note that the bundle $\pi^*E\otimes\mathcal{O}_\pi(1)$ can be described as the unique non-trivial extension $0\longrightarrow\mathcal{O}_{\mathbb{P}(E)}\longrightarrow\pi^*E\otimes\mathcal{O}_\pi(1)\longrightarrow\mathcal{T}_{\mathbb{P}(E)}\longrightarrow0$ and thus depends only on the Brauer–Severi variety $\pi:\mathbb{P}(E)\longrightarrow X$.

Isomorphism classes of $\mathbb{P}^r$-bundles are also parametrized by $H^1(X,\mathrm{PGL}_r)$. Locally a $\mathbb{P}^r$-bundle is of the form $\mathbb{P}(E_i)$ for some locally free sheaf E_i. They glue to a $\delta(\mathbb{P})$-twisted sheaf.

4. Twisted Chern characters

In order to study twisted K3 surfaces and twisted sheaves by cohomological methods, one needs a good cohomology theory and the notion of twisted Chern characters.

Let us begin by introducing the weight two Hodge structure $\widetilde{H}(X,\alpha,\mathbb{Z})$ of a twisted K3 surface (X,α). The exponential sequence shows that any element $\alpha\in H^2(X,\mathcal{O}^*)$ can be written as $\exp(B^{0,2})$ for some B-field $B\in H^2(X,\mathbb{R})$. If α is a torsion class, we may choose B to be rational. The exponential sequence also shows that a given B may be changed by integral B-fields $B_0 \in H^2(X,\mathbb{Z})$ without changing the Brauer class α. Once a B-field lift $B \in H^2(X,\mathbb{R})$ of a class α is chosen, one considers the generalized Calabi–Yau form $\exp(B)\sigma$ and its natural weight two Hodge structure $\widetilde{H}(X,B,\mathbb{Z})$. For $B_0 \in H^2(X,\mathbb{Z})$ multiplication with the integral class $\exp(B_0)$ defines a Hodge isometry

$$\widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X,B+B_0,\mathbb{Z}).$$

This allows us to introduce

$$\widetilde{H}(X,\alpha,\mathbb{Z})$$

as the Hodge isometry type of $\widetilde{H}(X,B,\mathbb{Z})$ with B an arbitrary B-field lift of α. We emphasize that this is an abstract Hodge structure which can only be realized if a concrete B-field lift of α has been chosen.

There are various approaches towards twisted Chern characters, e.g. [**19, 29, 60**]. The one introduced in [**29**] seems not very canonical, as it depends on the additional choice of a B-field. It is, however, the one that works best in the context of twisted K3 surfaces, as it allows us to work with integral(!) Hodge structures.

Let $B \in H^2(X,\mathbb{Q})$ and $\alpha \in H^2(X,\mathcal{O}^*)$ the induced Brauer class, i.e. the image of $B^{0,2} \in H^2(X,\mathcal{O})$ under the exponential $H^2(X,\mathcal{O}) \longrightarrow H^2(X,\mathcal{O}^*)$. Equivalently, α is the image of $B \in H^2(X,\mathbb{Q})$ under the composition of the exponential map $H^2(X,\mathbb{Q}) \longrightarrow H^2(X,\mathbb{C}^*)$ and the natural inclusion $\mathbb{C}^* \subset \mathcal{O}^*$. In addition, choose a Čech cocycle $B_{ijk} \in \Gamma(U_{ijk},\mathbb{Q})$ representing B and let $\alpha_{ijk} := \exp(B_{ijk})$ be the induced Čech cocycle representing α.

Once this $\{\alpha_{ijk}\}$ is fixed, we can speak of $\{\alpha_{ijk}\}$-twisted sheaves and we aim at defining a twisted Chern character for those. Before we can do this in practice we need to make yet another choice. Viewing the B_{ijk} as differentiable functions allows us to write them as $B_{ijk} = -a_{ij} + a_{ik} - a_{jk}$ for certain differentiable functions $a_{ij} : U_{ij} \longrightarrow \mathbb{R}$. (We use $H^2(X,\mathcal{C}^\infty) = 0$ and might have to refine the covering.)

Let now $E = (\{E_i\},\{\varphi_{ij}\})$ be an $\{\alpha_{ijk}\}$-twisted sheaf. Then one defines

$$E_B := (\{E_i\},\{\varphi'_{ij} := \varphi_{ij} \cdot \exp(a_{ij})\}) \text{ and } \mathrm{ch}^B(E) := \mathrm{ch}(E_B).$$

Note that E_B describes an untwisted sheaf, for the φ'_{ij} satisfy the usual cocycle condition. First observe that this definition is independent of the choice of $\{a_{ij}\}$. Indeed, passing to $a_{ij} + a'_{ij}$ with $-a'_{ij} + a'_{ik} - a'_{jk} = 0$ would change the bundle E_B by a twist with the line bundle L corresponding to the cocycle $\{\exp(a'_{ij})\}$. Since $\mathrm{c}_1(L) = \{-a'_{ij} + a'_{ik} - a'_{jk}\} = 0$, this has no effect on $\mathrm{ch}(E_B)$.

On the other hand, although not reflected by our notation, the choice of $\{B_{ijk}\}$ and the resulting $\{\alpha_{ijk}\}$ is necessary in order to be able to define $\mathrm{ch}^B(E)$, as we could otherwise not speak about $\{\alpha_{ijk}\}$-twisted sheaf. However, as was explained earlier, there is a non-canonical bijection between $\{\alpha_{ijk}\}$-twisted sheaves and $\{\alpha'_{ijk}\}$-twisted sheaves for two Čech cocycles representing α and and our Chern character ch^B is compatible with it.

Indeed, if $B'_{ijk} := B_{ijk} + (b_{ij} - b_{ik} + b_{jk})$, then $\{\alpha_{ijk}\}$ and $\{\alpha'_{ijk}\}$ differ by the boundary of $\{\lambda_{ij} := \exp(b_{ij})\}$ and we may send $E = (\{E_i\},\{\varphi_{ij}\})$ to the $\{\alpha'_{ijk}\}$-twisted sheaf $E' = (\{E_i\},\{\varphi_{ij} \cdot \lambda_{ij}\})$. (The given modification of $\{B_{ijk}\}$ by the boundary of $\{b_{ij}\}$ induces a canonical bijection between $\{\alpha_{ijk}\}$-twisted sheaves and $\{\alpha'_{ijk}\}$-twisted sheaves, which otherwise does not exist.) Clearly, E_B and E'_B

are defined by the same cocycle. Thus, ch^B does not depend on the Čech cocycle representing B.

The following properties of the twisted Chern character ch^B have been observed in [**29**]:

i) $\mathrm{ch}^B(E_1 \oplus E_2) = \mathrm{ch}^B(E_1) + \mathrm{ch}^B(E_2)$.

ii) If $B = \mathrm{c}_1(L) \in H^2(X, \mathbb{Z})$, then $\mathrm{ch}^B(E) = \exp(\mathrm{c}_1(L)) \cdot \mathrm{ch}(E)$.

iii) $\mathrm{ch}^{B_1}(E_1) \cdot \mathrm{ch}^{B_2}(E_2) = \mathrm{ch}^{B_1+B_2}(E_1 \otimes E_2)$.

Remark 4.1. i) In the note [**19**] Heinloth explains the relation between the twisted Chern character ch^B and the usual Chern character on the gerbe. He first proves that

$$H^*(\mathcal{M}_\alpha, \mathbb{Q}) \cong H^*(X, \mathbb{Q})[z], \tag{4.1}$$

where $z = \mathrm{c}_1(E)$ with E some vector bundle of weight one on $\mathcal{M}_\alpha$. In particular, the isomorphism in (4.1) depends on this choice. He furthemore explains that the choice of a differentiable line bundle L of weight one on $\mathcal{M}_\alpha$ allows to define $\mathrm{ch}_L(E)$ as $\mathrm{ch}(E \otimes L^*)$, which makes sense as $E \otimes L^*$ has weight zero, i.e. comes from X. The choice of L corresponds to the choice of the B-field B and one obtains $\mathrm{ch}^B(E) = \mathrm{ch}_L(E)$.

ii) Yoshioka uses yet other conventions to descend from the derived category to cohomology. A detailed comparison of the various twisted Chern characters can be found in [**30**].

We are primarily interested in twisted K3 surfaces (X, α) and their natural weight two Hodge structure $\widetilde{H}(X, \alpha, \mathbb{Z})$. Unfortunately, I don't know of any elegant way to define directly $\mathrm{ch}^\alpha : \mathbf{Coh}(X, \alpha) \longrightarrow \widetilde{H}(X, \alpha, \mathbb{Z})$. Twisted Chern character and twisted cohomology can physically only be realized after choosing a B-field lift.

The twisted as well as the untwisted Chern character has to be modified to the Mukai vector. Only working with the Mukai vector allows one to descend from equivalences of derived categories to isomorphisms of cohomologies.

With the above notation one defines the *Mukai vector* as

$$v^B(\ ) := \mathrm{ch}^B(\ ) \cdot \sqrt{\mathrm{td}(X)}.$$

It can be applied to (twisted) sheaves as well as to complexes of sheaves and maps coherent (twisted) sheaves to classes in $\widetilde{H}^{1,1}(X, B, \mathbb{Z})$. The definition in the untwisted case, i.e. $B = 0$, is due to Mukai and we write simply $v(\ )$ in this case. Note that for K3 surfaces $\sqrt{\mathrm{td}(X)} = (1, 0, 1)$.

If $E, F \in \mathbf{Coh}(X, \alpha)$, then the Hirzebruch–Riemann–Roch formula reads

$$\chi(E, F) := \sum (-1)^i \dim \mathrm{Ext}^i(E, F) = -\langle v^B(E), v^B(F) \rangle.$$

Examples 4.2. The following (untwisted) Mukai vectors of sheaves on K3 surfaces are used frequently: i) $v(\mathcal{O}) = (1, 0, 1)$, ii) $v(\mathcal{O}_C(-1)) = (0, [C], 0)$, iii) $v(k(x)) = (0, 0, 1)$.

5. The bounded derived category of a K3 surface

Let X be a K3 surface and let $\alpha \in H^2(X, \mathcal{O}^*)$ be a class represented by a Čech cocycle $\{\alpha_{ijk}\}$. One associates to X and (X, α) the abelian categories $\mathbf{Coh}(X)$ and $\mathbf{Coh}(X, \{\alpha_{ijk}\})$ of coherent sheaves on X and $\{\alpha_{ijk}\}$-twisted coherent sheaves, respectively.

Remark 5.1. These categories are defined for any X and any α. However, for X not projective or α not torsion, there exist usually very few interesting (twisted) coherent sheaves on X. The categories $\mathbf{Coh}(X)$ and $\mathbf{Coh}(X,\alpha)$ will simply be too small to be interesting. Thus, studying these categories is a sensible thing to do only under these 'projectivity' assumptions.

The following theorem in the untwisted case is a special case of a result of Gabriel [**17**].

Theorem 5.2. *Suppose (X,α) and (X',α') are two twisted K3 surfaces with X,X' projective and α,α' torsion. Then $(X,\alpha)\cong(X',\alpha')$ if and only if there exists an equivalence $\mathbf{Coh}(X,\alpha)\cong\mathbf{Coh}(X',\alpha')$.*

The basic idea is very simple. Consider the *minimal objects* in $\mathbf{Coh}(X,\alpha)$, i.e. those objects that do not contain any proper non-trivial sub-objects. This is certainly a notion that is preserved under any equivalence. On the other hand, it is straightforward to prove that minimal objects in $\mathbf{Coh}(X,\alpha)$ are just the skyscraper sheaves $k(x)$ of closed points $x\in X$. Hence, any equivalence will induce a bijection $X\cong X'$. It remains to show that this natural bijection is a morphism. In the case of surfaces the topology is determined by points and curves. Since curves have trivial Brauer group, their structure sheaves can be viewed as objects in the category of twisted sheaves.

Remark 5.3. i) The result holds in much broader generality. Gabriel proves the untwisted case for arbitrary schemes. For the untwisted case see [**46**].

ii) If X is not projective, then the abelian category $\mathbf{Coh}(X)$ does not, in general, encode the variety completely. E.g. as shown in [**58**] two very general complex tori have equivalent abelian categories.

The upshot is that passing from (twisted) K3 surfaces to their abelian categories no information is lost. We still can formulate a Global Torelli Theorem:

Corollary 5.4. Suppose (X,α) and (X',α') are two twisted K3 surfaces with X,X' projective and α,α' torsion classes. Then $\mathbf{Coh}(X,\alpha)\cong\mathbf{Coh}(X',\alpha')$ if and only if there exists a Hodge isometry $H^2(X,\mathbb{Z})\cong H^2(X',\mathbb{Z})$ such that the induced map $\mathrm{Br}(X)\longrightarrow\mathrm{Br}(X')$ sends α to α'.

One can be a little more specific: Any equivalence $\mathbf{Coh}(X,\alpha)\cong\mathbf{Coh}(X',\alpha')$ is of the form $(M\otimes\ )\circ f_*$, where $f:X\cong X'$ is an isomorphism and $M\in\mathrm{Pic}(X')$. Note that a priori it is not clear how to pass from an equivalence of the abelian categories directly to a Hodge isometry. In fact, the natural Hodge isometry associated to an equivalence of the form $(M\otimes\ )\circ f_*$ would not be f_*, but $\exp(\mathrm{c}_1(M))\circ f_*$. Here, $\exp(\mathrm{c}_1(M))$ means multiplication with the Chern character $\mathrm{ch}(M)$.

Remark 5.5. We have chosen to work with $\{\alpha_{ijk}\}$-twisted sheaves, but rephrasing everything in terms of $\mathcal{A}$-modules or sheaves on the gerbes $\mathcal{M}_{\mathcal{A}}$ or $\mathcal{M}_{\{\alpha_{ijk}\}}$ or on the Brauer–Severi variety $\mathbb{P}(E)$ is possible. One would then consider the abelian categories $\mathbf{Coh}(X,\mathcal{A})$, $\mathbf{Coh}(\mathcal{M}_{\mathcal{A}})_1$, $\mathbf{Coh}(\mathcal{M}_{\{\alpha_{ijk}\}})_1$, or $\mathbf{Coh}(\mathbb{P}(E)/X)$ (see Section 3). It follows from the discussion there that all these categories are equivalent if $\delta[\mathcal{A}]=\alpha$ and $E\in\mathbf{Coh}(X,\{\alpha_{ijk}\})$. It is largely a matter of taste which one is preferable.

Let us emphasize, however, that there is no $\mathbb{G}_m$-gerbe $\mathcal{M}_\alpha$ naturally associated to a Brauer class $\alpha\in\mathrm{Br}(X)$ but only an isomorphism class of $\mathbb{G}_m$-gerbes. In particular, before being able to introduce the abelian category of the twisted K3 surface (X, α) one has to make a choice, either of a cocycle $\{\alpha_{ijk}\}$ representing α, of an Azumaya algebra $\mathcal{A}$ with $\delta[\mathcal{A}] = \alpha$, of a $\mathbb{G}_m$-gerbe in the isomorphism class determined by α, or of a Brauer–Severi variety $\pi : \mathbb{P}(E) \longrightarrow X$ realizing α.

Let us stick for the rest of this section to ordinary K3 surfaces. The twisted case will be discussed in the next section. This is historically correct and makes, I hope, the most general case easier to digest

Definition 5.6. The *derived category* $\mathrm{D}^{\mathrm{b}}(X)$ of a K3 surface X is the bounded derived category of the abelian category $\mathbf{Coh}(X)$, i.e.

$$\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X)).$$

The category $\mathrm{D}^{\mathrm{b}}(X)$ is a $\mathbb{C}$-linear triangulated category and equivalences between such categories will always assumed to be $\mathbb{C}$-linear and exact, i.e. shifts and distinguished triangles are respected. Two K3 surfaces are called *derived equivalent* if there exists an equivalence $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X')$.

Why passing from the abelian category $\mathbf{Coh}(X)$ to its derived category might change things is explained by Mukai's celebrated example, which marked the beginning of the theory of Fourier–Mukai transforms (see [**38**]). Let A be an abelian variety and let $\widehat{A}$ be its dual abelian variety. In general, A and $\widehat{A}$ are non-isomorphic. Indeed, they are isomorphic if and only if A is principally polarized. Nevertheless, there always exists an exact equivalence

$$\mathrm{D}^{\mathrm{b}}(A) \cong \mathrm{D}^{\mathrm{b}}(\widehat{A}).$$

Mukai not only proves the equivalence of the two derived categories, but suggests how to produce geometrically interesting equivalences in general. This has led to the concept of Fourier–Mukai transformations.

Definition 5.7. Let X and X' be any two smooth projective varieties and $\mathcal{P} \in \mathrm{D}^{\mathrm{b}}(X \times X')$. The *Fourier–Mukai* transformation with *Fourier–Mukai kernel* $\mathcal{P}$ is the exact functor:

$$\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X') \ , \quad E^{\bullet} \longmapsto Rp_*(q^*E^{\bullet} \otimes^L \mathcal{P}),$$

where q and p denote the two projections from $X \times X'$.

In general, a Fourier–Mukai transformation will not define an equivalence, but due to a deep theorem of Orlov the converse holds, see [**44**]:

Theorem 5.8. *Let $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X')$ be an exact equivalence. Then there exists a unique object $\mathcal{P} \in \mathrm{D}^{\mathrm{b}}(X \times X')$ (up to isomorphism) such that $\Phi \cong \Phi_{\mathcal{P}}$.*

Note that this is somewhat equivalent to the fact that any equivalence between the abelian categories of coherent sheaves has a very special form. However, the object $\mathcal{P}$ might be difficult to describe explicitly and in general it will be a true complex (and not just a shifted sheaf).

In many situations the following criterion can be used to decide whether a given Fourier–Mukai transformation defines an equivalence. See the original articles [**4, 7**] or [**26**] for the proof and similar results.

THEOREM 5.9. *Suppose the Fourier–Mukai transform* $\Phi_{\mathcal{P}} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X')$ *satisfies the following two conditions:*

i) $\dim \mathrm{Hom}(\Phi(k(x)), \Phi(k(x))) = 1$ *for any* $x \in X$ *and*

ii) $\mathrm{Hom}(\Phi(k(x)), \Phi(k(y))[i]) = 0$ *for* $x \neq y$ *or* $i < 0$ *or* $i > \dim(X)$.

Then $\Phi_{\mathcal{P}}$ *defines an equivalence.*

EXAMPLES 5.10. i) Let A be an abelian variety and let $\widehat{A}$ be its dual. The Poincaré bundle $\mathcal{P}$ on $\widehat{A} \times A$ can be considered as an object in $\mathrm{D}^{\mathrm{b}}(\widehat{A} \times A)$. The famous result of Mukai alluded to before states that the induced Fourier–Mukai transform $\Phi_{\mathcal{P}} : \mathrm{D}^{\mathrm{b}}(\widehat{A}) \longrightarrow \mathrm{D}^{\mathrm{b}}(A)$ is an equivalence. Nowadays the result can be obtained as a direct consequence of Theorem 5.9.

ii) Any isomorphism $X \cong X'$ induces an equivalence $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X')$. The Fourier–Mukai kernel is the structure sheaf of its graph.

iii) Suppose M is a moduli space of stable sheaves on a K3 surface X. If M is complete and two-dimensional, then M is a K3 surface (see [**39**] or [**27**]). If M is fine, i.e. a universal sheaf $\mathcal{E}$ on $M \times X$ exists, then Theorem 5.9 again applies and yields an equivalence $\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(M) \cong \mathrm{D}^{\mathrm{b}}(X)$.

This is in analogy to i), where $\widehat{A}$ can be considered as a moduli space of line bundles on A and $\mathcal{P}$ as a universal family. For K3 surfaces however the 'dual' K3 surface provided by a moduli space M as above is not unique.

iv) If L is a line bundle on a projective variety, then $F^\bullet \longmapsto L \otimes F^\bullet$ defines an equivalence $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X)$ which can be described as the Fourier–Mukai transformation with kernel $\iota_* L$, where $\iota : X \longrightarrow X \times X$ is the diagonal embedding.

v) Suppose X is a K3 surface containing an irreducible smooth rational curve $C \subset X$. Consider $\mathcal{O}_C(-1)$ as an object in $\mathrm{D}^{\mathrm{b}}(X)$. The trace induces a natural morphism $\mathcal{O}_C(-1) \boxtimes \mathcal{O}_C(-1)^\vee \longrightarrow \mathcal{O}_\Delta$ in $\mathrm{D}^{\mathrm{b}}(X \times X)$, where $\mathcal{O}_C(-1)^\vee$ denotes the derived dual. The cone of this morphism shall be denoted $\mathcal{P}_{\mathcal{O}_C(-1)} \in \mathrm{D}^{\mathrm{b}}(X \times X)$ and the induced Fourier–Mukai functor is the *spherical twist* $T_{\mathcal{O}_C(-1)} := \Phi_{\mathcal{P}_{\mathcal{O}_C(-1)}}$, which is an equivalence.

vi) The sheaf $E := \mathcal{O}_C(-1)$ in v) is a *spherical object*, i.e. E satisfies

$$\mathrm{Ext}^*_X(E, E) = H^*(S^2, \mathbb{C}).$$

Kontsevich proposed to consider a spherical twist T_E associated to any spherical object E on a Calabi–Yau manifold and Seidel and Thomas were able to prove that T_E is indeed an equivalence. (This time Theorem 5.9 is of no use, another kind of spanning class is needed here, see [**55**] or [**26**, Ch.8].) Other examples of spherical objects on a K3 surface X are $\mathcal{O}_X$, or more generally any line bundle, and simple rigid sheaves.

Any Fourier–Mukai transformation $\Phi_{\mathcal{P}} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X')$ induces the cohomological Fourier–Mukai transformation

$$\Phi^H_{\mathcal{P}} : H^*(X, \mathbb{Q}) \longrightarrow H^*(X', \mathbb{Q}),$$

which is defined in terms of the Mukai vector $v(\mathcal{P}) \in H^*(X \times X', \mathbb{Q})$ as

$$\gamma \longmapsto p_*(v(\mathcal{P}).q^*(\gamma)).$$

If $\Phi_{\mathcal{P}}$ is an equivalence, then $\Phi^H_{\mathcal{P}}$ is bijective. This innocent looking statement is not trivial, as the part in $H^*(X, \mathbb{Q})$ that comes from objects in $\mathrm{D}^{\mathrm{b}}(X)$ may be small. Note that in general $\Phi_{\mathcal{P}}$ does neither respect the grading nor the algebra structure nor will it be defined over $\mathbb{Z}$.

Back to the case of K3 surfaces one finds that in Examples 5.10, v) and vi) the cohomological spherical shift T^H_E is given by the reflection $\gamma \longmapsto \gamma + (\gamma.v(E))v(E)$. In particular, $T^H_{\mathcal{O}_C(-1)} = s_{[C]}$. The tensor product $L\otimes$ acts by multiplication with $\exp(\mathrm{c}_1(L))$ on $H^*(X,\mathbb{Z})$.

Remark 5.11. In ii), Remark 1.4 we have pointed out that the Hodge isometry $s_{[C]}$ is, for trivial reasons, not induced by any automorphism of X. This is cured by the above observation which says that it can be lifted to an autoequivalence of $\mathrm{D^b}(X)$.

Mukai shows in [**39**] that $\Phi^H_{\mathcal{P}}$ of a derived equivalence $\Phi_{\mathcal{P}}$ between two K3 surfaces is defined over $\mathbb{Z}$ and that it defines a Hodge isometry $\widetilde{H}(X,\mathbb{Z}) \cong \widetilde{H}(X',\mathbb{Z})$. Combined with Orlov's result this becomes

Corollary 5.12. Any derived equivalence $\Phi : \mathrm{D^b}(X) \cong \mathrm{D^b}(X')$ between two K3 surfaces induces naturally a Hodge isometry

$$\Phi : \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X',\mathbb{Z}).$$

The other direction, namely how to deduce from the existence of a Hodge isometry of the Mukai lattices of two K3 surfaces the existence of a derived equivalence, was proved by Orlov. Both results together combine to

Theorem 5.13. **(Derived Global Torelli)** *Two projective K3 surfaces X and X' are derived equivalent if and only if there exists a Hodge isometry*

$$\widetilde{H}(X,\mathbb{Z}) \cong \widetilde{H}(X',\mathbb{Z}).$$

For the complete proof of the theorem the reader may consult the original article [**44**] or [**26**, Ch.10]. What is important to know for our purpose is that a given Hodge isometry is modified by the cohomological Fourier–Mukai transformations of the type iii)-v) in Example 5.10 such that the new Hodge isometry induces a Hodge isometry of the standard weight-two Hodge structure $H^2(X,\mathbb{Z})$ of X with the one of some moduli space Y of sheaves on X', which is again a K3 surface. Then the classical Global Torelli (see Theorem 1.3) applies and yields an isomorphism $X \cong Y$.

Remark 5.14. There is a minor, but annoying issue in the argument. At the very end one has to ensure that the image of a Kähler class is a Kähler class and not only up to sign. But of course, composing the given Hodge isometry with the Hodge isometry $\mathrm{id}_{H^0} \oplus -\mathrm{id}_{H^2} \oplus \mathrm{id}_{H^4}$ clears this problem.

The problem is reflected by the following more precise result that is the derived analogue of iii), Remark 1.4: Any orientation preserving Hodge isometry $\widetilde{H}(X,\mathbb{Z}) \cong \widetilde{H}(X',\mathbb{Z})$ lifts to a derived equivalence, i.e. is of the form $\Phi^H_{\mathcal{P}}$. (Note that the uniqueness of the classical Global Torelli Theorem does not hold. See the discussion in Section 7.)

In [**56**] Szendrői suggested that the mirror symmetry analogue of the result of Donaldson (1.1) should say that a Hodge isometry not preserving the natural orientation can not be lifted. In other words, one expects that Φ^H in Corollary 5.12 is always orientation preserving. That this is the case at least for all known examples was proven in [**29**]. The Fourier–Mukai equivalence induced by the universal family of stable sheaves is the only non-trivial case.

The issue becomes more serious in the twisted case. Composing a given Hodge isometry g with $g_0 := \mathrm{id}_{H^0} \oplus -\mathrm{id}_{H^2} \oplus \mathrm{id}_{H^4}$ in order to reverse the orientation is not allowed anymore. Indeed, only in the untwisted case is g_0 naturally a Hodge isometry.

If one prefers to work with the transcendental part of the Hodge structure, then the above theorem becomes

Corollary 5.15. Two projective K3 surfaces X and X' are derived equivalent if and only if there exists a Hodge isometry $T(X) \cong T(X')$.

Proof. Recall that the transcendental lattice T of a Hodge structure of weight two on a lattice H is the orthogonal complement of $H \cap H^{1,1}$ and that T is again a Hodge structure of weight two. In our geometric situation, $T(X)$ is the transcendental lattice of $H^2(X, \mathbb{Z})$ or, equivalently, of $\widetilde{H}(X, \mathbb{Z})$.

Clearly, any Hodge isometry $\widetilde{H}(X, \mathbb{Z}) \cong \widetilde{H}(X', \mathbb{Z})$ induces a Hodge isometry $T(X) \cong T(X')$. Conversely, due to a result of Nikulin [**41**], any Hodge isometry $T(X) \cong T(X')$ can be extended to a Hodge isometry of the Mukai lattices $\widetilde{H}(X, \mathbb{Z}) \cong \widetilde{H}(X', \mathbb{Z})$. The reason behind this is the existence of the hyperbolic plane $H^0 \oplus H^4$ in the orthogonal complement of $T(X) \subset \widetilde{H}(X, \mathbb{Z})$. Note that in general the orthogonal complement of $T(X) \subset H^2(X, \mathbb{Z})$ does not contain any hyperbolic plane, which explains why derived equivalent K3 surfaces are not necessarily isomorphic. □

As a consequence of the proof of the theorem, Orlov obtains

Corollary 5.16. Two projective K3 surfaces X and X' are derived equivalent if and only if X' is isomorphic to a moduli space of stable sheaves on X.

The polarization needs to be fixed appropriately and the sheaves might a priori be torsion, e.g. X itself is viewed as the moduli space of skyscraper sheaves $k(x)$ or, equivalently, of the ideal sheaves $\mathcal{I}_x$ of closed points $x \in X$. In fact, in the corollary one could replace 'stable sheaves' by 'torsion free stable sheaves'.

Remark 5.17. Let us also mention that a general conjecture stating that any smooth projective variety admits only finitely many Fourier–Mukai partners, i.e. smooth projective varieties with equivalent derived categories, can be proved for K3 surfaces. Once Corollary 5.12 is established, one uses lattice theory. A related natural question asks for the number of isomorphism types of K3 surfaces derived equivalent to a given K3 surface X in terms of the period of X. This question has been addressed in [**22, 54**].

One of the standard examples of Fourier–Mukai kernels defining a derived equivalence between K3 surfaces is provided by the universal sheaf $\mathcal{E}$ on $X \times M$, where M is a complete, fine moduli space of stable sheaves of dimension two (see iii), Examples 5.10).

There are examples of moduli spaces M of stable sheaves on a K3 surface X which are not fine, i.e. a universal sheaf $\mathcal{E}$ does not exist. Locally in the analytic (or étale) topology of M one finds universal sheaves, but the obstruction to glue those to a global universal sheaf might be non-trivial. Căldăraru observed in [**12**] that this obstruction can be considered as a Brauer class $\alpha \in H^2(M, \mathcal{O}^*)$ and that the local universal sheaves glue to a $(1 \times \alpha)$-twisted universal sheaf $\mathcal{E}$ on $X \times M$.

If M is complete and two-dimensional, then M is a K3 surface, but in general not derived equivalent to X. However, in [**12**] it is shown that the twisted universal sheaf $\mathcal{E}$ induces a Fourier–Mukai transformation that does define an equivalence

$$\mathrm{D^b}(M, \alpha^{-1}) \cong \mathrm{D^b}(X).$$

Here, $\mathrm{D^b}(M, \alpha^{-1})$ is the bounded derived category of $\mathbf{Coh}(M, \alpha^{-1})$ (see the next section).

Thus starting with classical untwisted K3 surfaces we naturally end up with twisted K3 surfaces. There are other reasons to consider twisted K3 surfaces, as has been alluded to in the introduction, but from the point of view of moduli spaces of sheaves on K3 surfaces this is absolutely necessary in order to fully understand the relation between K3 surfaces and their moduli spaces of sheaves.

Motivated by this example, Căldăraru formulated in [**12**] a conjecture that generalizes Corollary 5.15. to the case of twisted K3 surfaces. In fact, the conjecture could be verified in a number of other situations (see e.g. [**14, 32**]), but turned out to be wrong in general (see [**29**, Ex.4.11]).

When [**12**] was written, generalized Calabi–Yau structures had not been invented and the Hodge structure of twisted K3 surfaces had not been introduced. Only the transcendental part $T(X, \alpha)$ could be defined directly in terms of a Brauer class α and Orlov's result (see Corollary 5.15) suggested to conjecture $\mathrm{D^b}(X, \alpha) \cong \mathrm{D^b}(X', \alpha')$ if and only if $T(X, \alpha) \cong T(X', \alpha')$. However, in contrast to the untwisted case, a Hodge isometry between the transcendental lattices of two twisted K3 surface does not extend to a Hodge isometry of the full weight two Hodge structure on the Mukai lattice. Nikulin's result does not apply anylonger, as a hyperbolic plane in the orthogonal complement does not necessarily exist.

How the original conjecture of Căldăraru has to be modified will be explained next.

6. Twisted versions

The question we shall deal with in this section is the following. Suppose X and X' are two projective K3 surfaces endowed with Brauer classes α and α', respectively. When does there exist an equivalence

$$\mathrm{D^b}(X, \alpha) \cong \mathrm{D^b}(X', \alpha')?$$

We must confess that we are not able to deal with the question in this generality, as an analogue of Orlov's existence result (see Theorem 5.8) in the twisted case has not yet been proven. So, whenever we deal with equivalences between twisted derived categories in this section, we shall mean equivalences of Fourier–Mukai type. More precisely, we will consider

$$\Phi_{\mathcal{P}} : \mathrm{D^b}(X, \alpha) \xrightarrow{\sim} \mathrm{D^b}(X', \alpha')$$

which for an object $\mathrm{D^b}(X \times X', \alpha^{-1} \times \alpha')$ is defined by the usual formula

$$F^\bullet \longmapsto Rp_*(\mathcal{P} \otimes^L q^* F^\bullet).$$

In [**11**] it is explained that the usual formalism of derived functors goes through in the twisted case.

Let us first explain the 'easy' direction that has led to the definition of the twisted Chern character in [**29**] (see Section 4).

Proposition 6.1. Any equivalence of Fourier–Mukai type

$$\Phi_{\mathcal{P}} : \mathrm{D}^{\mathrm{b}}(X, \alpha) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X', \alpha')$$

induces a Hodge isometry

$$\Phi_{\mathcal{P}}^{H} : \widetilde{H}(X, \alpha, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X', \alpha', \mathbb{Z}).$$

The Mukai lattice $\widetilde{H}(X, \alpha, \mathbb{Z})$ of a twisted K3 surface (X, α), has been introduced in Section 3 as the Hodge structure $\widetilde{H}(X, B, \mathbb{Z})$ of the generalized K3 surface given by $\exp(B)\sigma = \sigma + B \wedge \sigma$, where $B \in H^2(X, \mathbb{Q})$ with $\exp(B^{0,2}) = \alpha$. The isomorphism type of the weight two Hodge structure $\widetilde{H}(X, \alpha, \mathbb{Z})$ is independent of the choice of B, but for the definition of it one B has to be picked.

Thus, in order to explain the idea behind the proposition, we need to fix B-field lifts $B \in H^2(X, \mathbb{Q})$ and $B' \in H^2(X', \mathbb{Q})$ of α and α', respectively. This allows us at the same time to consider $v^{-B \oplus B'}(\mathcal{P}) \in H^*(X \times X', \mathbb{Q})$. The claimed Hodge isometry is then provided by

$$\gamma \longmapsto p_*(v^{-B \oplus B'}(\mathcal{P}).q^*\gamma).$$

In [**29**] we explain how one has to modify the arguments of Mukai to make them work in the twisted case as well, e.g. why the Mukai vector is again integral etc. The new feature in the twisted case is that one has to make the additional choice of the B-field lifts. In general there is no canonical lift, but if the untwisted case is considered as a twisted case with trivial Brauer class, then one may use the canonical lift $B = 0$.

The converse of the above has been proved in [**30**]. Unlike the untwisted case, the orientation has to be incorporated in the assertion from the beginning (see Remark 5.14).

Theorem 6.2. *Suppose (X, α) and (X', α') are two projective twisted K3 surfaces. If there exists an orientation preserving Hodge isometry*

$$\widetilde{H}(X, \alpha, \mathbb{Z}) \cong \widetilde{H}(X', \alpha', \mathbb{Z}),$$

then one finds a Fourier–Mukai equivalence

$$\Phi_{\mathcal{P}} : \mathrm{D}^{\mathrm{b}}(X, \alpha) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X', \alpha').$$

Remark 6.3. (joint with P. Stellari) For a generic projective twisted K3 surface (X, α) there are no (-2)-classes in $\widetilde{H}^{1,1}(X, \alpha, \mathbb{Z})$. Hence $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ does not contain any spherical objects. In this case one can show that any Fourier–Mukai kernel $\mathcal{P} \in \mathrm{D}^{\mathrm{b}}(X \times X', \alpha^{-1} \times \alpha')$ inducing an equivalence $\Phi_{\mathcal{P}} : \mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha')$ is isomorphic to a shifted sheaf $E[k]$. This is enough to conclude that $\Phi_{\mathcal{P}}^{H}$ is orientation preserving. Thus, for a generic projective (X, α) the theorem reads: There exists a Fourier–Mukai equivalence $\mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha')$ if and only if there exists an orientation preserving Hodge isometry $\widetilde{H}(X, \alpha, \mathbb{Z}) \cong \widetilde{H}(X', \alpha', \mathbb{Z})$.

In order to prove that any orientation preserving Hodge isometry can be lifted to a derived equivalence one tries to imitate Orlov's proof of Theorem 5.13. Most of the arguments go through, but at a few crucial points the non-triviality of the Brauer class necessitates a different approach. One is the occasional absence of spherical objects for general twisted K3 surfaces and of the structure sheaf $\mathcal{O}$ as an object in $\mathrm{D^b}(X,\alpha)$ in particular. Another one is the non-emptiness and smoothness of certain moduli spaces of stable twisted sheaves that has to be assured. In particular, the following result due to Yoshioka's plays a central rôle in the argument (see [**60**]).

Theorem 6.4. *Let X be a projective K3 surface, $B \in H^2(X,\mathbb{Q})$ be a B-field and $v \in \widetilde{H}^{1,1}(X,B,\mathbb{Z})$ a primitive vector with $\langle v,v\rangle = 0$ and $v_0 \neq 0$. Then there exists a moduli space $M(v)$ of stable (with respect to a generic polarization) α_B-twisted sheaves E with $v^B(E) = v$ which is a (non-empty!) K3 surface.*

The existence of moduli spaces of stable twisted sheaves has been shown in broad generality by Lieblich [**33**] and Yoshioka [**60**]. Using the equivalence to sheaves over Azumaya algebras, it can also be deduced from the general results of Simpson [**51**]. From this theorem Yoshioka deduces by standard methods the existence of a universal $\alpha_B^{-1} \times \alpha'$-twisted sheaf $\mathcal{P}$ on $X \times M(v)$ which induces an equivalence $\mathrm{D^b}(X,\alpha_B) \cong \mathrm{D^b}(M(v),\alpha')$. The latter is needed in order to imitate iii), Example 5.10 in the twisted case. Eventually, the assertion is reduced to the classical Global Torelli Theorem.

Remark 6.5. The twisted version of Remark 5.17 holds true as well. E.g. for a given K3 surface X and a fixed Brauer class $\alpha_0 \in \mathrm{Br}(X)$ there are only finitely many classes $\alpha \in \mathrm{Br}(X)$ such that $\mathrm{D^b}(X,\alpha)$ is Fourier–Mukai equivalent to $\mathrm{D^b}(X,\alpha_0)$. See [**29**, Prop.3.4].

Remark 6.6. Twisted derived equivalences have also been considered for abelian varieties by Polishchuk (see e.g. [**31, 50**]). A complete analogue of the untwisted results of Mukai, Orlov, and Polishchuk has been obtained, although by methods different from the ones in [**38, 43**].

7. What's left

So far we have treated 'half' of the derived Global Torelli Theorem. Staying on one K3 surface, we have up to now only tried to determine the image of the natural representation

$$\mathrm{Aut}(\mathrm{D^b}(X,\alpha)) \longrightarrow \mathrm{O}_+(\widetilde{H}(X,\alpha,\mathbb{Z})). \tag{7.1}$$

The classical Global Torelli Theorem asserts that $\mathrm{Aut}(X) \longrightarrow \mathrm{O}_+(H^2(X,\mathbb{Z}))$ is injective. This is no longer true in the derived setting, e.g. the shift [2] is contained in the kernel of (7.1). So, the 'other half' of a derived (twisted) Global Torelli Theorem would be concerned with the kernel of (7.1). A similar question has been asked for abelian varieties and a beautiful answer has been given by Orlov in [**43**] (see [**26**, Ch.9] for an account).

For a long time the kernel of (7.1) for K3 surfaces seemed mysterious. Bridgeland's work [**9**] on stability conditions on derived categories of K3 surfaces has changed the situation completely. We now at least have a clear conjecture and an answer seems in reach. This would then yield the final form of the derived, twisted Global Torelli Theorem.

Without giving any background on stability conditions, we simply state Bridgeland's conjecture (generalized to the case of twisted K3 surfaces).

Conjecture 7.1. For any projective twisted K3 surface (X,α) there exists a natural short exact sequence

$$0 \longrightarrow \pi_1(\mathcal{P}_0(X,\alpha)) \longrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X,\alpha)) \longrightarrow \mathrm{O}_+(\widetilde{H}(X,\alpha,\mathbb{Z})) \longrightarrow 1.$$

Before explaining what $\mathcal{P}_0(X,\alpha)$ is, let us once more recall that we actually only know that $\mathrm{O}_+(\widetilde{H}(X,\alpha,\mathbb{Z}))$, which denotes the group of all orientation preserving Hodge isometries, is contained in the image, but we are unable, for the time being, to show that the image is not bigger.

Consider the set $\Delta(X,\alpha) \subset \widetilde{H}^{1,1}(X,\alpha,\mathbb{Z})$ of all classes δ with $\langle\delta,\delta\rangle = -2$. Let $\mathcal{P}(X,\alpha) \subset \widetilde{H}^{1,1}(X,\alpha,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ be the open subset of all vectors whose real and imaginary parts (in this order) span a positive oriented plane. Then

$$\mathcal{P}_0(X,\alpha) := \mathcal{P}(X,\alpha) \setminus \bigcup_{\delta\in\Delta(X,\alpha)} \delta^\perp.$$

Bridgeland constructs a natural map $\pi_1(\mathcal{P}_0(X,\alpha)) \longrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$. (Adapting [**9**] to the twisted case is rather straightforward.)

We conclude with a few observation in the case of generic twisted K3 surfaces (X,α) (joint work with P. Stellari).

If X is a generic projective K3 surface and α is a generic Brauer class, then $\Delta(X,\alpha) = \varnothing$. Thus, $\mathcal{P}_0(X,\alpha) = \mathcal{P}(X,\alpha)$, whose fundamental group is $\mathbb{Z}$. This group is mapped onto the subgroup of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ that is spanned by the shift [2]. Due to Remark 6.3 every Fourier–Mukai autoequivalence $\Phi_{\mathcal{P}}$ of $\mathrm{D}^{\mathrm{b}}(X,\alpha)$ has a kernel of the form $\mathcal{P} \cong E[\ell]$ for some twisted sheaf E on $X \times X$ and some $\ell \in \mathbb{Z}$.

Suppose $\Phi_{\mathcal{P}}^H = \mathrm{id}$. Then $E_x := E|_{\{x\}\times X}$ has Mukai vector $(0,0,1)$. Therefore, $E_x \cong k(y)$ for some $y \in X$ and ℓ must be even. As line bundle twists and automorphisms of X are all detected on cohomology, this yields

Proposition 7.2. For a generic twisted K3 surface (X,α) one has an exact sequence

$$0 \longrightarrow \mathbb{Z}[2] \longrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X,\alpha)) \longrightarrow \mathrm{O}_+(\widetilde{H}(X,\alpha,\mathbb{Z}))) \longrightarrow 1.$$

We are not able to exclude the existence of exotic components of the moduli space of stability conditions in the generic case, which might be expected in general.

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