Moduli Spaces of Tori and the Narain Moduli Space

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The aim of this talk is to describe both moduli spaces mentioned in the title and to explain their relation. The moduli space of tori is a moduli space of geometric structures, whereas the Narain moduli space should be considered as the moduli space of toroidal CFT's, although in this talk it will be introduced as a moduli space of certain linear data. In the subsequent talks it will be explained how to associate to any point in the Narain moduli space a vertex algebra in the sense of Kapustin and Orlov, which should be thought of as an approximation of a CFT.

1 Definition of the moduli spaces

A *torus* in the context of this talk consists of a triple (T, G, B) where T is a real manifold diffeomorphic to a compact torus of fixed dimension d, G is a *Ricci-flat* metric on T, and B is a *two-form* on T which is parallel with respect to G. Thus, T is diffeomorphic to $(S^1)^d$. A metric G on such a manifold is Ricci-flat if and only if there exists a diffeomorphism $\varphi : \mathbb{R}^d/\mathbb{Z}^d \cong T$ such that φ^*G is a constant metric, i.e. its pull-back to $\mathbb{R}^d$ is constant. The two-form B can be regarded as a two-form which is harmonic with respect to the metric G. We will thus think of B as an element in $H^2(T, \mathbb{R}) = \Lambda^2 H^1(T, \mathbb{R})$. If G is a constant metric on $\mathbb{R}^d/\mathbb{Z}^d$ then the B-fields are just constant two-forms on $\mathbb{R}^d/\mathbb{Z}^d$. If $T = \mathbb{R}^d/\mathbb{Z}^d$, then the constant metrics on T are uniquely determined by elements $G \in S^2H^1(T, \mathbb{Z})$ that define a positive definite quadratic form on $H_1(T, \mathbb{R})$.
Two tori are equivalent, $(T, G, B) \sim (T', G', B')$, if and only if there exists a diffeomorphism $\varphi : T \cong T'$ such that $\varphi^*G' = G$ and $\varphi^*B' = B$. Note that any torus (T, G, B) is equivalent to a torus of the form $(\mathbb{R}^d/\mathbb{Z}^d, G', B')$ with G' and B' are constant (i.e. their pull-backs to $\mathbb{R}^d$ are constant). The *moduli space of tori* of dimension d is the set of tori modulo this equivalence relation:

$$\boxed{\mathcal{M}_d^{tori} := \{(T, G, B)\}/\sim}$$

Let U^d be the direct sum of d copies of the standard hyperbolic lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We will consider pairs (Γ, Π), where Γ is a lattice, which is abstractly isomorphic to U^d, and $\Pi \subset \Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ is a space-like d-subspace. Recall that a space-like d-subspace is a d-dimensional subspace $\Pi \subset \Gamma_{\mathbb{R}}$ such that the quadratic form on $\Gamma_{\mathbb{R}}$, obtained by extending the quadratic form on Γ linearly to $\Gamma_{\mathbb{R}}$, is positive definite on Π.

Two such pairs are called equivalent, $(\Gamma,\Pi) \sim (\Gamma',\Pi')$, if and only if there exists a lattice isomorphism $\varphi : \Gamma \cong \Gamma'$ with $\varphi_{\mathbb{R}}(\Pi) = \Pi'$. Here, $\varphi_{\mathbb{R}} : \Gamma_{\mathbb{R}} \cong \Gamma'_{\mathbb{R}}$ is the $\mathbb{R}$-linear extension of φ. The *Narain moduli space* is the set of these pairs modulo equivalence:

$$\boxed{\mathcal{M}_d^{Narain} := \{(\Gamma,\Pi)\}/\sim}$$

The aim of this talk is to describe $\mathcal{M}_d^{tori}$ and $\mathcal{M}_d^{Narain}$. We will see that there exists a natural surjective map

$$\boxed{\mathcal{M}_d^{tori} \twoheadrightarrow \mathcal{M}_d^{Narain}}$$

and we will describe the fibres of this map. In the process we will encounter integral shifts of the B-field, T-duality, and the $R \to 1/(4\pi^2 R)$ circle duality. We try to give all the details, although many of the arguments are straightforward.

2 Description of $\mathcal{M}_d^{tori}$

In order to describe $\mathcal{M}_d^{tori}$ we will introduce the moduli space of marked tori, of which $\mathcal{M}_d^{tori}$ is a discrete quotient. Fix a free abelian group Λ of rank d and denote the vector space $\Lambda\otimes_{\mathbb{Z}}\mathbb{R}$ by H. E.g. $\Lambda = \mathbb{Z}^d$ and $H = \mathbb{R}^n$.
A *marked torus* (T,G,B,ϕ) consists of a torus (T,G,B) and a group isomorphism $\phi : H_1(T,\mathbb{Z}) \cong \Lambda$. Two marked tori (T,G,B,ϕ) and (T',G',B',ϕ') are equivalent if and only if there exists a diffeomorphism $\varphi : T \cong T'$ such that $\varphi^*G' = G$, $\varphi^*B' = B$, and $\phi' \circ \varphi_* = \phi$. The *moduli space of marked tori* is the set

$$\boxed{\widetilde{\mathcal{M}}_d^{tori} := \{(T,G,B,\phi)\}/\sim}$$

The group $\mathrm{Gl}(\Lambda)$ of automorphisms of Λ acts on $\widetilde{\mathcal{M}}_d^{tori}$ by

$$\begin{array}{ccccc} \mathrm{Gl}(\Lambda) & \times & \widetilde{\mathcal{M}}_d^{tori} & \to & \widetilde{\mathcal{M}}_d^{tori} \\ (\varphi & , & (T,G,B,\phi)) & \mapsto & (T,G,B,\varphi\circ\phi) \end{array}$$

Clearly, $\mathcal{M}_d^{tori}$ is the quotient of this action, i.e.

$$\mathcal{M}_d^{tori} = \mathrm{Gl}(\Lambda) \setminus \widetilde{\mathcal{M}}_d^{tori}.$$

Next, $\widetilde{\mathcal{M}}_d^{tori}$ can be explicitly described by identifying it with the set of all space-like d-subspaces of $H \oplus H^*$. This will be done in two steps.
Let $E \in (H^*)^{\otimes 2}$. Then E can be decomposed in its symmetric part G and its skew-symmetric part B. Let $\mathcal{U} \subset (H^*)^{\otimes 2}$ be the open subset of all those tensors E, such that G is positive definite. Then, the map $(T,G,B,\phi) \mapsto (\phi_{\mathbb{R}}^{-1})^*(G+B)$ defines an isomorphism

$$\widetilde{\mathcal{M}}_d^{tori} \cong \mathcal{U}.$$

Indeed, the map is clearly well-defined and surjective, for $E = G + B \in \mathcal{U}$ is the image of $(T = H/\Lambda, G, B, \phi = \mathrm{id} : H_1(T, \mathbb{Z}) = \Lambda \to \Lambda)$.
The injectivity follows from the fact that for any torus (T, G, B) there exists a diffeomorphism $\varphi : \mathbb{R}^d/\mathbb{Z}^d \cong T$ such that $\varphi^* G$ and $\varphi^* B$ are constant. Thus, (T, G, B, ϕ) is equivalent to $(\mathbb{R}^d/\mathbb{Z}^d, \varphi^* G, \varphi^* B, \phi \circ \varphi_*)$. But the latter is equivalent to $(H/\Lambda, ((\phi \circ \varphi_*)^{-1})^* \varphi^* G, ((\phi \circ \varphi_*)^{-1})^* \varphi^* B, \mathrm{id})$.
The set $\mathcal{U}$ should be thought of as the set of all metrics and B-fields on the torus H/Λ. Using the identification $\widetilde{\mathcal{M}}_d^{tori} \cong \mathcal{U}$ one obtains a $\mathrm{Gl}(\Lambda)$-action on $\mathcal{U}$. This action is given by $(\varphi, E) \mapsto (\varphi_{\mathbb{R}}^{-1})^* E$.
We now can go on to identify $\mathcal{U}$ with a certain space $\mathcal{N}$ which will also be used later to describe the Narain moduli space. Here $\mathcal{N}$ is the set of all space-like d-subspaces $\Pi \subset H \oplus H^*$, i.e. of all subspaces $\Pi \subset H \oplus H^*$ of dimension d, such that the natural quadratic form $\| (\alpha, \beta) \|^2 = 2\beta(\alpha)$ on $H \oplus H^*$ restricted to Π is positive definite. Thus $\mathcal{N}$ is an open subset of the Grassmannian. We claim that the map $E \mapsto \Pi := \{(\alpha, E(\alpha)) | \alpha \in H\}$ defines an isomorphism $\mathcal{U} \cong \mathcal{N}$. Here $E \in (H^*)^{\otimes 2}$ is considered as a linear map $H \to H^*, \alpha \mapsto E(\alpha, \ )$.
In order to prove this statement let us write $E = G + B$ as before. If $\alpha \in H$ is non-zero then $\| (\alpha, G(\alpha) + B(\alpha)) \|^2 = 2G(\alpha)(\alpha) + 2B(\alpha)(\alpha) = 2G(\alpha, \alpha) + 2B(\alpha.\alpha) > 0$, as B is alternating and hence $B(\alpha, \alpha) = 0$ and G is positive definite. Hence, the subspace Π associated to $E \in \mathcal{U}$ is space-like and obviously d-dimensional.
Conversely, if $\Pi \subset H \oplus H^*$ in $\mathcal{N}$ is given, then $\Pi \cap H^* = \{0\}$, since H^* is isotropic, and thus the projection $\Pi \to H$ is bijective. Let ψ denote the inverse map $H \to \Pi \subset H \oplus H^*$. Then ψ can be written as $\alpha \mapsto (\alpha, E(\alpha))$ and the above calculation shows that the symmetric part of E is positive definite, for Π is space-like.
Eventually, we obtain an identification $\widetilde{\mathcal{M}}_d^{tori} \cong \mathcal{N}$. Let us now describe $\mathcal{N}$ as an homogeneous space. In order to do so, let us consider the left action

$$\begin{array}{ccccc} \mathrm{O}(H \oplus H^*) & \times & \mathcal{N} & \to & \mathcal{N} \\ (\varphi & , & \Pi) & \mapsto & \varphi(\Pi) \end{array}$$

Any $\Pi \in \mathcal{N}$ induces an orthogonal decomposition $\Pi \oplus \Pi^\perp = H \oplus H^*$. If two such decompositions $\Pi_1 \oplus \Pi_1^\perp = H \oplus H^*$ and $\Pi_2 \oplus \Pi_2^\perp = H \oplus H^*$ are given, then there exist isometries $g : \Pi_1 \cong \Pi_2$ and $g^\perp : \Pi_1^\perp \cong \Pi_2^\perp$. Hence, $(g, g^\perp) \in \mathrm{O}(H \oplus H^*)$ maps $\Pi_2 \in \mathcal{N}$ to Π_1. Thus, the action is transitive. Fix an arbitrary point $\Pi_0 \in \mathcal{N}$. Then the stabilizer of this point is the subgroup $\mathrm{O}(\Pi_0) \times \mathrm{O}(\Pi_0^\perp) \subset \mathrm{O}(H \oplus H^*)$ which leaves invariant the decomposition $\Pi_0 \oplus \Pi_0^\perp = H \oplus H^*$. Thus,

$$\mathcal{N} \cong \mathrm{O}(H \oplus H^*) / \mathrm{O}(\Pi_0) \times \mathrm{O}(\Pi_0^\perp).$$

Choosing bases of Λ and Π_0 yields the following description of the moduli space of tori:

$$\boxed{\mathcal{M}_d^{tori} \cong \mathrm{Gl}(d, \mathbb{Z}) \setminus \mathrm{O}(d, d, \mathbb{R}) / \mathrm{O}(d) \times \mathrm{O}(d)}$$

The action of $\mathrm{Gl}(d,\mathbb{Z})$ is not apparent in this description.

Before going on to describe the Narain moduli space in a similar fashion we will explicitly describe the action of $\mathrm{Gl}(\Lambda)$ on $\mathcal{N}$. An automorphism $\varphi \in \mathrm{Gl}(\Lambda)$ acts on $\mathcal{U}$ by $E \mapsto (\varphi_{\mathbb{R}}^{-1})^*(E)$. Hence the corresponding space $\Pi = \{(\alpha, E(\alpha)) | \alpha \in H\}$ is mapped to $\{(\alpha, (\varphi^{-1})^*(E)(\alpha)) | \alpha \in H\} = \{(\alpha, (\varphi^{-1})^t(E(\varphi^{-1}(\alpha)))) | \alpha \in H\} = \{(\varphi(\alpha), (\varphi^{-1})^t(E\alpha)) | \alpha \in H\}$. Thus, Π is mapped to $\begin{pmatrix} \varphi & 0 \\ 0 & (\varphi^{-1})^t \end{pmatrix}(\Pi)$. The action of $\mathrm{Gl}(\Lambda)$ can also be described via the action of $\mathrm{O}(H \oplus H^*)$ by embedding $\mathrm{Gl}(\Lambda)$ into $\mathrm{O}(H \oplus H^*)$ with $\varphi \mapsto \begin{pmatrix} \varphi & 0 \\ 0 & (\varphi^{-1})^t \end{pmatrix}$.

3 Description of $\mathcal{M}_d^{Narain}$

In order to describe $\mathcal{M}_d^{Narain}$ we have to pass to its 'marked' version. This is very similar to what we have seen for the torus moduli space. Let us define $\widetilde{\mathcal{M}}_d^{Narain}$ as $\{(\Gamma, \Pi, \psi)\}/\sim$, where (Γ, Π, ψ) consists of a lattice Γ isomorphic to $U^{\oplus d}$, a space-like d-subspace $\Pi \subset \Gamma_{\mathbb{R}}$ and a lattice isomorphism $\Gamma \cong \Lambda \oplus \Lambda^*$. Recall, that $\Lambda \oplus \Lambda^*$ is endowed with the quadratic form $\| (\alpha, \beta) \|^2 = 2\beta(\alpha)$. Two triples (Γ, Π, ψ) and (Γ', Π', ψ') are equivalent if there exists a lattice isomorphism $\varphi : \Gamma \cong \Gamma'$ such that $\varphi_{\mathbb{R}}(\Pi) = \Pi'$ and $\psi' \circ \varphi = \psi$, where $\varphi_{\mathbb{R}} : \Gamma_{\mathbb{R}} \cong \Gamma'_{\mathbb{R}}$ is the $\mathbb{R}$-linear extension of φ.
Then $\mathcal{M}_d^{Narain}$ is the quotient of $\widetilde{\mathcal{M}}_d^{Narain}$ by the obvious action of the automorphism group of the lattice $\Lambda \oplus \Lambda^*$:

$$\begin{array}{ccccc} \mathrm{O}(\Lambda \oplus \Lambda^*) & \times & \widetilde{\mathcal{M}}_d^{Narain} & \to & \widetilde{\mathcal{M}}_d^{Narain} \\ (\varphi & , & (\Gamma, \Pi, \psi)) & \mapsto & (\Gamma, \Pi, \varphi \circ \psi) \end{array}$$

Using $(\Gamma, \Pi, \psi) \mapsto \psi_{\mathbb{R}}(\Pi)$ one can identify $\widetilde{\mathcal{M}}_d^{Narain}$ with $\mathcal{N}$. Indeed, the map is well-defined and surjective, as $\Pi \in \mathcal{N}$ is the image of $(\Lambda \oplus \Lambda^*, \Pi, \psi = \mathrm{id})$. Since $(\Gamma, \Pi, \psi) \sim (\Lambda \oplus \Lambda^*, \psi_{\mathbb{R}}(\Pi), \mathrm{id})$, the map is also injective. Hence,

$$\boxed{\mathcal{M}_d^{Narain} \cong \mathrm{O}(\Lambda \oplus \Lambda^*) \setminus \mathrm{O}(H \oplus H^*) / \mathrm{O}(\Pi_0) \times \mathrm{O}(\Pi_0^{\perp})}$$

Before comparing the descriptions of $\mathcal{M}_d^{tori}$ and $\mathcal{M}_d^{Narain}$, we study the action of $\mathrm{O}(\Lambda \oplus \Lambda^*)$ on $\mathcal{N}$ and $\mathcal{U}$.
Since $\varphi \in \mathrm{O}(\Lambda \oplus \Lambda^*)$ acts on $\widetilde{\mathcal{M}}_d^{Narain}$ by $(\Gamma, \Pi, \psi) \mapsto (\Gamma, \Pi, \varphi \circ \psi)$, the induced action on $\mathcal{N}$ is given by $\Pi \mapsto \varphi_{\mathbb{R}}(\Pi)$. Something more interesting happens on $\mathcal{U}$. Let $\Pi = \{(\alpha, E(\alpha)) | \alpha \in H\}$ and $\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{O}(\Lambda \oplus \Lambda^*)$. Here $a : \Lambda \to \Lambda$, $d : \Lambda^* \to \Lambda^*$, $b : \Lambda^* \to \Lambda$, and $c : \Lambda \to \Lambda^*$. Then $\varphi_{\mathbb{R}}(\Pi) = \{(a\alpha + bE(\alpha), c\alpha + dE(\alpha)) | \alpha \in H\} = \{((a + bE)(\alpha), (c + dE)(\alpha)) | \alpha \in H\} = \{(\alpha, (c + dE)(a + bE)^{-1}(\alpha)) | \alpha \in H\}$. Thus, the induced action on $\mathcal{U}$ is given by $E \mapsto (c + dE)(a + bE)^{-1}$.

We conclude this section with an alternative moduli description of the Narain moduli space. Let us consider the space of all triples (V, W, Γ), that consist of two euclidean real vector spaces of the same dimension d and a discret subgroup $\Gamma \subset V \oplus W$ of rank $2d$ such that the restriction of $\langle\ ,\ \rangle_V \oplus (-\langle\ ,\ \rangle_W)$ to Γ makes Γ isomorphic to the standard unimodular lattice $U^{\oplus d}$. Two such triples (V, W, Γ), (V', W', Γ') are called equivalent if and only if there exist isometries $\varphi : V \cong V'$ and $\psi : W \cong W'$ with $(\varphi \oplus \psi)(\Gamma) = \Gamma'$. Modulo this equivalence the space of all triples (V, W, Γ) can naturally be identified with the Narain moduli space via the maps $(\Gamma, \Pi) \mapsto (\Pi, \Pi^\perp, \Gamma)$, where $\Pi^\perp$ is endowed with the negative product naturally given on $\Gamma \otimes \mathbb{R}$, and $(V, W, \Gamma) \mapsto (\Gamma, V)$. These maps are clearly inverse to each other. And one easily checks that they are compatible with the two equivalence relations.

4 Comparison of $\mathcal{M}_d^{tori}$ and $\mathcal{M}_d^{Narain}$

The natural inclusion $\mathrm{Gl}(\Lambda) \subset \mathrm{O}(\Lambda \oplus \Lambda^*)$, $\varphi \mapsto \begin{pmatrix} \varphi & 0 \\ 0 & (\varphi^{-1})^t \end{pmatrix}$ is compatible with the two actions on $\mathcal{N}$. Hence there is a natural surjective map

$$\boxed{\mathcal{M}_d^{tori} \twoheadrightarrow \mathcal{M}_d^{Narain}}$$

This map is the quotient by the action of $\mathrm{O}(\Lambda \oplus \Lambda^*)/\mathrm{Gl}(\Lambda)$.
Since for $\mathrm{Gl}(\Lambda)$ is strictly smaller than $\mathrm{O}(\Lambda \oplus \Lambda^*)$, the map is not injective. One particular class of elements in $\mathrm{O}(\Lambda \oplus \Lambda^*)$ is given by 'integral B-fields'. Let $B_0 \in \Lambda^2\Lambda^*$ and consider B_0 as a map $\Lambda \to \Lambda^*$. Then $\begin{pmatrix} \mathrm{id}_\Lambda & 0 \\ B_0 & \mathrm{id}_{\Lambda^*} \end{pmatrix}$ is an element of $\mathrm{O}(\Lambda \oplus \Lambda^*)$. Here one needs that B_0 is skew-symmetric. On $\mathcal{U}$ this automorphism acts by $(G, B) \mapsto (G, B + B_0)$. Identifying $\Lambda^* \cong H^1(T, \mathbb{Z})$ and $H^* \cong H^1(T, \mathbb{R})$ and using $H^2(T, \mathbb{R}) = \Lambda^2 H^1(T, \mathbb{R})$ this corresponds to the way the Kähler cone $\mathcal{K}_X \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$ of an arbitrary Calabi-Yau manifold is complexified by incorporating B-fields to $(H^2(X, \mathbb{R}) + i\mathcal{K}_X)/H^2(X, \mathbb{Z})$.
If g is an isomorphism $\Lambda \cong \Lambda^*$, then $\begin{pmatrix} 0 & (g^{-1})^t \\ g & 0 \end{pmatrix}$ belongs to another class of automorphisms of the lattice $\Lambda \oplus \Lambda^*$ that is not contained in $\mathrm{Gl}(\Lambda)$. In the subsequent two sections we will describe these two classes of elements and their action in more detail.

5 Charge lattices

It will turn out that taking the quotient by integral shifts of the B-field describes the map that associates to a torus its charge lattice.
Let $E = G + B \in \mathcal{U}$, then the *charge lattice* associated to E is the subgroup $\Gamma_{G+B} := \Phi_{G+B}(\Lambda \oplus \Lambda^*) \subset H^* \oplus H^*$ together with the quadratic form induced by the morphism $\Phi_{G+B} : \Lambda \oplus \Lambda^* \cong \Gamma_{G+B}$. Here $\Phi_{G+B} : H \oplus H^* \to H^* \oplus H^*$ is the map $(\alpha, \beta) \mapsto (1/\sqrt{2})(\beta + G(\alpha) - B(\alpha), \beta - G(\alpha) - B(\alpha))$.

There is a number of useful facts about this map:
i) If $H^* \oplus H^*$ is endowed with the quadratic form $G^* \oplus (-G^*)$, then Φ_{G+B} is an isometry.
ii) If $\Pi \subset H \oplus H^*$ is the space-like d-space associated with (G, B), then Φ_{G+B} is compatible with the decompositions $\Pi \oplus \Pi^\perp$ and $H^* \oplus H^*$.

Proof. That Φ_{G+B} is an isometry can be best seen by decomposing it into three easier isometries:

- $H \oplus H^* \to H \oplus H^*$, $(\alpha, \beta) \mapsto (\alpha, \beta - B(\alpha))$, where both sides have the canonical hyperbolic quadratic form.
- $H \oplus H^* \to H^* \oplus H^*$, $(\alpha, \beta) \mapsto (G(\alpha), \beta)$, where on the right hand side $\| (\beta_1, \beta_2) \|^2 := 2G^*(\beta_1, \beta_2)$.
- $H^* \oplus H^* \to H^* \oplus H^*$, $(\beta_1, \beta_2) \mapsto (1/\sqrt{2})(\beta_1 + \beta_2, \beta_2 - \beta_1)$, where on the right hand side $\| (\beta_1, \beta_2) \|^2 := G^*(\beta_1, \beta_1) - G^*(\beta_2, \beta_2)$.

The second assertion can be seen as follows. If $\Pi = \{(\alpha, G(\alpha) + B(\alpha)) | \alpha \in H\}$, then $\Phi_{G+B}(\Pi) = \{(G(\alpha) + B(\alpha) + G(\alpha) - B(\alpha), G(\alpha) + B(\alpha) - G(\alpha) - B(\alpha)) | \alpha \in H\} = \{(G(\alpha), 0) | \alpha \in H\} = H^* \oplus \{0\}$. $\square$

The first assertion says that the quadratic form on Γ_{G+B} can be extended linearly to the quadratic form $G^* \oplus (-G^*)$ on $H^* \oplus H^*$.

The integral shifts of the B-field come in by the following fact:
iii) The charge lattices Γ_{G+B} and $\Gamma_{G'+B'}$ coincide if and only if $G = G'$ and $B - B' \in \Lambda^2\Lambda^*$.

Proof. Note that equality of the charge lattices means equality of the subgroups of $H^* \oplus H^*$ and equality of the quadratic forms. Hence, if $\Gamma_{G+B} = \Gamma_{G'+B'}$, then $G = G'$. Let $\alpha \in \Lambda$. Then there exist $\alpha' \in \Lambda$ and $\beta' \in \Lambda^*$ such that $\Phi_{G+B}(\alpha, 0) = \Phi_{G'+B'}(\alpha', \beta')$. Hence, $(G(\alpha) - B(\alpha), -G(\alpha) - B(\alpha)) = (\beta' + G(\alpha') - B'(\alpha'), \beta' - G(\alpha') - B'(\alpha'))$. This yields $G(\alpha') = G(\alpha)$, i.e. $\alpha = \alpha'$, and $B(\alpha) = B'(\alpha) - \beta'$. The latter implies $B - B' \in \Lambda^2\Lambda^*$.
Conversely, any B can be changed by $B_0 \in \Lambda^2\Lambda^*$ without changing the charge lattice. $\square$

iv) Let $\varphi \in \mathrm{Gl}(\Lambda)$. Then $\Gamma_{(\varphi^{-1})^*(G+B)} = \begin{pmatrix} (\varphi^{-1})^t & \\ 0 & (\varphi^{-1})^t \end{pmatrix} \Gamma_{G+B}$.

Proof. $\Gamma_{(\varphi^{-1})^*(G+B)}$
$= \frac{1}{\sqrt{2}}\{(\beta + (\varphi^{-1})^*G(\alpha) - (\varphi^{-1})^*B(\alpha), \beta - (\varphi^{-1})^*G(\alpha) - (\varphi^{-1})^*B(\alpha)) | \alpha \in \Lambda, \beta \in \Lambda^*\}$
$= \frac{1}{\sqrt{2}}\{((\varphi^{-1})^t\beta + (\varphi^{-1})^tG\alpha - (\varphi^{-1})^tB\alpha, (\varphi^{-1})^t\beta - (\varphi^{-1})^tG(\alpha) - (\varphi^{-1})^tB(\alpha)) | \alpha \in \Lambda, \beta \in \Lambda^*\}$
$= \begin{pmatrix} (\varphi^{-1})^t & 0 \\ 0 & (\varphi^{-1})^t \end{pmatrix} \Gamma_{G+B}$ $\square$

In particular, this formula shows that the $\mathrm{Gl}(\Lambda)$-action on $\mathcal{U}$ descends to an action on the set of all charge lattices. We say that Γ_{G+B} and $\Gamma_{(\varphi^{-1})^*(G+B)}$ are equivalent charge lattices and define the moduli space of charge lattices as

$$\boxed{\mathcal{M}_d^{cl} := \{\Gamma_{G+B} | G+B \in \mathcal{U}\}/\sim}$$

Hence, the map $\mathcal{M}_d^{tori} \twoheadrightarrow \mathcal{M}_d^{Narain}$ factorizes naturally over $\mathcal{M}_d^{cl}$, i.e. one has a commutative diagram

$$\begin{tikzcd} \mathcal{M}_d^{tori} \arrow[rr, two heads] \arrow[dr] & & \mathcal{M}_d^{Narain} \\ & \mathcal{M}_d^{cl} \arrow[ur] & \end{tikzcd}$$

Here, $\mathcal{M}_d^{tori} \twoheadrightarrow \mathcal{M}_d^{cl}$ divides by integral shifts of the B-field and $\mathcal{M}_d^{cl} \twoheadrightarrow \mathcal{M}_d^{Narain}$ corresponds to T-duality which will be explained in the next section.
The moduli space $\mathcal{M}_d^{cl}$ has only been introduced in order to separate integral shifts of the B-field from T-duality. As a moduli space itself it seems not very meaningful.

6 T-Duality

Let us begin with the one-dimensional case, i.e. $d = 1$. Thus we are looking at circles and the metric G is determined by the radius $R \in \mathbb{R}_{>0}$. Clearly, in this situation there are no B-fields. If we fix isomorphisms $\Lambda \cong \Lambda^* \cong \mathbb{Z}$, then any element $\varphi \in \mathrm{O}(\Lambda \oplus \Lambda^*)$ can be written as $\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$. In this case $\mathrm{Gl}(\Lambda) = \{\pm 1\} = \{\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\} \subset \mathrm{O}(\Lambda \oplus \Lambda^*)$. One easily checks that $\mathrm{O}(\Lambda \oplus \Lambda^*)$ contains two more elements, namely $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$. So the only 'T-duality' in this situation is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This matrix acts on G by $G \mapsto G^{-1}$, i.e. under $\mathcal{M}_1^{tori} \cong \mathcal{M}_1^{cl} \twoheadrightarrow \mathcal{M}_1^{Narain}$ the circles of radius R and of radius $1/(4\pi^2 R)$ are identified. This is the so-called $R \to 1/(4\pi^2 R)$ circle duality. Indeed, the map $f : \mathbb{R} \to \mathbb{R}^2, t \mapsto Re^{2\pi i t}$ identifies the quotient $\mathbb{R}/\mathbb{Z}$ with a circle of radius R. This map induces an isometrie of the tangent spaces, where G is a fixed scalar product on $\mathbb{R}$ considered as a number such that 1 has length $\sqrt{G}$, if and only if $(df/dt)_{t=0}(1) = 2\pi R(0,1)$ is a vector of length $\sqrt{G}$. Thus, $2\pi R = \sqrt{G}$. Hence, the involution $G \mapsto G^{-1}$ corresponds to $R \mapsto 1/(4\pi^2 R)$.
To understand the higher dimensional case let us identify $\Lambda \cong \mathbb{Z}^d$. In addition to $\mathrm{Gl}(\Lambda)$ and the integral shifts of the B-field one has the following matrices in $\mathrm{O}(\Lambda \oplus \Lambda^*)$:

$$\varphi_i := \begin{pmatrix} \mathrm{id} - e_i & e_i \\ e_i & \mathrm{id} - e_i \end{pmatrix},$$

where the entries in e_i are zero except for the (i,i)-component which is one. For $d = 1$ this gives back the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The action of φ_i on $\mathcal{U}$ generalizes the $R \to 1/R$ duality to higher dimensions. This can be explicitly seen in the case that the torus $(T, G, 0)$ is the product of circles with radii $R_1, \ldots, R_d$. Then φ_i maps $(T, G, 0)$ to the product of circles of radii $R_1, \ldots, R_{i-1}, 1/R_i, R_{i+1}, \ldots R_d$. The proof of the following assertion should be a matter of linear algebra, but I have not worked out the details [1]:

The group $\mathrm{O}(\Lambda \oplus \Lambda^*)$ is generated by $\mathrm{Gl}(\Lambda)$, $\Lambda^2\Lambda^*$, and the matrices φ_i.

7 World sheet parity

There is yet another 'symmetry' that seems very natural. Namely, $(T, G, B) \mapsto (T, G, -B)$. The fixed points of this symmetry are the tori with vanishing B-field. In general, (T, G, B) and $(T, G, -B)$ will give different CFT's, i.e. different points in the Narain moduli space. On the charge lattices this induces the following action

$$\Gamma_{G+B} \mapsto \Gamma_{G-B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\Gamma_{G+B}).$$

Indeed, $\Gamma_{G-B} = (1/\sqrt{2})\{(\beta + G(\alpha) + B(\alpha), \beta - G(\alpha) + B(\alpha)) | \alpha \in \Lambda, \beta \in \Lambda^*\} = (1/\sqrt{2})\{(\beta - G(\alpha) - B(\alpha), \beta + G(\alpha) - B(\alpha)) | \alpha \in \Lambda, \beta \in \Lambda^*\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\Gamma_{G+B})$.
Hence, the coordinates p_L and p_R of $\Gamma \subset H^* \oplus H^*$ are interchanged. Here, p_L and p_R denote the two projection $H^* \oplus H^* \to H^*$. On the level of the Narain moduli space, this yields the action $(\Gamma, \Pi) \mapsto (-\Gamma, \Pi^\perp)$, where $-\Gamma$ means Γ with the negative quadratic form.

References

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