Framed modules and their moduli

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This paper is a continuation of our previous work [HL]. There we discussed pairs consisting of a coherent sheaf and a homomorphism to a fixed reference sheaf. This notion of a pair is wide enough to comprise various other notions of pairs as in [Se, Le, Th, Ga, Be, BDW, Lü, BD]: these differ essentially in the choice of the reference sheaf only. Despite, or rather, because of the abundance of existing names for these objects (framed vector bundles, bundles with level structure, stable Higgs pairs, k-tuples), and in order to avoid the somewhat meaningless term 'pair', we chose the name 'framed module' for the objects discussed in this paper. In [HL] we gave a stability condition for framed modules, which depends on a polynomial valued parameter δ. We constructed moduli spaces for framed modules that are defined over curves or surfaces, but failed to do so for higher dimensional varieties, mainly because of difficulties with proving boundedness properties of the involved families of sheaves. In this paper we use techniques of C. Simpson to deal with the problem in full generality. As a supplement to the construction we give a description of the compactification of the moduli space of stable pairs and its tangent space and prove a smoothness criterion.

Let $(X, \mathcal{O}_X(1))$ denote a smooth projective variety, D a coherent $\mathcal{O}_X$-sheaf and δ a rational polynomial with positive leading coefficient. A framed module is a pair consisting of a coherent sheaf E and a nonzero homomorphism $\alpha : E \to D$. A framed module of positive rank r is said to be stable (with respect to δ), if for all nontrivial proper submodules E' of rank r' the Hilbert polynomials of E and E' satisfy the following conditions: $P_{E'} < \frac{r'}{r}P_E + \frac{r-r'}{r}\delta$, and $P_{E'} < \frac{r'}{r}P_E - \frac{r'}{r}\delta$ if $E' \subset \ker(\alpha)$. A flat family of framed modules, parametrized by a scheme T of finite type over the ground field k, is a T-flat coherent $\mathcal{O}_{T\times X}$-sheaf E together with a homomorphism $\alpha : E \to \mathcal{O}_T \otimes D$ such that $\alpha_t \neq 0$ for all $t \in T$. We call $P_E - \delta$ the 'Hilbert polynomial' of the framed module (E, α).

The main theorem then is this:

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Theorem 0.1 — *Let $\delta \in \mathbf{Q}[m]$ be a polynomial with positive leading coefficient and of degree $< \dim(X)$. There is a projective scheme $\mathcal{M}_\delta^{ss}(X; D, P)$ which is a coarse moduli space for the functor which associates to a scheme T the set of isomorphism classes of flat families of semistable framed modules defined over T with Hilbert polynomial P. Moreover, there is an open subscheme $\mathcal{M}_\delta^{s}(X; D, P)$ which represents the subfunctor of families of stable framed modules, i.e. $\mathcal{M}_\delta^{s}(X; D, P)$ is a fine moduli space. A closed point in $\mathcal{M}_\delta^{ss}(X; D, P)$ represents an S-equivalence class of semistable framed modules.*

For the notion of S-equivalence see 1.14.

The paper is organized as follows: Section 1 contains the basic definitions and properties of framed modules. In section 2 we prove the boundedness result which is needed in the construction. Section 3 contains the construction of the moduli schemes by means of geometric invariant theory and the proof of the main theorem. Finally, section 4 discusses the infinitesimal deformation theory of framed modules.

Throughout the paper we will use the following convention: If the word '(semi)-stable' occurs in any statement in combination with the symbol '$(\leq)$', then in fact two variants of the statement are asserted at the same time: A 'semistable' one involving the relation '$\leq$', and a 'stable' one involving the relation '$<$'. This allows a more concise presentation and will hopefully not lead to confusion.

1 Semistable framed modules

Let X be a nonsingular projective variety defined over an algebraically closed field k of characteristic zero endowed with a very ample line bundle $\mathcal{O}_X(1)$. We denote by d the dimension of X and by g its degree with respect to the embedding given by $\mathcal{O}_X(1)$. Let D be a coherent $\mathcal{O}_X$-module and δ a polynomial with rational coefficients and positive leading term. Then a *framed module* is a pair consisting of a coherent $\mathcal{O}_X$-module E and a homomorphism $\alpha : E \to D$, called the framing of E. We refer to $\ker(\alpha)$ as the kernel of the framed module. Let $\varepsilon(\alpha) = 1$, if $\alpha \neq 0$, and equal to 0 else. We denote by $P_E(n) = \chi(E(n))$ the Hilbert polynomial of E and by $P_{(E,\alpha)} = P_E - \varepsilon(\alpha)\delta$ the *Hilbert polynomial of the pair* (E, α). Similarly, we put $h^0((E, \alpha)(m)) = h^0(E(m)) - \varepsilon(\alpha)\delta(m)$.

If E' is a coherent submodule of E with quotient $E'' = E/E'$, then a framing $\alpha : E \to D$ induces framings $\alpha' = \alpha|E'$ on E' and α'' on E'': α'' is 0, if $\alpha' \neq 0$, and is the induced homomorphism on E'', if α' vanishes. E' is said to be saturated if (E'', α'') has torsion free kernel. With this convention the Hilbert polynomial of framed modules behaves additively

$$P_{(E,\alpha)} = P_{(E',\alpha')} + P_{(E'',\alpha'')}.$$

Definition 1.1 — *A framed module (E, α) of rank r is said to be (semi)stable with respect to δ with reduced Hilbert polynomial p, if $P_{(E,\alpha)} = rp$ and for all submodules E', $0 \neq E' \neq E$, of rank r' with induced framing α' the inequality $P_{(E',\alpha')}(\leq)r'p$ holds.*

The ring of polynomials with rational coefficients is given the lexicographic order. Here are some immediate consequences of the definition:

Lemma 1.2 — *If (E, α) is semistable, then its kernel is torsion free, i.e. α embeds the torsion $T(E)$ of E as a submodule of D.*

Proof. If T is the torsion part of the kernel of α, then in the inequality of the definition $r' = 0$ and $\alpha' = 0$, so that $P_{(T,\alpha')} = P_T$ and the inequality reads: $rP_T \leq 0$, which implies $T = 0$. □

Lemma 1.3 — *Suppose E is a torsion module. If (E, α) is semistable, then it is already stable, which in turn is equivalent to the assertion that α is injective and $P_E = \delta$.*

Proof. Semistability for a nontrivial torsion module requires $P_{(E,\alpha)} = P_E - \delta = 0$. □

Definition 1.4 — *A homomorphism $\varphi : (E, \alpha) \to (E', \alpha')$ of framed modules is a homomorphism of the underlying modules $\varphi : E \to E'$ for which there is an element $\lambda \in k$ such that $\alpha' \circ \varphi = \lambda\alpha$.*

Lemma 1.5 — *The set* $\mathrm{Hom}((E, \alpha), (E', \alpha'))$ *of homomorphisms of framed modules is a linear subspace of* $\mathrm{Hom}(E, E')$. *If $\varphi : (E, \alpha) \to (E', \alpha')$ is an isomorphism, then the factor λ in the definition can be taken in k^*. In particular, the isomorphism $\varphi_0 := \lambda^{-1}\varphi$ satisfies $\alpha' \circ \varphi_0 = \alpha$.* □

Lemma 1.6 — *If (E, α) and (E', α') are stable with the same reduced Hilbert polynomial p, then any nontrivial homomorphism $\varphi : (E, \alpha) \to (E', \alpha')$ is an isomorphism. Moreover, in this case* $\mathrm{Hom}((E, \alpha), (E', \alpha')) \cong k$. *If in addition $\alpha \neq 0$, or equivalently, $\alpha' \neq 0$, then there is a unique isomorphism φ_0 with $\alpha' \circ \varphi_0 = \alpha$.*

Proof. Suppose $\varphi : (E, \alpha) \to (E', \alpha')$ is nontrivial. The image $F := \mathrm{im}(\varphi)$ inherits framings β and β' when considered as a quotient of E and as a submodule of E', respectively. If $\beta' \neq 0$ then $\beta \neq 0$ and $\beta' = \lambda\beta$ for some $\lambda \neq 0$. In any case one has:

$$\mathrm{rk}(F)p \leq P_{(F,\beta)} \leq P_{(F,\beta')} \leq \mathrm{rk}(F)p.$$

Therefore equality holds at all places. This implies because of the stability assumptions: $E \cong F \cong E'$, $\alpha = \beta \circ \varphi$, $\beta' = \alpha'$, and β and β' differ by a nontrivial factor. Hence φ is an isomorphism of framed modules. In order to prove the remaining statements it is enough to show $\mathrm{Aut}(E, \alpha) = k \cdot \mathrm{id}_E$. Suppose φ is an automorphism of (E, α). Choose $x \in \mathrm{Supp}(E)$ and let μ be an eigenvalue of φ restricted to the fibre $E(x)$. Then $\varphi - \mu \cdot \mathrm{id}_E$ is not surjective at x and hence not an isomorphism, which implies $\varphi - \mu \cdot \mathrm{id}_E = 0$. $\square$

Lemma 1.7 — *If $\deg(\delta) \geq d$, then in any semistable framed module (E, α) the framing α is injective or zero. Conversely, if α is the inclusion homomorphism of a submodule E of D of positive rank, then (E, α) is stable.*

Proof. Assume that $\alpha \neq 0$. If E' is any nontrivial submodule of the kernel of α, then the semistability of (E, α) says:

$$rP_{E'} - r'P_E \leq -r'\delta.$$

The two polynomials on the left hand side are of degree d and have the same leading coefficient. If $\deg(\delta) \geq d$, this yields a contradiction. Similarly, if α is injective, the inequality of the definition is strictly satisfied because of the dominance of δ. $\square$

The last lemma shows that the discussion of semistable framed modules reduces to the study of submodules of D, which is covered by Grothendieck's theory of the Hilbert scheme, if $\deg(\delta) \geq d$. For that reason we assume henceforth that δ has degree less than d and write:

$$\delta(m) = \delta_1 \frac{m^{d-1}}{(d-1)!} + \delta_2 \frac{m^{d-2}}{(d-2)!} + \ldots + \delta_d,$$

the first nonzero coefficient being positive.

By the assumption on δ the reduced Hilbert polynomial $P_{(E,\alpha)}/r$ of any nontorsion framed module (E, α) has the same leading coefficient $\deg(X)/d!$. Hence the dominating terms in the stability inequality are the degrees of the modules. This leads to a linearized stability definition which is related to the one given above in the same way as the Mumford-Takemoto stability is related to that of Gieseker and Maruyama. Let

$$\mu(E, \alpha) = (\deg(E) - \varepsilon(\alpha)\delta_1)/\mathrm{rk}(E).$$

Definition 1.8 — *A framed module (E, α) of positive rank r is said to be μ-(semi)stable with respect to δ_1, if it has torsion free kernel and if for all submodules E' with induced framing α' and rank r' satisfying $0 < r' < r$, one has $\mu(E', \alpha')(\leq)\mu(E, \alpha)$.*

Obviously one has the following implications between properties of a framed module of positive rank:

$$\mu\text{-stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu\text{-semistable}.$$

Remark 1.9 — In fact, even stable framed modules may contain torsion submodules. The following simple trick allows to make use of results about torsion free sheaves in the study of framed modules: Choose once and for all a fixed locally free coherent module $\hat{D}$ and a surjective homomorphism $\varphi : \hat{D} \to D$, its kernel being denoted by B. Then to each framed module (E, α) we can associate a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \uparrow & & \uparrow \\
0 & \to & \ker(\alpha) & \to & E & \to & D \\
 & & \| & & \uparrow & & \uparrow \\
0 & \to & \ker(\alpha) & \to & \hat{E} & \to & \hat{D} \\
 & & & & \uparrow & & \uparrow \\
 & & & & B & = & B \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
\end{array}$$

The second row of the diagram shows that $\hat{E}$ is torsion free if the kernel of α is torsion free, hence in particular if (E, α) is μ-semistable. This construction works as well with a flat family (E_T, α_T) of framed modules parametrized by a noetherian k-scheme T if we replace D by $D_T = D \otimes \mathcal{O}_T$. Then $\hat{E}_T$ is again flat, and for every point $t \in T$ the kernel of α_t is torsion free if and only if $\hat{E}_t$ is torsion free. By [M1, Prop. 2.1] torsion freeness is an open property and we get as a corollary:

Corollary 1.10 — *If (E_T, α_T) is a flat family of framed modules, parametrized by a noetherian k-scheme T, then the subset of points t of T for which $\ker(\alpha_t)$ is torsion free is open in T.* □

Lemma 1.11 — *If (E, α) is a framed module that can be deformed to a framed module with torsion free kernel, then there is a morphism $\varphi : (E, \alpha) \to (G, \beta)$ of framed modules, such that (G, β) has torsion free kernel, $P_E = P_G$, $P_{(E,\alpha)} = P_{(G,\beta)}$, and finally $\ker(\varphi) = T(\ker(\alpha))$.*

This lemma logically corresponds to Lemma 4.2 in [Gi] and Lemma 1.17 in [Si]. Indeed, if $\alpha = 0$, the lemma of Simpson provides us with the required homomorphism. We make use of the trick above and follow Simpson's proof closely:

Proof. The assumption in the lemma means that there is a flat family (E_T, α_T) of framed modules with the same Hilbert polynomial, parametrized by a smooth curve T, such that $(E_0, \alpha_0) \cong (E, \alpha)$ for some closed point $0 \in T$ and such that (E_t, α_t) has torsion free kernel for all $t \neq 0$. We may assume that $\alpha_t \neq 0$ for all $t \in T$. Let $\hat{\alpha}_T : \hat{E}_T \to \hat{D}_T = \hat{D} \otimes \mathcal{O}_T$ be the flat family associated to (E_T, α_T) by the process above. Then $\hat{E}_t$ is torsion free for all $t \in T$ except $t = 0$. Let $U \subset T \times X$ denote the complement of the support Y of $T(\ker(\alpha_0)) = T(\hat{E}_0)$ and $j : U \to T \times X$ the inclusion morphism. Let $E' := j_*(\hat{E}_T|_U)$. Then E' contains no T-torsion and therefore is T-flat. In particular, the fibre E'_0 has the same Hilbert polynomial as $\hat{E}_0$. The canonical homomorphism $\hat{E}_T \to E'$ induces a homomorphism $\hat{\varphi} : \hat{E}_0 \to E'_0$, which is an isomorphism outside Y. Since $\hat{D}_T$ is normal, $\hat{\alpha}_T$ defines a framing $\alpha' : E' \to \hat{D}_T$ which coincides with $\hat{\alpha}_T$ on U. As in the proof of Simpson's lemma, E'_0 is torsion free. Finally, B maps injectively to E'_0, and setting $G = E'_0/B$ with the induced homomorphisms $\varphi : E_0 \to G$ and $\beta : G \to D$, we are done. □

In analogy to the study of torsion free semistable coherent sheaves we will define Jordan-Hölder filtrations and the notion of S-equivalence for framed modules. We begin with the following observation:

Lemma 1.12 — *Let $F \subset G \subset E$ be coherent modules and α a framing of E. Then the framings induced on G/F as a quotient of G and as a submodule of E/F agree.*

Proof. If $\alpha|_F = 0$, then all framings of the modules in the commutative diagram

$$\begin{array}{ccc} G & \longrightarrow & E \\ \downarrow & & \downarrow \\ G/F & \longrightarrow & E/F \end{array}$$

are induced by a framing of E/F. If $\alpha|_F \neq 0$, then both the framings of G/F (as a quotient of G) and of E/F are zero, so that again there is no ambiguity. □

This lemma allows to endow any subquotient of a framed module with a canonical framing.

Proposition 1.13 — *Let (E, α) be a semistable framed module with reduced Hilbert polynomial p. Then there is a filtration*

$$E_\bullet : \qquad 0 = E_0 \subset E_1 \subset \ldots \subset E_s = E$$

such that all the factors $gr_i(E_\bullet) := E_i/E_{i-1}$ together with the induced framings α_i are stable with respect to δ with reduced Hilbert polynomial p. Any such filtration is called a Jordan-Hölder filtration of (E, α). The framed module

$$(gr(E), gr(\alpha)) := \bigoplus_i (gr_i(E_\bullet), \alpha_i)$$

does not depend on the choice of the Jordan-Hölder filtration.

Proof. If (E, α) is not stable, then there is a proper submodule (E', α') with reduced Hilbert polynomial p, i.e. $P_{(E',\alpha')} = \mathrm{rk}(E')p$. Let (E', α') be maximal with this property. Then (E', α') is semistable and E/E' with the induced framing is stable. Inductively, we can construct a descending sequence of submodules such that the factors are stable with reduced Hilbert polynomial p. Note that at most one of these factors carries a nonzero framing. In particular all but possibly one of the factors are torsion free. For rank reasons the descending sequence must be finite. This gives the existence of a Jordan-Hölder filtration. Now suppose $E_\bullet$ and $E'_\bullet$ are two such filtrations. Let j be the smallest index such that $E_1 \subset E'_j$. Then the canonical homomorphism

$$\varphi : E_1 \longrightarrow E'_j \longrightarrow E'_j/E'_{j-1}$$

is nontrivial and is in fact a homomorphism of framed modules. Now $(gr_1(E_\bullet), \alpha_1)$ and $(gr_j(E'_\bullet), \alpha'_j)$ are stable. Hence by lemma 1.6 φ is an isomorphism of framed modules. Moreover, there is a short exact sequence of framed modules

$$0 \to (E'_{j-1}, \alpha) \to (E/E_1, \alpha) \to (E/E'_j, \alpha) \to 0,$$

abusing α as a generic notation for the induced framings. The filtrations of E/E'_j and E'_{j-1} give rise to a filtration of E/E_1, whose graded object by induction on the rank of E is isomorphic to the graded object of the filtration $E_\bullet/E_1$. □

Definition 1.14 — *Two semistable framed modules (E, α) and (E', α') with reduced Hilbert polynomial p are called S-equivalent, if their associated graded objects $(gr(E), gr(\alpha))$ and $(gr(E'), gr(\alpha'))$ are isomorphic.*

Obviously, if an S-equivalence class contains a stable framed module then it contains no other modules.

2 Boundedness

The first step in the construction of moduli spaces of semistable framed modules is to get a boundedness result for the family of semistable framed modules. In fact the application of the Geometric Invariant Theory machinery in the following section requires a slightly different notion of stability. We shall prove boundedness and equivalence of the various notions at the same time.

Throughout this section let P be a polynomial and let $r > 0$ and μ_P be the rank and the slope of any coherent $\mathcal{O}_X$-module with Hilbert polynomial P.

Theorem 2.1 — *There is an integer m_0 such that the following three properties of a framed module (E, α) with Hilbert polynomial P and torsion free kernel are equivalent for all $m \geq m_0$:*

i) (E,α) *is (semi)stable.*

ii) $P(m) \leq h^0((E,\alpha)(m))$ *and* $h^0((E',\alpha')(m))(\leq) r'P(m)/r$ *for all submodules* (E',α') *of rank* r', $0 \neq E' \neq E$.

iii) $h^0((E'',\alpha'')(m))(\geq) r''P(m)/r$ *for all quotient modules* (E'',α'') *of rank* r'', $E \neq E'' \neq 0$.

Moreover, for any framed module satisfying these conditions, E *is* m*-regular.*

The families of framed modules having torsion free kernel and satisfying the weak version of one of the conditions i) – iii) are denoted by $\mathcal{S}^s$, $\mathcal{S}'_m$ and $\mathcal{S}''_m$, respectively.

We shall prove this theorem by reducing it to well-known results in the theory of semistable torsion free coherent sheaves. In particular we will need the following results due to Maruyama and Simpson:

Lemma 2.2 (Simpson) [Si, Lemma1.5]— *Let* r *be a positive integer. Then there is a positive constant* c *such that for every* μ*-semistable coherent* $\mathcal{O}_X$*-module* F *of positive rank* $< r$ *and slope* μ *one has*

$$\frac{h^0(F)}{\mathrm{rk}(F)} \leq \frac{1}{g^{d-1}d!}([\mu + c]_+)^d,$$

where $[x]_+ = \max\{x, 0\}$ *for any real number* x. □

See also [LeP, lemme 2.4]. Please note the difference in the notation: In Le Potier's paper the slope μ of a d-dimensional coherent sheaf is defined as the quotient b/a where

$$P(n) = a\binom{n+d-1}{d} + b\binom{n+d-2}{d-1} + \ldots$$

is the Hilbert polynomial. Even though this makes computations more elegant, we stick to the conventional definition that the slope is the quotient of degree by rank.

If F is no longer μ-semistable, let $0 \subset F_1 \subset \ldots \subset F_s = F$ be the Harder-Narasimhan filtration of F with μ-semistable factors $G_i = F_i/F_{i-1}$ of rank r_i. Then $h^0(F) \leq \sum_i h^0(G_i)$, and if one applies Simpson's formula to each of the factors G_i one gets:

$$\frac{h^0(F)}{\mathrm{rk}(F)} \leq \frac{1}{g^{d-1}d!}\left((1 - \frac{1}{\mathrm{rk}(F)})([\mu_{\max}(F) + c]_+)^d + \frac{1}{\mathrm{rk}(F)}([\mu_{\min}(F) + c]_+)^d\right).$$

Here $\mu_{\max}(F)$ and $\mu_{\min}(F)$ denote the maximum and minimum value of $\mu(G_i)$.

Theorem 2.3 (Maruyama) [M3]— *Let P be a polynomial and C a constant. Then the family of torsion free coherent $\mathcal{O}_X$-modules F with Hilbert polynomial $P_F = P$ and $\mu_{\max}(F) \le C$ is bounded.* $\square$

Lemma 2.4 — *There are integers C and m_1 such that for all framed modules (E,α) in the family $\mathcal{S} = \mathcal{S}^s \cup \bigcup_{m\ge m_1} \mathcal{S}''_m$ and for all saturated submodules (E',α') the following holds: $\deg(E') - r'\mu_P \le C$, and either $-C \le \deg(E') - r'\mu_P$ or*

$$\begin{array}{ll} h^0((E',\alpha')(m)) < r'P(m)/r & \text{, if } (E,\alpha) \text{ is in } \mathcal{S}^s \text{ and } m \ge m_1\text{; and} \\ r''P/r < P_{(E'',\alpha'')} & \text{, if } (E,\alpha) \text{ is in } \mathcal{S}''_m \text{ for some } m \ge m_1. \end{array}$$

Here r' and r'' denote the rank of E' and $E'' = E/E'$, respectively, as usual.

Proof. Let c denote the same constant as in lemma 2.2 and L the coefficient of $m^{d-1}/(d-1)!$ in $P(m)/r$. Choose C sufficiently large so that $C > \delta_1 + \deg(T(D))$ and

$$\mu_P + c + (1 - \frac{1}{r})\delta_1 - \frac{1}{r^2}C + \deg(T(D)) - L < 0.$$

Up to the factor $(d-1)!$ this is just the leading coefficient of the polynomial $G + P_{T(D)} - P/r$, where

$$G(m) = \frac{1}{g^{d-1}d!}\left((1-\frac{1}{r})(mg + \mu_P + \delta_1 + c)^d + \frac{1}{r}(mg + \mu_P - \frac{1}{r}C + c)^d\right).$$

Let m_1 be an integer large enough so as to satisfy the following conditions: $T(D)$ is m_1-regular,

$$m_1 g + \mu_P + c - \frac{C}{r} > 0$$

and for all $m \ge m_1$ one has $G(m) + P_{T(D)}(m) < P(m)/r$ and $\delta(m) \ge 0$. Now let (E,α) be a framed module in either of the families $\mathcal{S}^s$ or $\mathcal{S}''_m$, $m \ge m_1$, and assume that (E',α') is a saturated submodule of rank r'. Then any torsion of E' or E'' is embedded into $T(D)$ by α' and α'', respectively. If $r' = 0$, then $0 \le \deg(E') \le \deg(T(D)) \le C$; if $r'' = 0$, then

$$r'\mu_P - C < \deg(E) - \deg(T(D)) \le \deg(E') \le \deg(E) \le r'\mu_P + C.$$

Thus we can restrict ourselves to the case $0 < r' < r$.

a) Suppose (E,α) is μ-semistable. Then by definition $\deg(E') \le r'\mu_P + \delta_1 \le r'\mu_P + C$ and $\mu_{\max}(E'/T(E')) \le \mu_P + \delta_1$. If E' fails to satisfy $-C \le \deg(E') - r'\mu_P$, then there is also a bound for the minimal slope: $\mu_{\min}(E'/T(E')) \le \mu_P - C/r$. We have $h^0((E',\alpha')(m)) \le h^0(E'(m)) \le h^0(T(E')(m)) + h^0(E'/T(E')(m))$. The first term can be roughly bounded by $h^0(T(D)(m)) = P_{T(D)}(m)$, the second one by iterated application of lemma 2.2 to the factors of a Harder-Narasimhan filtration of $E'/(T(E'))$:

$$\begin{aligned}\frac{h^0((E'/T(E'))(m))}{r'} &\leq \frac{1}{g^{d-1}d!}\left((1-\frac{1}{r'})([\mu_{\max}+mg+c]_+)^d+\frac{1}{r'}([\mu_{\min}+mg+c]_+)^d\right)\\ &\leq \frac{1}{g^{d-1}d!}\left((1-\frac{1}{r})([\mu_P+\delta_1+mg+c]_+)^d+\frac{1}{r}([\mu_P-\frac{C}{r}+mg+c]_+)^d\right)\\ &= G(m)\end{aligned}$$

for all $m \geq m_1$. It follows: $h^0((E',\alpha')(m))/r' \leq G(m)+P_{T(D)}(m) < P(m)/r$.

b) Suppose now that (E,α) belongs to $\mathcal{S}''_m$. Let $\mu_{\max}$ and $\mu_{\min}$ denote maximal and minimal slope of $E''/T(E'')$. Then we have for all $m \geq m_1$

$$\begin{aligned}G(m) &< \frac{P(m)}{r}-P_{T(D)}(m)\\ &\leq \frac{h^0((E'',\alpha'')(m))}{r''}-\frac{h^0(T(D)(m))}{r''} \leq \frac{h^0((E''/T(E''))(m))}{r''}\\ &\leq \frac{1}{g^{d-1}d!}\left((1-\frac{1}{r''})([\mu_{\max}+mg+c]_+)^d+\frac{1}{r''}([\mu_{\min}+mg+c]_+)^d\right).\end{aligned}$$

This must hold in particular, if E'' is replaced by the last factor of the Harder-Narasimhan filtration of $E''/T(E'')$, showing that $\mu_{\min} \geq \mu_P+(1-\frac{1}{r})\delta_1-\frac{C}{r^2}$. From this one infers:

$$\begin{aligned}\deg(E') &= \deg(E)-\deg(E'')\\ &\leq r\mu_P+\delta_1-r''\left(\mu_P+(1-\frac{1}{r})\delta_1-\frac{C}{r^2}\right)\\ &\leq r'\mu_P+\frac{\delta_1}{r}+\frac{C}{r} \leq r'\mu_P+C.\end{aligned}$$

And if E' fails to satisfy $-C \leq \deg(E')-r'\mu_P$, then

$$\deg(E'') \geq \deg(E)-\deg(E') > r''\mu_P+C \geq r''\mu_P+\delta_1,$$

which implies $P_{(E'',\alpha'')}/r'' > P/r$. □

Lemma 2.5 — *Let $\mathcal{S}_K$ denote the family of kernels of framed modules in $\mathcal{S}$. The families $\mathcal{S}$ and $\mathcal{S}_K$ are bounded.*

Proof. Assume that (E,α) belongs to $\mathcal{S}$, and let $\hat{E}$ be the torsion free module obtained as fibred sum of α and $\hat{\varphi}: \hat{D} \to D$. Then $P_{\hat{E}} = P_E+P_B$ does not depend on (E,α). Moreover, if $\hat{F}$ is any nontrivial submodule of $\hat{E}$, let F denote its image in E and $F_B = \hat{F} \cap B$. Then by lemma 2.4:

$$\frac{\deg(\hat{F})}{\mathrm{rk}(\hat{F})} \leq \frac{\deg(F)}{\mathrm{rk}(\hat{F})}+\frac{\mathrm{rk}(F_B)}{\mathrm{rk}(\hat{F})}\mu_{\max}(B) \leq C+\mu_P+\mu_{\max}(B).$$

Therefore by Maruyama's theorem 2.3 the family of modules $\hat{E}$ is bounded. Since the modules E are quotients of the modules $\hat{E}$ with fixed Hilbert polynomial P, they form a bounded family, too. Hence there is a k-scheme T of finite type and a framed module $(E_T, \alpha_T : E_T \to \mathcal{O}_T \otimes D)$, such that the restrictions to the fibres of $X \times T \to T$ contain the family $\mathcal{S}$. Cutting T into smaller pieces if necessary we may assume that E_T as well as the cokernel of α_T are flat over T, so that the operation of taking the kernel of a framed module commutes with base change. This shows that the family $\mathcal{S}_K$ is also bounded. □

The last ingredient for the proof of theorem 2.1 is the following lemma:

Lemma 2.6 (Grothendieck) — *If $\mathcal{D}$ is a bounded family of modules F, then the family of torsion free quotients F'' of the modules F satisfying a uniform estimate $|\deg(F'')| \le C''$ is also bounded.*

Proof. [Gr, lemme 2.5] □

Lemma 2.7 — *The family $\mathcal{S}_0$ of saturated submodules (E', α') of any framed module (E, α) in $\mathcal{S}$ with the property $|\deg(E') - r'\mu_P| \le C$ is bounded. In particular, the set $\mathcal{H}$ of Hilbert polynomials of any such submodule is finite.*

Proof. Let (E, α) belong to $\mathcal{S}$ and let (E', α') be a submodule satisfying the properties of the lemma. If we associate to (E', α') the module $F = E/E'$, if $\alpha' \neq 0$, and $F = \ker(\alpha)/E'$ otherwise, then these modules F are torsion free quotients of modules of the bounded family $\mathcal{S} \cup \mathcal{S}_K$ and have degrees absolutely bounded by

$$|\deg(F)| \le C + r|\mu_P| + \max_{E \in \mathcal{S} \cup \mathcal{S}_K} \{\deg(E)\}$$

According to Grothendieck's lemma they form a bounded family, and a posteriori the same is true for the modules E' themselves. □

Lemma 2.8 — *There is an integer m_2 such that for any $m \ge m_2$ and for any module (E', α') in $\mathcal{S}_0$ one has: $rP_{(E',\alpha')}(\le)r'P$ if and only if $rP_{(E',\alpha')}(m)(\le)r'P(m)$.*

Proof. The set that consists of P, and of all polynomials of the form p or $p - \delta$ for $p \in \mathcal{H}$ is finite and contains all possible Hilbertpolynomials $P_{(E',\alpha')}$. □

Proof of theorem 2.1. Let m_0 be an integer greater than $\max\{m_1, m_2\}$ and such that all modules in any of the families $\mathcal{S}$, $\mathcal{S}_0$ or $\mathcal{S}_K$ are m_0-regular.

First assume that (E, α) is a framed module belonging to $\mathcal{S}'_m$ for some $m \ge m_0$. If (E'', α'') is a nontrivial quotient and (E', α') the corresponding submodule, then

$$h^0((E'', \alpha'')(m)) \ge h^0((E, \alpha)(m)) - h^0((E', \alpha')(m))(\ge)P(m) - r'P(m)/r = r''P(m)/r.$$

Hence obviously ii) $\Rightarrow$ iii).

Assume now that (E, α) belongs to $\mathcal{S}^s$ and that (E', α') is a saturated submodule. If (E', α') belongs to $\mathcal{S}_0$ then

$$h^0((E', \alpha')(m)) = P_{(E',\alpha')}(m)(\leq)r'P(m)/r = r'h^0((E, \alpha)(m))/r.$$

If it does not belong to $\mathcal{S}_0$, then the second alternative of lemma 2.4 applies and gives the even stronger relation

$$h^0((E', \alpha')(m)) < r'P(m)/r.$$

The condition that (E', α') be saturated can obviously be dropped immediately without any harm. This proves i) $\Rightarrow$ ii).

Finally, let (E, α) be in $\mathcal{S}''_m$, $m \geq m_0$, and let (E'', α'') be a quotient module. First assume that (E'', α'') has torsion free kernel. Then either $r''P/r < P_{(E'',\alpha'')}$ or the other alternative of lemma 2.4 applies and, since E,E' and E'' then are m-regular,

$$r''P(m)/r(\leq)h^0((E'', \alpha'')(m)) = P_{(E'',\alpha'')}(m).$$

By lemma 2.8 this implies $r''P/r(\leq)P_{(E'',\alpha'')}$. Again the condition that $\ker(\alpha'')$ be torsion free can be dropped. This proves iii) $\Rightarrow$ i) and finishes the proof. $\Box$

As a corollary to the proof we note the following lemma, keeping the notations of the theorem:

Lemma 2.9 — *If (E, α) is a semistable framed module, $m \geq m_0$ an integer and (E', α') a submodule of rank r' such that $h^0((E', \alpha')(m)) = r'P(m)/r$, then (E', α') is semistable with reduced Hilbert polynomial P/r.* $\Box$

Remark 2.10 — It might seem that the choice of m_0 depends in a controlled manner on δ in the sense that m_0 could be chosen to work simultaneously for framed modules that are semistable with respect to any polynomial δ' with $\max_i\{|\delta_i - \delta'_i|\}$ sufficiently small. But in fact, the proof of lemma 2.8 shows that the number of polynomials δ that we can simultaneously deal with must be finite. For a finite set of δ's the proof does indeed go through.

3 Constructions

In this section we will give a construction for the moduli spaces of semistable framed modules. If the framing is trivial, these are just the ordinary moduli spaces of semistable torsion free sheaves. Therefore in this chapter we will always assume that the framings are nontrivial unless the contrary is explicitely stated. Let P_0

denote a numerical polynomial of degree d, $P = P_0 - \delta$, and let $r > 0$ and μ_P denote the rank and slope of any coherent $\mathcal{O}_X$-module with Hilbert polynomial P. Choose some integer $m \geq m_0$ (notations of theorem 2.1) and let V be a vector space of dimension $P_0(m)$. For sufficiently large ℓ the standard morphisms

$$\begin{array}{rcl} \mathbf{H} := \mathrm{Hilb}(V \otimes \mathcal{O}_X(-m), P_0) & \to & \mathrm{Grass}(V \otimes H^0(\mathcal{O}_X(\ell - m)), P_0(\ell)) \\ & \to & \mathbb{P}(\Lambda^{P_0(\ell)}(V \otimes H^0(\mathcal{O}_X(\ell - m)))) \end{array}$$

are well-defined closed immersions. Let $\mathcal{L}$ denote the corresponding very ample line bundle on $\mathbf{H}$. Finally, let $\mathbf{P} := \mathbb{P}(\mathrm{Hom}(V, H^0(D(m)))^{\vee})$ and let $Z' \subset \mathbf{H} \times \mathbf{P}$ denote the closed subscheme of points

$$([q : V \otimes \mathcal{O}_X(-m) \to F], [a : V \to H^0(D(m))])$$

for which the homomorphism $a : V \otimes \mathcal{O}_X(-m) \to D$ factors through q and induces a framing $\alpha : F \to D$. The group $\mathrm{SL}(V)$ acts diagonally on Z' and the line bundles

$$\mathcal{O}_{Z'}(n_1, n_2) := p_{\mathbf{H}}^* \mathcal{L}^{\otimes n_1} \otimes p_{\mathbf{P}}^* \mathcal{O}_{\mathbf{P}}(n_2)$$

carry natural $\mathrm{SL}(V)$-linearizations. In the following discussion only the ratio n_2/n_1 matters, and we choose it to be

$$\frac{n_2}{n_1} = P(\ell) \frac{\delta(m)}{P(m)} - \delta(\ell),$$

assuming, of course, that ℓ is chosen large enough so as to make this term positive. With these notations we have the following proposition:

Proposition 3.1 — *For sufficiently large ℓ the point $([q], [a]) \in Z'$ is (semi)stable with respect to the linearization of $\mathcal{O}_{Z'}(n_1, n_2)$ if and only if the following holds: If V' is a nontrivial proper linear subspace of V and $F' \subset F$ the submodule generated by $V' \otimes \mathcal{O}_X(-m)$, then*

$$\dim V' \cdot (n_1 P_0(\ell) + n_2) (\leq) \dim V \cdot (n_1 P_{F'}(\ell) + n_2 \varepsilon(\alpha|_{F'})).$$

Proof. A 1-parameter subgroup $\lambda : \mathbf{G}_m \to \mathrm{SL}(V)$ is determined by giving a basis $\{v_1, \ldots, v_p\}$ of V, a weight vector γ, i.e. a nonzero element $(\gamma_1, \ldots, \gamma_p) \in \mathbf{Z}^p$, that satisfies the conditions $\gamma_1 \leq \ldots \leq \gamma_p$, $\sum \gamma_i = 0$, and by setting $\lambda(t) \cdot v_i = t^{\gamma_i} v_i$ for all $i \leq p$. Let $q : V \otimes \mathcal{O}_X(-m) \to F$, $a : V \to H^0(D(m))$ be homomorphisms representing the point $([q], [a]) \in Z'$ and let $\alpha : F \to D$ denote the corresponding framing. The appropriate value of ℓ will be determined in the course of the proof. For the moment let $W = H^0(\mathcal{O}_X(\ell - m))$ and $\varrho = h^0(F(\ell))$ for convenience sake. q induces homomorphisms $q' : V \otimes W \to H^0(F(\ell))$ and

$q'' : \Lambda^\varrho(V \otimes W) \to \det H^0(F(\ell))$. If $\{w_1, \ldots, w_t\}$ is a basis for W, then a basis for $\Lambda^\varrho(V \otimes W)$ is given by the elements of the form

$$u_{IJ} = (v_{i_1} \otimes w_{j_1}) \wedge \ldots \wedge (v_{i_\varrho} \otimes w_{j_\varrho}),$$

where I and J are multiindices satisfying $i_\alpha \leq i_{\alpha+1}$ and $j_\alpha < j_{\alpha+1}$ if $i_\alpha = i_{\alpha+1}$. Then $\mathbf{G}_m$ acts via λ on $\Lambda^\varrho(V \otimes W)$ by

$$\lambda(t) \cdot u_{IJ} = t^{\gamma_I} u_{IJ}, \qquad \gamma_I := \sum_\alpha \gamma_{i_\alpha}.$$

Now let $\mu(q'', \lambda) := -\min\{\gamma_I | \exists I, J \text{ with } q''(u_{IJ}) \neq 0\}$. This number can be computed as follows. Let φ denote the function $t \mapsto \dim q'(\langle v_1, \ldots, v_t\rangle \otimes W)$. Then

$$\mu(q'', \lambda) = -\sum_{i=1}^p \gamma_i(\varphi(i) - \varphi(i-1)).$$

Similarly, if we put $\mu(a, \lambda) = -\min\{\gamma_i | a(v_i) \neq 0\}$, then $\mu(a, \lambda) = -\gamma_\tau$ where $\tau = \min\{i | a(\langle v_1, \ldots, v_i\rangle) \neq 0\}$. Now the Hilbert-Mumford criterion [MF, Thm. 2.1] says:

$([q], [a])$ *is a (semi)stable point if and only if for all 1-parameter subgroups* λ *one has*

$$n_1 \cdot \mu(q'', \lambda) + n_2 \cdot \mu(a, \lambda)\, (\geq)\, 0,$$

or, equivalently,

$$n_1 \cdot \sum_{i=1}^p \gamma_i(\varphi(i) - \varphi(i-1)) + n_2 \cdot \gamma_\tau (\leq) 0.$$

The left hand side is a linear form on the set of weight vectors whose coefficients are determined only by the choice of the basis. Keeping such a basis fixed for a moment, it is enough to check the inequality for the special weight vectors

$$\gamma^{(i)} = (\underbrace{i-p, \ldots, i-p}_{i}, \underbrace{i, \ldots, i}_{p-i}), \qquad i = 1, \ldots, p-1,$$

which span the cone of all weight vectors. For $\gamma^{(i)}$ the inequality above is equivalent to

$$i \cdot (n_1\varrho + n_2) \quad (\leq) \quad p \cdot (n_1\varphi(i) + \varepsilon(i)),$$

where $\varepsilon(i) := 1$ if $a(\langle v_1, \ldots, v_i\rangle) \neq 0$ and 0 otherwise. Having got rid of the weights we can now vary the bases, and since the inequality depends on the flag of subspaces generated by any given basis rather than the basis itself, the criterion takes the following form:

$([q], [a])$ *is a semistable point if and only if for all nontrivial proper subspaces* V' *of* V *one has*

$$\dim V' \cdot (n_1\varrho + n_2)\, (\leq)\, \dim V \cdot (n_1 \dim(q'(V' \otimes W)) + n_2\varepsilon(V')),$$

where $\varepsilon(V') = 1$ *if* $a(V') \neq 0$ *and* 0 *otherwise.*

Now let F' denote the submodule $q(V' \otimes \mathcal{O}_X(-m))$ of F. The family of all such submodules, including F itself, for varying F and V', is bounded. Hence for sufficiently large ℓ, all these F' will be ℓ-regular, the equality $q'(V' \otimes W) = h^0(F'(\ell))$ holds and this vector space has dimension $P_{F'}(\ell)$. In this case the framing $\alpha : F \to D$ vanishes when restricted to F' if and only if $a(V') = 0$. Hence $\varepsilon(V') = \varepsilon(\alpha|_{F'})$. This finishes the proof. $\square$

By lemma 1.10 there is an open subscheme $U \subset Z'$ consisting precisely of those points that represent framed modules with torsion free kernel. If there are any semistable framed modules with the given Hilbert polynomial at all (and otherwise the present discussion is void), then U is nonempty and we denote by Z its closure in Z'.

Proposition 3.2 — *For sufficiently large* ℓ*, a point* $([q],[a]) \in Z$ *is (semi)stable with respect to the* $\mathrm{SL}(V)$*-action on* Z *if and only if the corresponding framed module* (F,α) *is (semi)stable and* q *induces an isomorphism* $V \to H^0(F(m))$*.*

Proof. We keep the notations of the proof of the previous proposition. First of all, observe that, if $([q],[a])$ is a semistable point, the homomorphism $V \to H^0(F(m))$ must be injective. For if V' denotes its kernel, then $q'(V' \otimes W) = 0$ and $\varepsilon(V') = 0$, so the previous proposition shows that $\dim(V') \leq 0$. Hence we can think of V as a subspace in $H^0(F(m))$. Since the family of modules F' generated by an arbitrary subspace V' of V is bounded, the set of polynomials $P_{F'}$ is finite. Hence choosing ℓ large enough and thinking of the quotient n_2/n_1 as a function of ℓ, the inequality in the previous proposition will hold for some specific value of ℓ if and only if it holds as an inequality between polynomials in ℓ. Substitute $P \cdot \frac{\delta(m)}{P(m)} - \delta(m)$ for $\frac{n_2}{n_1}$. At this point we can drop the restriction on the submodules F' to be generated by subspaces of V, and define $V' = V \cap H^0(F'(m))$ for any nontrivial proper submodule $F' \subset F$ instead. Now we can rewrite the stability criterion as follows:

$([q],[a])$ *is a semistable point if and only if for all nontrivial proper submodules* F' *of* F *with induced framings* $\alpha' = \alpha|_{F'}$ *the following inequality of polynomials in* ℓ *holds:*

$$\dim V' \left(1 + \frac{\delta(m)}{P(m)}\right) P \ (\leq)\ P_0(m) \left(P_{(F',\alpha')} + \varepsilon(\alpha')\frac{\delta(m)}{P(m)}P\right).$$

Passing to the leading coefficients of the polynomials one can derive the inequality:

$$\dim V' - \varepsilon(\alpha')\delta(m)) \leq \mathrm{rk}(F')P(m)/r.$$

Thus for any quotient module $F'' = F/F'$ with the induced framing we get

$$\begin{aligned} h^0((F'',\alpha'')(m)) &\geq \dim(V/V') - \varepsilon(\alpha'')\delta(m) \\ &= (\dim V - \varepsilon(\alpha)\delta(m)) - (\dim V' - \varepsilon(\alpha')\delta(m)) \\ &\geq P(m) - \mathrm{rk}(F')P(m)/r = \mathrm{rk}(F'')P(m)/r. \end{aligned}$$

By the definition of Z the framed module (F,α) deforms into a framed module with torsion free kernel, so that we can apply lemma 1.11 to F and conclude that there is a morphism of framed modules $\varphi : (F,\alpha) \to (G,\beta)$ such that (G,β) has torsion free kernel, $\ker(\varphi)$ is torsion and such that $P_F = P_G$, $\varepsilon(\alpha) = \varepsilon(\beta)$. If (G'',β'') is any quotient of (G,β), F'' the image of F under φ and the projection map, and α'' the framing of F'' induced from α, then one has

$$h^0((G'',\beta'')(m)) \geq h^0((F'',\alpha'')(m)) \geq \mathrm{rk}(F'')P(m)/r = \mathrm{rk}(G'')P(m)/\mathrm{rk}(G).$$

According to theorem 2.1 (G,β) is semistable. Applying this argument to (G,β) itself, one sees that in fact equality must hold at all places of this chain of inequalities so that the image of φ has as many sections as G, and since the latter is globally generated the image is in fact equal to G. Since F and G have the same Hilbert polynomial, φ is an isomorphism. In particular, (F,α) is semistable and $V \to H^0(F(m))$ is an isomorphism for dimension reasons.

Conversely, theorem 2.1 and lemma 2.9 state that if (F,α) is (semi)stable (and q given by some isomorphism $V \to H^0(F(m))$), then for any nontrivial proper submodule F' of rank r' one has $h^0((F',\alpha')(m)) < r'P(m)/r$ unless both (F,α) and (F',α') are semistable, in which case only equality holds. In the first case a strict inequality of the leading coefficients will also give a strict inequality of polynomials

$$h^0(F'(m))\left(1 + \frac{\delta(m)}{P(m)}\right)P \,<\, P_0(m)\left(P_{(F',\alpha')} + \varepsilon(\alpha')\frac{\delta(m)}{P(m)}P\right).$$

Hence if (F,α) is stable, then $([q],[a])$ is stable, too. If (F,α) is semistable but not stable, again strict inequality will hold except the case of a destabilizing semistable submodule (F',α'). In which case $h^0(F'(m)) = P_{F'}(m)$ and

$$h^0(F'(m))\left(1 + \frac{\delta(m)}{P(m)}\right)P = P_0(m)\left(P_{(F',\alpha')} + \varepsilon(\alpha')\frac{\delta(m)}{P(m)}P\right).$$

This proves the proposition. □

Let $Z^s \subset Z^{ss} \subset Z$ denote the open subschemes of stable and semistable points of Z, respectively. By the proposition above a point in $Z^{(s)s}$ corresponds, roughly speaking, to a (semi)stable framed module (F,α) together with the choice of a basis in $H^0(F(m))$.

Proposition 3.3 — *There exists a projective scheme $\mathcal{M}^{ss}$ and a morphism $\pi : Z^{ss} \to \mathcal{M}^{ss}$ which is a good quotient for the action of* $\mathrm{SL}(V)$ *on Z^{ss}. Moreover there is an open subscheme $\mathcal{M}^s \subset \mathcal{M}^{ss}$ such that $Z^s = \pi^{-1}(\mathcal{M}^s)$ and $\pi : Z^s \to \mathcal{M}^s$ is a geometric quotient. Two points $([q],[a])$ and $([q',\alpha'])$ are mapped to the same point in $\mathcal{M}^{ss}$ if and only if the corresponding framed modules are S-equivalent.*

Proof. The first two statements follow from proposition 3.2 and theorem 1.10 of [MF]. As for the third it is easy to see that any semistable framed module (F,α) corresponding to a closed point $([q],[a])$ can be deformed into its graded object: Suppose that (F',α') is a destabilizing submodule, (F'',α'') the quotient module. Consider the following pull-back diagram of extensions parametrized by the affine line $\mathbf{A}^1$:

$$\begin{array}{ccccccccc} 0 & \to & F'\otimes\mathcal{O}_{\mathbf{A}^1} & \to & \mathcal{F} & \to & F''\otimes\mathcal{O}_{\mathbf{A}^1} & \to & 0 \\ & & \| & & \downarrow & & \downarrow t & & \\ 0 & \to & F'\otimes\mathcal{O}_{\mathbf{A}^1} & \to & F\otimes\mathcal{O}_{\mathbf{A}^1} & \to & F''\otimes\mathcal{O}_{\mathbf{A}^1} & \to & 0 \end{array},$$

where t denotes the multiplication with the parameter of $\mathbf{A}^1$. $\mathcal{F}$ inherits a framing $a : \mathcal{F} \to D \otimes \mathcal{O}_{\mathbf{A}^1}$ from $F''\otimes\mathcal{O}_{\mathbf{A}^1}$ if $\alpha'' \neq 0$ and from $F\otimes\mathcal{O}_{\mathbf{A}^1}$ otherwise. Then $(\mathcal{F}_0, a_0) \cong (F,\alpha') \oplus (F'',\alpha'')$ and $(\mathcal{F}_t, a_t) \cong (F,\alpha)$ for all $t \neq 0$. Moreover, $\mathcal{V} := (pr_{\mathbf{A}^1})_*\mathcal{F}$ is locally free of rank $P(m)$ and we can choose an isomorphism $V\otimes\mathcal{O}_{\mathbf{A}^1} \cong \mathcal{V}$. These data provide us with a morphism $\psi : \mathbf{A}^1 \to Z^{ss}$ such that $\psi(\mathbf{A}^1 - \{0\})$ lies in the orbit of $([q],[a])$ and $\psi(0)$ corresponds to $(F',\alpha') \oplus (F'',\alpha'')$. Proceeding this way we see that the closure of the orbit determined by a semistable framed module contains points that correspond to its graded object. To finish the proof it is enough to show that the orbits determined by polystable framed modules, i.e. those which are direct sums of stable framed modules, are closed in Z^{ss}; for closed orbits are separated by good quotient maps. It suffices to show that if (F_T, β_T) is a flat family of semistable framed modules parametrized by some smooth curve T such that all fibres (F_t, β_t), $t \neq 0$, are isomorphic to a given polystable framed module, then the same holds for the special fibre (F_0, β_0). We will need the following result:

Lemma 3.4 — *Let (E,α) and (F,β) be flat families of semistable framed modules with the same reduced Hilbert polynomial p, parametrized by a scheme T of finite type over k. Then the function*

$$t \mapsto \dim_{k(t)} \mathrm{Hom}((E_t,\alpha_t),(F_t,\beta_t))$$

is semicontinuous in $t \in T$.

Before proving the lemma, we finish the proof of proposition 3.3: Suppose, $(F_t,\beta_t) \cong \bigoplus_{i=0}^{s}(E_i,\alpha_i)^{\oplus\nu_i}$ for $t \neq 0$. Note that precisely one of the α_i is nonzero, say for $i = 0$, and that then $\nu_0 = 1$. We have

$$\dim \mathrm{Hom}((E_t,\alpha_t),(F_i,\beta_i)) = \nu_i$$

for all $t \neq 0$, hence the previous lemma implies

$$\dim \operatorname{Hom}((E_0, \alpha_0), (F_i, \beta_i)) \geq \nu_i.$$

Choose ν_i independent homomorphisms for each i and using these define a homomorphism of framed modules

$$\varphi : (E_0, \alpha_0) \oplus \bigoplus_{i=1}^{s} (E_i, 0)^{\oplus \nu_i} \to (F_0, \beta_0).$$

Using lemma 1.6 it is easy to verify that φ is an isomorphism. □

Proof of the lemma. This semicontinuity statement can be proved with the usual techniques. The corresponding statement for flat families of coherent sheaves can be found in [BPS] for complex spaces and in [La] for projective varieties. We give a selfcontained proof. Since the problem is local, we may assume that $T = \operatorname{Spec} A$, where A is a k-algebra of finite type.

1^{st} Step. Let E be any flat family over T. Choose a locally free resolution $L_\bullet \to E$ of finite length. Let $\mathcal{U} = \{U_i\}$ be a finite affine open cover of $X \times T$. For any $\mathcal{O}_{X\times T}$-module G consider the double complex

$$C^{p,q} = C^{p,q}(E, G) = \prod_{|I|=q+1} \operatorname{Hom}(L_p|_{U_I}, G|_{U_I})$$

for $p, q \geq 0$ (here $I = \{i_0, \ldots, i_q\}$ is an ordered multiindex, $U_I = U_{i_0} \cap \ldots \cap U_{i_q}$) with the canonical differentials $d' : C^{p,q} \to C^{p+1,q}$, $d'' : C^{p,q} \to C^{p,q+1}$ induced from the resolution of E and the covering $\mathcal{U}$. Let $C^\bullet(G) = C^\bullet(E, G)$ denote the corresponding total complex. There is a spectral sequence with $E_1^{pq} = Ext^q(L_p, G)$ converging to $h^n(C^\bullet)$. For any G one has $h^n(C^\bullet) = 0$ for all $n < 0$ and $h^0(C^\bullet) = \operatorname{Hom}(E, G)$. If G is also injective, then $h^n(C^\bullet) = 0$ for all $n > 0$. Hence $G \mapsto h^i(C^\bullet(E, G))$ is a universal δ-functor. Therefore $h^i(C^\bullet(E, G)) = \operatorname{Ext}^n(E, G)$. Assume now, that G is A-flat. Then $C^\bullet(G)$ is a complex of finite length consisting of A-flat modules and with cohomology groups which are finitely generated as A-modules. For any A-module M, one has $C^\bullet(G \otimes M) = C^\bullet(G) \otimes M$. Hence $h^i(C^\bullet(G) \otimes M) = \operatorname{Ext}^i_A(E, G \otimes_A M)$. Note that if $M = B$ is an A-algebra, then $\operatorname{Ext}^i_B(E \otimes B, G \otimes B) = \operatorname{Ext}^i_A(E, G \otimes B)$. By lemma [H2, III 12.3] there exist a complex $M^\bullet_{E,G}$ of finite free A-modules bounded from above and a quasiisomorphism $M^\bullet_{E,G} \to C^\bullet(E, G)$. Following the arguments in [H2, III 12], one can see that

$$t \mapsto \dim(\operatorname{Ext}^p(E_t, G_t))$$

is a semicontinuous function.

2^{nd} Step. Suppose $\beta : F \to D$ is a homomorphism of flat families. Then there is an induced homomorphism of complexes $C^\bullet(E, F) \to C^\bullet(E, D)$ which computes the induced homomorphisms of relative Ext groups. Let $M_D^\bullet := M_{E,D}^\bullet \to C^\bullet(E,D)$ be a quasiisomorphism as above, and let

$$\begin{array}{ccc} N^\bullet & \to & M_D^\bullet \\ \downarrow & & \downarrow \\ C^\bullet(E,F) & \to & C^\bullet(E,D) \end{array}$$

be the fibred product. Then $N^\bullet \to C^\bullet(E,F)$ is a quasiisomorphism and $N^\bullet$ is a complex satisfying the conditions of lemma [H2, III 12.3] as above, so that there is an approximation $M_F^\bullet \to N^\bullet$ by a complex of finite free A-modules which is bounded from above. The composite homomorphism $M_F^\bullet \to M_D^\bullet$ has the property that the diagram

$$\begin{array}{ccc} h^i(M_F^\bullet \otimes M) & \to & h^i(M_D^\bullet \otimes M) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^i(E, F \otimes_A M) & \to & \mathrm{Ext}^i(E, D \otimes_A M) \end{array}$$

commutes for any A-module M.

3rd Step. Suppose (E,α) and (F,β) are flat families of framed modules as in the lemma. Let $a \in M_D^0$ be a cycle that represents the framing $\alpha \in \mathrm{Hom}_A(E, D \otimes \mathcal{O}_T) = h^0(M_D^\bullet)$. Then a gives a chain homomorphism $a : A^\bullet \to M_D^\bullet$, where $A^\bullet$ is the complex with $A^0 = A$ and $A^i = 0$ for $i \neq 0$. Consider the homomorphism $\psi := (\beta, -a) : M_F^\bullet \oplus A^\bullet \to M_D^\bullet$ and let $C(\psi)$ denote its mapping cone. From the short exact sequence

$$0 \to M_D^\bullet \to C(\psi)^\bullet \to (M_F^\bullet \oplus A^\bullet)[1] \to 0$$

one gets the exact sequence

$$0 = Ext^{-1}(E, D \otimes M) \to h^{-1}(C(\psi)^\bullet \otimes M) \to \mathrm{Hom}(E, F \otimes M) \oplus M \to \mathrm{Hom}(E, D \otimes M).$$

In particular, for any $t \in \mathrm{Spec}A$ and $M = k(t)$ there is a pull-back diagram

$$\begin{array}{ccc} h^{-1}(C(\psi)^\bullet \otimes k(t)) & \longrightarrow & k(t) \\ \downarrow & & \downarrow \alpha \\ \mathrm{Hom}(E_t, F_t) & \xrightarrow{\beta} & \mathrm{Hom}(E_t, D \otimes k(t)). \end{array}$$

Hence $\dim \mathrm{Hom}((E_t, \alpha_t), (F_t, \beta_t)) = \dim h^{-1}(C(\psi)^\bullet \otimes k(t)) - 1 + \varepsilon(\alpha_t)$ is a semicontinuous function. By assumption α_t is zero either for all $t \in T$ or for none. This finishes the proof. □

Proof of the main theorem 0.1. Suppose T is a scheme parametrizing a flat family (F_T, α_T) of semistable framed modules. Let m be still the number chosen at the beginning of this section. Then $\mathcal{V} := p_{T*}(F_T \otimes p_X^* \mathcal{O}_X(m))$ is a locally free sheaf of rank $P_0(m)$ on T, and $p_T^* \mathcal{V} \to F_T$ is surjective. Moreover, the framing α_T induces a homomorphism $a_T : \mathcal{V} \to \mathcal{O}_T \otimes H^0(D(m))$. Covering T by small enough open subschemes T_i, we can find trivializations $V \otimes \mathcal{O}_{T_i} \to \mathcal{V}|_{T_i}$. The composition with these trivializations gives homomorphisms $q_i : V \otimes \mathcal{O}_{T_i \times X} \to F_{T_i}$ and $a_i : V \otimes \mathcal{O}_{T_i} \to H^0(D(m)) \otimes \mathcal{O}_{T_i}$, which in turn lead to morphisms $f_i : T_i \to Z' \subset \mathbf{H} \times \mathbf{P}$. Moreover, by proposition 3.2 $f_i(T_i) \subset Z^{ss} \subset Z'$. The trivializations of $\mathcal{V}$ over the intersection T_{ij} of two open sets T_i and T_j differ by a morphism $g : T_{ij} \to \mathrm{GL}(V)$, in the sense that $f_i|_{T_{ij}} = g \cdot f_j|_{T_{ij}}$. Therefore, if π denotes the geometric quotient $Z^{ss} \to \mathcal{M}^{ss}$, the morphisms $\pi \circ f_i$ and $\pi \circ f_j$ coincide on T_{ij} and thus glue to give a morphism $\bar{f} : T \to \mathcal{M}^{ss}$. If the family (F_T, α_T) consists of stable framed modules, then obviously $\bar{f}(T) \subset \mathcal{M}^s$. This gives a natural transformation $\underline{\mathcal{M}}_\delta^{ss}(X; D, P) \to \mathcal{M}^{ss}$. If N is any other scheme with a natural transformation $\underline{\mathcal{M}}_\delta^{ss}(X; D, P) \to N$, then the tautological family over Z^{ss} defines a $\mathrm{SL}(V)$-invariant morphism $Z^{ss} \to N$, which must factor through π and a morphism $\mathcal{M}^{ss} \to N$. This shows that $\mathcal{M}^{ss}$ is a coarse moduli scheme. By taking etal slices to the $\mathrm{SL}(V)$ action on Z^s we can find an etal cover $\mathcal{M}' \to \mathcal{M}^s$ over which a universal family (F', α') exists. Let $\mathcal{M}'' = \mathcal{M}' \times_{\mathcal{M}^s} \mathcal{M}'$. Take an isomorphism $\Phi : p_1^*(F', \alpha') \to p_2^*(F', \alpha')$ which is normalized by the requirement that $p_1^*\alpha' \circ \Phi = p_2^*\alpha'$. The uniqueness result of lemma 1.6 implies that Φ satisfies the cocycle condition of descend theory [Mu, Ch VII].Therefore, (F', α') descends to a framed module on $\mathcal{M}^s$. Finally, the assertion about the closed points of $\mathcal{M}^{ss}$ is proved in proposition 3.3. □

4 The deformation theory of framed modules

There are several ways to describe the tangent space of our moduli space $\mathcal{M}_\delta^s(P, D)$ at a point (E, α). One possible method is to identify the infinitesimal deformations with a certain cohomology group by using a cocycle calculation. In this case the framed module does not have to be stable. Regarding the notation this approach tends to be rather messy and gives little insight. Therefore, we prefer to work with a different technique. Starting out with the description of the Zariski tangent space of the Hilbert scheme $\mathrm{Hilb}(V \otimes \mathcal{O}(-m), P)$ due to Grothendieck and its modification in our situation, we will describe the tangent space of $\mathcal{M}_\delta^s(P, D)$ regarding it as a geometric quotient of a subscheme of $\mathrm{Hilb}(V \otimes \mathcal{O}(-m), P) \times \mathbb{P}(\mathrm{Hom}(V, H^0(D(m)))^{\mathrm{v}})$. In order to obtain a smoothness criterion by using methods of Ran and Kawamata ([Ka, R]) we explain the infinitesimal deformations of framed modules over $X_A := X \times \mathrm{Spec}(A)$ where A is an Artinian ring.

Theorem 4.1 — *Let $[E, \alpha]$ be a point in $\mathcal{M}_\delta^s(P, D)$. Consider E and $E \xrightarrow{\alpha} D$ as complexes which are concentrated in dimensions zero, and (zero,one), respectively.*

i) The Zariski tangent space of $\mathcal{M}_\delta^s(P, D)$ at a point (E, α) is naturally isomorphic to the hyper-Ext group $\mathbb{E}\mathrm{xt}_X^1(E, E \xrightarrow{\alpha} D)$.

ii) If the second hyper-Ext group $\mathbb{E}\mathrm{xt}^2_X(E, E \xrightarrow{\alpha} D)$ *vanishes, then* $\mathcal{M}^s_\delta(P,D)$ *is smooth at* $[E,\alpha]$.

Proof. i) **The Zariski tangent space of the Hilbert scheme**

In order to introduce some notations and to make the whole proof more accessible we recall the description of the tangent space of $\mathrm{Hilb}(\mathcal{H}, P_0)$ at a point $[q : \mathcal{H} \to \to \mathcal{E}]$, where $\mathcal{H}$ is an $A-$flat coherent sheaf on X_A and $\mathcal{E}$ is an $A-$flat quotient of $\mathcal{H}$ with Hilbert polynomial P_0. Let $A[\varepsilon] := A[x]/(x^2)$ and $S := \mathrm{Spec}(A[\varepsilon])$. Then by definition the Zariski tangent space of $\mathrm{Hilb}(\mathcal{H}, P_0)$ at q is isomorphic to $\mathrm{Hom}_q(S, \mathrm{Hilb}(\mathcal{H}, P_0))$, the space of all $A-$morphisms $S \to \mathrm{Hilb}(\mathcal{H}, P_0)$ such that $\mathrm{Spec}(A[\varepsilon]/(\varepsilon))$ is mapped to q. By the universal property of the Hilbert scheme it can also be described as the set of all $S-$flat quotients $\tilde{q} : \mathcal{H}_S \twoheadrightarrow \tilde{\mathcal{E}}$ with $\tilde{q} \equiv q\, mod(\varepsilon)$. By $\mathcal{H}_S$ and $q_S : \mathcal{H}_S \twoheadrightarrow \mathcal{E}_S$ we denote the pull-back of $\mathcal{H}$ and $q : \mathcal{H} \twoheadrightarrow \mathcal{E}$, respectively, under the natural projection $X_S \to X_A$. Grothendieck showed that there is a natural isomorphism

$$T_q\mathrm{Hilb}(\mathcal{H}, P_0) \cong \mathrm{Hom}_{X_S}(\mathcal{K}, \varepsilon \cdot \mathcal{E}_S) \cong \mathrm{Hom}_{X_A}(\mathcal{K}, \mathcal{E}),$$

where $\mathcal{K} := \ker q$. The second isomorphism follows from the facts that $\varepsilon \in \mathrm{Ann}_S(\mathcal{K})$ and $\varepsilon \cdot \mathcal{E}_S \cong \mathcal{E}$ as $\mathcal{O}_{X_S}-$modules. The first isomorphism is established as follows: If $\tilde{q} : \mathcal{H}_S \twoheadrightarrow \tilde{\mathcal{E}}$ is a quotient over X_S and $\tilde{\mathcal{K}}$ is its kernel, then the image of the composition $\tilde{\mathcal{K}} \subset \mathcal{H}_S \xrightarrow{q_S} \mathcal{E}_S$ is contained in $\ker(\mathcal{E}_S \twoheadrightarrow \mathcal{E}) \cong \varepsilon \cdot \mathcal{E}_S$. Since $\varepsilon^2 = 0$, this map factors through $\tilde{\mathcal{K}} \twoheadrightarrow \mathcal{K}$. Hence we obtain an $\mathcal{O}_{X_A}$-homomorphism $\gamma : \mathcal{K} \to \mathcal{E}$. The inverse of this map is given by

$$\gamma \longmapsto \tilde{\mathcal{K}} := \rho^{-1}(\mathcal{K}) \cap \ker(q_S + \gamma \circ \rho),$$

where ρ is the natural surjection $\mathcal{H}_S \twoheadrightarrow \mathcal{H}$. First, one checks that this is in fact inverse to the map defined above. Then one shows, that the induced quotient $\mathcal{H}_S \twoheadrightarrow \mathcal{H}_S/\tilde{\mathcal{K}}$ is flat over S and extends $\mathcal{H} \twoheadrightarrow \mathcal{E}$.

ii) **The Zariski tangent space of** Z'

We recall that $Z'_A \subset \mathrm{Hilb}(V \otimes \mathcal{O}_{X_A}(-m), P_0) \times \mathbb{P}(\mathrm{Hom}(V \otimes \mathcal{O}_{X_A}(-m), D_A)^{\mathrm{v}})$ is the subscheme which represents the functor which associates to each $A-$scheme T the set

$$\{(V \otimes \mathcal{O}_{X_T}(-m) \xrightarrow{\tilde{q}} \tilde{\mathcal{E}}, \tilde{\mathcal{E}} \xrightarrow{\tilde{\alpha}} D_T) | \tilde{\mathcal{E}} \text{ is } T-\text{flat with Hilbert polynomial } P_0\}$$

Write $\mathcal{H} := V \otimes \mathcal{O}_{X_A}(-m)$ for short. Since any homomorphism $\mathcal{H}_S \to D_S$ which extends $a : \mathcal{H} \to D_A$ can be written in the form

$$h + h'\varepsilon \mapsto a(h) + \varepsilon(a(h') + \delta(h)),$$

it naturally induces an element $\delta \in \mathrm{Hom}(\mathcal{H}, D_A)$, and vice versa. An element of $\mathbb{P}(\mathrm{Hom}(\mathcal{H}_S, D_S)^{\mathrm{v}})$ defines a homomorphism $\mathcal{H}_S \to D_S$ up to multiplication with units. These act by $(\lambda+\mu\varepsilon)(a+\delta\varepsilon)=\lambda a+\varepsilon(\mu a+\lambda\delta)$. Thus the Zariski tangent space of $\mathbb{P}(\mathrm{Hom}(\mathcal{H},D_A)^{\mathrm{v}})$ at a point $[a]$ can be described as $\mathrm{Hom}(\mathcal{H},D_A)/A\cdot a$. Now we claim that the Zariski tangent space of Z'_A at a point $(q, a=\alpha\circ q)$ can be naturally identified with the space

$$W:=\left\{(\gamma,\delta)\in\mathrm{Hom}(\mathcal{K},\mathcal{E})\oplus\mathrm{Hom}(\mathcal{H},D_A)/A\cdot a\,\middle|\,\alpha\circ\gamma=\delta|_{\mathcal{K}}\right\}.$$

This can be seen as follows: Since $a|_{\mathcal{K}}=0$, this subspace is well defined. A pair $(\gamma,\delta)\in\mathrm{Hom}(\mathcal{K},\mathcal{E})\oplus\mathrm{Hom}(\mathcal{H},D_A)$ defines an extension

$$0\longrightarrow\tilde{\mathcal{K}}\longrightarrow\mathcal{H}_S\xrightarrow{\tilde{q}}\tilde{\mathcal{E}}\longrightarrow 0$$

of

$$0\longrightarrow\mathcal{K}\longrightarrow\mathcal{H}\xrightarrow{q}\mathcal{E}\longrightarrow 0$$

and a homomorphism $\tilde{a}:\mathcal{H}_S\to D_S$ extending a. It defines a point in the Zariski tangent space of Z_A if and only if $\tilde{a}|_{\tilde{\mathcal{K}}}=0$. The elements $h+h'\varepsilon\in\mathcal{H}_S$ contained in $\tilde{\mathcal{K}}$ are described by the two conditions $q(h)=0$ and $\gamma(h)+q(h')=0$ (cf. i)). The homomorphism $\tilde{a}$ is given by

$$h+h'\varepsilon\mapsto a(h)+\varepsilon(a(h')+\delta(h)).$$

Since $a|_{\mathcal{K}}=0$, the condition $\tilde{a}|_{\tilde{\mathcal{K}}}=0$ is satisfied if and only if

$$a(h')=\delta(h)\text{ for all }h,h'\text{ with }\gamma(h)=q(h').$$

Composing with α, using $\alpha\circ q=a$ and the surjectivity of q, we conclude that $\tilde{a}|_{\tilde{\mathcal{K}}}=0$ is equivalent to $\alpha\circ\gamma=\delta|_{\mathcal{K}}$.

iii) **The Zariski tangent space of the quotient**

In order to describe the tangent space of the quotient we first have a look at the orbits of the $\mathrm{Aut}(\mathcal{H})$-action. Again we use $\mathcal{H}:=V\otimes\mathcal{O}_{X_A}(-m)$. Let us start with the $\mathrm{Aut}(\mathcal{H})$-action on $\mathrm{Hilb}(\mathcal{H},P_0)$. It is given by

$$\begin{array}{ccccc}\mathrm{Aut}(\mathcal{H}) & \times & \mathrm{Hilb}(\mathcal{H},P_0) & \longrightarrow & \mathrm{Hilb}(\mathcal{H},P_0)\\ (\lambda & , & q:\mathcal{H}\twoheadrightarrow\mathcal{E}) & \mapsto & (\mathcal{H}\xrightarrow{\lambda}\mathcal{H}\xrightarrow{q}\mathcal{E}).\end{array}$$

Analogously, the $\mathrm{Aut}(\mathcal{H})$-action on Z'_A is given by

$$(\lambda,q:\mathcal{H}\twoheadrightarrow\mathcal{E},a:\mathcal{H}\longrightarrow D_A)\mapsto(\mathcal{H}\xrightarrow{\lambda}\mathcal{H}\xrightarrow{q}\mathcal{E},\mathcal{H}\xrightarrow{\lambda}\mathcal{H}\xrightarrow{a}D_A).$$

The tangent map of the orbit map $f_q:\mathrm{Aut}(\mathcal{H})\longrightarrow Z'_A$ at a point (q,a) can be described as the composition

$$\begin{array}{cccccc}Tf_q: & \mathrm{End}(\mathcal{H}) & \longrightarrow & \mathrm{Hom}(\mathcal{H},\mathcal{E}) & \longrightarrow & W\\ & \psi & \mapsto & q\circ\psi & \mapsto & (q\circ\psi|_{\mathcal{K}},a\circ\psi).\end{array}$$

Thus the tangent space $T_{(q,a)}$ of $Z_A^s/\mathrm{Aut}(\mathcal{H})$ at a point (q,a) is naturally isomorphic to the quotient $W/\mathrm{im}(Tf_q)$. If, as we assume, $[q,a]$ is a stable point in Z_A, then proposition 3.2 shows that $H^0(q(m)) : H^0(\mathcal{H}(m)) \to H^0(\mathcal{E}(m))$ is surjective, hence the natural homomorphism $\mathrm{End}(\mathcal{H}) \to \mathrm{Hom}(\mathcal{H},\mathcal{E})$ is surjective, too. Therefore $T_{(q,a)}$ is naturally isomorphic to $\mathrm{Hom}_K(\mathcal{K} \to \mathcal{H}, \mathcal{E} \xrightarrow{\alpha} D_A)$, the set of homomorphisms of complexes from $\mathcal{K} \to \mathcal{H}$ to $\mathcal{E} \to D_A$ up to homotopy. We know that $\mathrm{Ext}^1(\mathcal{H},\mathcal{E}) \cong V^{\mathrm{v}} \otimes H^1(X_A, \mathcal{E}(m)) = 0$ for a stable point (q,a) and, using this fact, want to show that

$$\mathrm{Hom}_K(\mathcal{K} \to \mathcal{H}, \mathcal{E} \xrightarrow{\alpha} D_A) \cong \mathbb{E}\mathrm{xt}^1(\mathcal{E}, \mathcal{E} \xrightarrow{\alpha} D_A).$$

The general reference for the hyper-Ext groups is [H1]. Since $\mathcal{E}$, considered as a complex concentrated in zero, is quasi-isomorphic to $(\mathcal{K} \to \mathcal{H})[-1]$, the group $\mathbb{E}\mathrm{xt}^1(\mathcal{E}, \mathcal{E} \xrightarrow{\alpha} D_A)$ is isomorphic to $\mathbb{H}\mathrm{om}(\mathcal{K} \to \mathcal{H}, \mathcal{E} \xrightarrow{\alpha} D_A)$, the group of homomorphisms in the derived category. Thus it is enough to show that the natural homomorphism

$$\mathrm{Hom}_K(\mathcal{K} \to \mathcal{H}, \mathcal{E} \xrightarrow{\alpha} D_A) \longrightarrow \mathbb{H}\mathrm{om}(\mathcal{K} \to \mathcal{H}, \mathcal{E} \xrightarrow{\alpha} D_a)$$

is bijective.

We begin with the surjectivity. An element $\varphi \in \mathbb{H}\mathrm{om}(\mathcal{K} \to \mathcal{H}, \mathcal{E} \xrightarrow{\alpha} D_A)$ is given by a quasi-isomorphism $M^\bullet \to (\mathcal{K} \to \mathcal{H})$ and a homomorphism $M^\bullet \to (\mathcal{E} \xrightarrow{\alpha} D_A)$. Obviously, we can assume that $M^\bullet$ is concentrated in zero and one. From the diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 \to & \mathcal{K} & \to & \mathcal{H} & \to & \mathcal{E} & \to 0 \\ & \uparrow & & \uparrow & & =\uparrow & \\ 0 \to & M^0 & \to & M^1 & \to & \mathcal{E} & \to 0 \end{array}$$

we deduce

$$\begin{array}{ccccc} \mathrm{Hom}(\mathcal{K},\mathcal{E}) & \twoheadrightarrow & \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) & \longrightarrow \mathrm{Ext}^1(\mathcal{H},\mathcal{E}) = 0 \\ =\uparrow & & \uparrow & \\ \mathrm{Hom}(\mathcal{K},\mathcal{E}) & \longrightarrow & \mathrm{Hom}(M^0,\mathcal{E}) & & , \\ & & \uparrow & \\ & & \mathrm{Hom}(M^1,\mathcal{E}) & \end{array}$$

so that $\mathrm{Hom}(\mathcal{K},\mathcal{E}) \to \mathrm{Hom}(M^0,\mathcal{E})/\mathrm{Hom}(M^1,\mathcal{E})$ is surjective. Therefore the given homomorphism $M^\bullet \to (\mathcal{E} \xrightarrow{\alpha} D_A)$ is homotopic to another one with the property that $M^0 \to \mathcal{E}$ is composed of $M^0 \to \mathcal{K}$ and a homomorphism $\mathcal{K} \to \mathcal{E}$. Similarly, we get the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}(\mathcal{E},D_A) & \to & \mathrm{Hom}(\mathcal{H},D_A) & \to & \mathrm{Hom}(\mathcal{K},D_A) & \to & \mathrm{Ext}^1(\mathcal{E},D_A) \\ =\downarrow & & \downarrow & & \downarrow & & =\downarrow \\ \mathrm{Hom}(\mathcal{E},D_A) & \to & \mathrm{Hom}(M^1,D_A) & \to & \mathrm{Hom}(M^0,D_A) & \to & \mathrm{Ext}^1(\mathcal{E},D_A) \end{array}$$

and can find a homomorphism $\mathcal{H} \to D_A$ which makes the diagram

$$\begin{array}{ccccc} M^0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow & & \alpha\downarrow \\ M^1 & \longrightarrow & \mathcal{H} & \longrightarrow & D_A \end{array}$$

commutative and such that the composition $M^1 \to \mathcal{H} \to D_A$ is the given homomorphism. In other words, we have found a preimage of φ under

$$\mathrm{Hom}_K(\mathcal{K} \to \mathcal{H}, \mathcal{E} \xrightarrow{\alpha} D_A) \longrightarrow \mathbb{H}\mathrm{om}(\mathcal{K} \to \mathcal{H}, \mathcal{E} \xrightarrow{\alpha} D_a).$$

Finally, we prove the injectivity. Let $(\gamma, \delta) : (\mathcal{K} \to \mathcal{H}) \longrightarrow (\mathcal{E} \xrightarrow{\alpha} D_A)$ be a homomorphism which vanishes in the derived category, i.e. there is a quasi-isomorphism $(M^0 \to M^1) \longrightarrow (\mathcal{K} \to \mathcal{H})$ such that the composition with (γ, δ) is homotopic to zero. Using the diagram

$$\begin{array}{ccccccc} \mathrm{Hom}(\mathcal{E}, \mathcal{E}) & \to & \mathrm{Hom}(M^1, \mathcal{E}) & \to & \mathrm{Hom}(M^0, \mathcal{E}) & \to & \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \\ =\uparrow & & \uparrow & & \uparrow & & =\uparrow \\ \mathrm{Hom}(\mathcal{E}, \mathcal{E}) & \to & \mathrm{Hom}(\mathcal{H}, \mathcal{E}) & \to & \mathrm{Hom}(\mathcal{K}, \mathcal{E}) & \to & \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \end{array}$$

we can lift the homotopy $M^1 \longrightarrow \mathcal{E}$ to a homomorphism $\mathcal{H} \longrightarrow \mathcal{E}$ which makes (γ, δ) homotopic to zero. Hence we have shown the injectivity.

iv) **The smoothness of the moduli space**

Here we want to use recent results by Kawamata and Ran [Ka, R]. They proved in a very general context that the T^1 lifting property implies the smoothness of the deformation space. In our situation this means the following: Let $A_n := k[t]/(t^{n+1})$ and let $(\mathcal{E}_n, \alpha_n)$ be a framed module over X_{A_n}. Let $T^1((\mathcal{E}_n, \alpha_n)/A_n)$ be the set of infinitesimal deformations of $(\mathcal{E}_n, \alpha_n)$ and

$$T^1_n : T^1((\mathcal{E}_n, \alpha_n)/A_n) \to T^1((\mathcal{E}_{n-1}, \alpha_{n-1})/A_{n-1})$$

be the natural map induced by

$$\begin{array}{ccc} A_n \otimes k[\varepsilon] & \longrightarrow & A_{n-1} \otimes k[\varepsilon] \\ \cong k[t, \varepsilon]/(t^{n+1}) & & \cong k[t, \varepsilon]/(t^n), \end{array}$$

where $(\mathcal{E}_{n-1}, \alpha_{n-1}) \equiv (\mathcal{E}_n, \alpha_n) \quad mod(t^n)$. The T^1 lifting property is satisfied if all these maps T^1_n are surjective. Let $(\mathcal{E}_n, \alpha_n)$ be a stable framed module, i.e. $(\mathcal{E}_n, \alpha_n) \in \underline{\mathcal{M}}^s_\delta(P, D)(A_n)$, and $(\mathcal{E}_n, \alpha_n) \equiv (E, \alpha) \quad mod(t)$. Then we have seen that

$$T^1((\mathcal{E}_n, \alpha_n)/A_n) \cong \mathbb{E}\mathrm{xt}^1(\mathcal{E}_n, \mathcal{E}_n \xrightarrow{\alpha_n} D_{A_n}).$$

Therefore, in order to prove that $\mathcal{M}^s_\delta(P, D)$ is smooth at (E, α) it is enough to show the surjectivity of the natural map

$$\mathbb{E}\mathrm{xt}^1_{X_{A_n}}(\mathcal{E}_n, \mathcal{E}_n \xrightarrow{\alpha_n} D_{A_n}) \longrightarrow \mathbb{E}\mathrm{xt}^1_{X_{A_{n-1}}}(\mathcal{E}_{n-1}, \mathcal{E}_{n-1} \xrightarrow{\alpha_{n-1}} D_{A_{n-1}}).$$

As in [Ka] we apply the functor $\mathbb{E}\mathrm{xt}^1_{X_{A_n}}(\mathcal{E}_n,\ .\ )$ to the exact sequence

$$\begin{array}{ccccccc} 0 \to & E & \to & \mathcal{E}_n & \to & \mathcal{E}_{n-1} & \to 0 \\ & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \alpha_n}\downarrow & & {\scriptstyle \alpha_{n-1}}\downarrow & \\ 0 \to & D & \to & D_{A_n} & \to & D_{A_{n-1}} & \to 0 \end{array}$$

and use $\mathbb{E}\mathrm{xt}^1_{X_{A_n}}(\mathcal{E}_n, \mathcal{E}_{n-1} \xrightarrow{\alpha_{n-1}} D_{A_{n-1}}) \cong \mathbb{E}\mathrm{xt}^1_{X_{A_{n-1}}}(\mathcal{E}_{n-1}, \mathcal{E}_{n-1} \xrightarrow{\alpha_{n-1}} D_{A_{n-1}})$ to see that the cokernel of T^1_n is contained in $\mathbb{E}\mathrm{xt}^2(E, E \xrightarrow{\alpha} D)$. Hence T^1_n is surjective if this Ext-group vanishes. □

We complete this section by presenting a couple of examples and comparing our result with the known ones in these cases.

Examples:

- Obviously, if $D = 0$, then $\mathbb{E}\mathrm{xt}^i(E, E \xrightarrow{\alpha} D) \cong \mathrm{Ext}^i(E, E)$. This is the infinitesimal description of the moduli space due to Grothendieck and Maruyama.
- In the case $D \cong \mathcal{O}_X$ and E locally free the tangent space $\mathbb{E}\mathrm{xt}^1(E, E \xrightarrow{\alpha} D)$ is isomorphic to $\mathbb{H}^1(\mathcal{E}nd(E) \longrightarrow E^{\mathrm{v}})$. Such a framed module corresponds to a Higgs pair $(E^{\mathrm{v}}, \alpha^{\mathrm{v}} \in H^0(X, E^{\mathrm{v}}))$. The description of the infinitesimal deformations of such pairs was given by Thaddeus for rank two vector bundles on curves [Th] and by Welters for line bundles [We].
- Let D be the trivial vector bundle on a hypersurface Y, E be locally free and $\alpha : E \longrightarrow D$ be induced by an isomorphism $E|_Y \cong D$. The tangent space of the moduli space in this case is naturally isomorphic to $\mathbb{E}\mathrm{xt}^1(E, E \xrightarrow{\alpha} D) \cong H^1(X, \mathcal{E}nd(E)(-Y))$. This was computed in [Le].

5 Some examples

This section is devoted to the study of two examples of the moduli spaces of framed modules as constructed in the preceeding sections. In particular, we want to show how these moduli spaces for varying δ can be used in order to resolve certain rational maps naturally occuring in the theory of moduli spaces of pure bundles. For the first example we restrict our attention to the case when X is of dimension two, i.e. X is an algebraic surface. Let E be a $\mu-$stable bundle of rank r and Chern classes c_1 and c_2, hence defining a point in the moduli space $\mathcal{M}(r, c_1, c_2)$ of semistable torsion free sheaves on X with these invariants, and $x \in X$ a closed point in the surface. The kernel E_x of a surjection $E \twoheadrightarrow k(x)$ is a torsion free sheaf of rank r with Chern classes c_1 and $c_2 + 1$. For generic choices E_x is expected to be semistable. This way one obtains a rational map

$$\mathcal{M}(r, c_1, c_2) \longrightarrow \mathcal{M}(r, c_1, c_2 + 1).$$

The second example will show how to resolve the rational map

$$\mathcal{M}(r, c_1, c_2) \longrightarrow \mathcal{M}_Y(r, c_1|_Y, c_2|_Y)$$

given by the restriction of a $\mu-$stable vector bundle to an ample hypersurface of high degree. $\mathcal{M}_Y(r, c_1|_Y, c_2|_Y)$ denotes the moduli space of torsion free semistable sheaves on Y.

Both examples require the existence of a certain relative moduli spaces $\mathcal{M}_\delta^{ss}(X \times S \to S, \mathcal{D}, P)$, i.e. moduli spaces of framed modules associate to a family of reference sheaves $\mathcal{D}$ over $X \times S$. A straightforward generalization of the techniques in section 3 show that $\mathcal{M}_\delta^{ss}(X \times S \to S, \mathcal{D}, P)$ exists as a projective $S-$scheme. The fibre of $\mathcal{M}_\delta^{ss}(X \times S \to S, \mathcal{D}, P) \to S$ over a closed point $s \in S$ is naturally isomorphic to $\mathcal{M}_\delta^{ss}(X, \mathcal{D}_s, P)$.

Example 1: Let X be an algebraic surface and $\mathcal{D} = \mathcal{O}_\Delta$ the structure sheaf of the diagonal $\Delta \subset X \times X$. Considered as a family of reference sheaves parametrized by X itself we can associate to every choice of a rank r, Chern classes c_1, c_2 and a parameter δ a relative moduli space $\mathcal{M}_\delta^{ss}(\mathcal{O}_\Delta, r, c_1, c_2) \to X$ whose fibre over a closed point $x \in X$ is naturally isomorphic to $\mathcal{M}_\delta^{ss}(k(x), r, c_1, c_2)$. We want to write down the stability conditions explicitely.

$(E, E \xrightarrow{\alpha} k(x))$ is (semi)stable if and only if

i) $P_L/\mathrm{rk}L (\leq) P_E/r + (r - \mathrm{rk}L)\delta/r \cdot \mathrm{rk}L$ for all subsheaves $L \subset E$ with $0 < \mathrm{rk}L < r$.

ii) $P_L/\mathrm{rk}L (\leq) P_E/r - \delta/r$ for all subsheaves $L \subset \ker\alpha$ with $0 < \mathrm{rk}L < r$.

iii) $P_{\ker\alpha} (\leq) P_E - \delta$.

iv) α is injective on the torsion $T(E)$ of E and $P_{T(E)} (\leq) \delta$.

By iii) the moduli space $\mathcal{M}_\delta^{ss}(k(x), r, c_1.c_2)$ is empty if $\delta > 1$, i.e. for the moduli space to be nonempty δ has to be a constant polynomial less than or equal to 1.

The critical values of δ, i.e. values where there a priori could exist properly semistable framed modules, are $s/(r, s)$ where $s = 0, ..., r - 1$. For the rank two case we have only two critical values 0 and 1. For simplicity we want to restrict to this case. In particular, all the moduli spaces for $0 < \delta < 1$ are mutually isomorphic. For $\delta \equiv 0$ the moduli space $\mathcal{M}_\delta^{ss}(k(x), 2, c_1.c_2)$ is naturally isomorphic to $X \times \mathcal{M}(2, c_1, c_2)$. If $\delta = 1$ the third conditon implies that a semistable framed module (E, α) is S-equivalent to $(\ker\alpha \oplus k(x), pr_2)$ (cf. 1.13) and condition ii) says that $\ker\alpha$ is semistable, i.e. $[\ker\alpha] \in \mathcal{M}(2, c_1, c_2 + 1)$. Hence

$$\mathcal{M}_1^{ss}(\mathcal{O}_\Delta, 2, c_1, c_2) \cong X \times \mathcal{M}(2, c_1, c_2 + 1).$$

The natural diagram of morphisms

$$
\begin{array}{ccc}
 & \mathcal{M}^{ss}_{1/2}(\mathcal{O}_\Delta, r, c_1.c_2) & \\
p \swarrow & & \searrow q \\
\mathcal{M}^{ss}_0(\mathcal{O}_\Delta, r, c_1, c_2) & & \mathcal{M}^{ss}_1(\mathcal{O}_\Delta, r, c_1, c_2) \\
\cong X \times \mathcal{M}(r, c_1, c_2) & & \cong X \times \mathcal{M}(r, c_1, c_2 + 1)
\end{array}
$$

in fact "resolves" the natural rational map from $\mathcal{M}(r, c_1, c_2)$ to $\mathcal{M}(r, c_1, c_2 + 1)$ we alluded to before.

This is due to the fact that p dominates the open set of stable sheaves. Another interesting fact (cf. 0.1) is that the moduli space in the middle is in fact fine, i.e. there exists a universal family, though the other two may very well be only coarse moduli spaces. Note that in the higher rank case the rational map is not "resolved" by a single diagram as above, but by a sequence of such going-ups and going-downs.

The reader will recognize the above construction as a two dimensional analogue of the Hecke-correspondence on curves. In fact, moduli spaces of framed modules can be used to describe the Hecke-correspondence between moduli spaces of vector bundles on a curve quite similar to what can be done with moduli spaces of parabolic bundles.

Example 2: This example is again motivated by the study of stable vector bundle on surfaces. There the restriction of vector bundles to ample curves of high degree and the associated map between the moduli spaces has become an important tool. Since the fact that a given $\mu-$stable vector bundle will restrict to a $\mu-$stable vector bundle on a hypersurface of high degree is also valid in higher dimension, we want to treat the general case. The only simplifying assumption we make is that the Hilbert polynomial P_Y is chosen in such a way that the moduli space $\mathcal{M}(Y, P_Y)$ of semistable torsionfree sheaves with Hilbert polynomial P_Y is fine, i.e. there exists a universal family $\mathcal{F}$ over $Y \times \mathcal{M}(Y, P_Y)$.

We want to consider the moduli space of objects $(E, E \xrightarrow{\alpha} F)$, where E is a sheaf on X with Hilbert polynomial P_X, F is a semistable sheaf on Y with Hilbert polynomial $P(E_Y) = P_Y$ and α is a non-zero homomorphism. As above our construction works also in this relative setting, i.e. there exists a projective moduli space $\mathcal{M}^{ss}_\delta(\mathcal{F}, P_X) \to \mathcal{M}(Y, P_Y)$. The fibre over a stable sheaf $[F] \in \mathcal{M}(Y, P_Y)$ is naturally isomorphic to the moduli space $\mathcal{M}^{ss}_\delta(F, P_X)$ of semistable framed modules with respect to the coherent sheaf F on X and the parameter δ.

For small δ, say $\delta \equiv \text{const} < 1$, there is a natural morphism from $\mathcal{M}^{ss}_\delta(\mathcal{F}, P_X)$ to the moduli space $\mathcal{M}(X, P_X)$ of semistable torsionfree sheaves on X.

Now we assume that the $\mu-$stable sheaves are dense in $\mathcal{M}(X, P_X)$. By the theorem of Mehta-Ramanathan [MR] for a fixed $\mu-$stable sheaf E we can find a hypersurface Y such that E_Y is $\mu-$stable again. We can even find a Y which will work for finitely many E's. In fact, on surfaces results of Bogomolov and Tyurin show that there is a Y for all $\mu-$stable bundles of fixed Hilbert polynomial. Since $\mu-$stability is an open property we do in deed get a rational morphism

$$\mathcal{M}(X, P_X) \longrightarrow \mathcal{M}(Y, P_Y).$$

We claim that the moduli space $\mathcal{M}_\delta^{ss}(\mathcal{F}, P_X)$ with its natural morphisms to $\mathcal{M}(X, P_X)$ and $\mathcal{M}(Y, P_Y)$ resolves this rational map. The only thing to check is that $\mathcal{M}_\delta^{ss}(\mathcal{F}, P_X) \longrightarrow \mathcal{M}(X, P_X)$ is an isomorphism over the set of sheaves $[\mathcal{E}] \in \mathcal{M}(X, P_X)$ such that E_Y is stable. But this is clear, since for stable E_Y the only non-zero homomorphisms from E to a stable F on Y, which automatically factorizes over E_Y, are isomorphisms. They all differ by scalars only. In particular, the fibre of $\mathcal{M}_\delta^{ss}(\mathcal{F}, P_X) \longrightarrow \mathcal{M}(X, P_X)$ over E is a single point. By the universality properties of the involved moduli spaces one concludes that over these points the morphism is in fact an isomorphism.

In the surface case this resolution answers a question of Tyurin who asked for an extension of the restriction morphism $\mathcal{M}(X, 2, c_1, c_2) \longrightarrow \mathcal{M}(Y, 2, c_1|Y)$, where Y is an ample curve of high degree(cf. [H]). We hope that this approach to the problem will be of use for the study of this map and the relation of the two moduli spaces, e.g. if one is in the case where all three involved moduli spaces are irreducible one gets a fairly explicit description of the boundary of the image of $res : \mathcal{M}(X, 2, c_1, c_2) \longrightarrow \mathcal{M}(Y, 2, c_1|Y)$ (cf. corollary 1.2 of [H]).

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