Derived equivalences and stability conditions (mainly for K3 surfaces)

D. Huybrechts (Bonn)

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Notation

X = smooth projective variety (over $\mathbb{C}$)

$\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ = $\mathbb{C}$-linear triangulated category.

General questions

- $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(Y) \Leftrightarrow X \overset{?}{\leftrightarrow} Y$
- $\mathrm{Aut}(X) \subset \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) =?$
- $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X)) =?$

Notation

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General questions. Answers for curves $g > 1$

- $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(Y) \Leftrightarrow X \simeq Y$.
- $\mathrm{Aut}(X) \subset \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) = \mathbb{Z}[1] \oplus \mathrm{Aut}(X) \rtimes \mathrm{Pic}(X)$.
- $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X)) = \widetilde{\mathrm{Gl}}_{+}(2, \mathbb{R})$.

Some open problems

- Birational invariance of $\mathrm{D}^{\mathrm{b}}(X)$ ($\rightsquigarrow$ MMP)
- Finiteness of Fourier–Mukai partners ($\rightsquigarrow$ MMP)
- Numerical invariants of $\mathrm{D}^{\mathrm{b}}(X)$ ($\rightsquigarrow$ motivic integration)
- Construction of autoequivalences ($\rightsquigarrow$ mirror symmetry)
- Stability conditions for compact CY threefolds
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Birational invariance

Conjecture (Bondal, Orlov, Kawamata) $X \sim_K Y \Rightarrow X \sim_D Y$.

$X \sim_K Y$: $X \leftarrow Z \longrightarrow Y$ birational with $\pi_X^*\omega_X \simeq \pi_Y^*\omega_Y$.
$X \sim_D Y$: $\mathrm{D^b}(X) \simeq \mathrm{D^b}(Y)$.

Special case Birational CYs are derived equivalent.

- Surfaces: OK
- Standard flop: OK (Bondal, Orlov)
- CY-threefolds: OK (Bridgeland)
- Hyperkähler manifolds: special cases OK (Kawamata, Namikawa)

Remark

- $\exists\ X \sim_D Y$, but $X \not\sim_K Y$: abelian varieties, K3 surfaces, CY-threefolds (Borisov, Caldararu)
- $\exists\ X \sim Y$, $X \sim_D Y$, but not $X \sim_K Y$ (Uehara)

Finiteness of FM partners

Conjecture (Kawamata) For any X there exist only finitely many $Y \sim_D X$ (up to isomorphism).

- Surfaces: OK (Bridgeland, Maciocia)
- Abelian varieties: OK (Orlov)

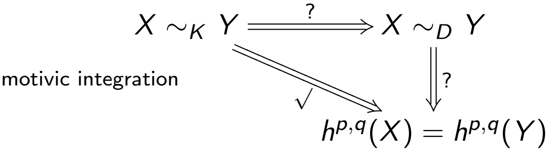
Conjecture (Morrison) For any CY-threefold X there exist only finitely many $Y \sim_K X$ (up to isomorphism).

Numerical invariants

Known $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(Y) \Rightarrow \sum_{p-q=k} h^{p,q}(X) = \sum_{p-q=k} h^{p,q}(Y)$

Conjecture (Kontsevich) $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(Y) \Rightarrow h^{p,q}(X) = h^{p,q}(Y)$

Theorem (Batyrev, Kontsevich) $X \sim_K Y \Rightarrow h^{p,q}(X) = h^{p,q}(Y)$



Definition

$\mathcal{T} = \mathbb{C}$-linear triangulated category (e.g. $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(X)$)
Fix $K(\mathcal{T}) \twoheadrightarrow K$ $(\mathrm{rk}(K) < \infty)$ (e.g. algebraic classes in $H^*(X)$).

Stability condition $\sigma \in \mathrm{Stab}(\mathcal{T})$:
$=$ bounded t-structure $+$ additive $Z : K \longrightarrow \mathbb{C}$ such that:

- $Z(E) = r(E)\exp(i\pi\phi(E))$ with $\phi(E) \in (0,1]$ and $r(E) > 0$, where $0 \neq E \in \mathcal{A} =$ heart of t-structure.
- Z satisfies the HN-property: For $E \in \mathcal{A}$ there exists a filtration $0 \subset E_n \subset \ldots \subset E_0 = E$ with $F_i := E_i/E_{i+1}$ semi-stable and $\phi(F_n) > \ldots > \phi(F_0)$, i.e. $\phi(G) \leq \phi(F_i)$ for all $0 \neq G \subset F_i$.
- 'locally finite', in particular $\mathcal{P}(\phi) := \{E \in \mathcal{A} \mid \text{semi-stable } \phi(E) = \phi\}$ of finite length.

Space of stability conditions

$\sigma \in \mathrm{Stab}(\mathcal{T}) \rightsquigarrow$ slicing $\{\mathcal{P}(t)\} \in \mathrm{Slice}(\mathcal{T})$: $\mathcal{P}(t) \subset \mathcal{T}$, $t \in \mathbb{R}$, full-additive with $\mathcal{P}(t+1) = \mathcal{P}(t)[1]$ and HN-property.

$$\mathrm{Stab}(\mathcal{T}) \subset \mathrm{Slice}(\mathcal{T}) \times K_{\mathbb{C}}^*$$

Topology - Metric topology on $\mathrm{Slice}(\mathcal{T})$: measuring the distance between

$$Z(F_1), \dots, Z(F_n) \text{ and } Z(F'_1), \dots, Z(F'_m)$$

for HNF (E_i) and (E'_j) of any $0 \in E \in \mathcal{T}$ wrt. σ resp. σ'
- Linear topology on $K_{\mathbb{C}}^*$.
- Product topology on connected components.

Theorem (Bridgeland) The projection

$$\pi : \mathrm{Stab}(\mathcal{T}) \longrightarrow K_{\mathbb{C}}^*$$

is a local homeomorphism from each connected component Σ to a linear subspace $V_\Sigma \subset K_{\mathbb{C}}^*$.

Group actions

- $\mathrm{Aut}(\mathcal{T}) \times \mathrm{Stab}(\mathcal{T}) \longrightarrow \mathrm{Stab}(\mathcal{T})$,
  $(\Phi, \sigma = (Z, \mathcal{A})) \longmapsto (Z \circ \Phi^{-1}, \Phi(\mathcal{A}))$.
- $\mathrm{Stab}(\mathcal{T}) \times \widetilde{\mathrm{Gl}}_+(2, \mathbb{R}) \longrightarrow \mathrm{Stab}(\mathcal{T})$ covering
  $K_{\mathbb{C}}^* \times \mathrm{Gl}_+(2, \mathbb{R}) \longrightarrow K_{\mathbb{C}}^*$, $(Z, g) \longmapsto g^{-1} \circ Z$.

$\widetilde{\mathrm{Gl}}_+(2, \mathbb{R}) \longrightarrow \mathrm{Gl}_+(2, \mathbb{R})$ universal cover.
$\pi_1(\widetilde{\mathrm{Gl}}_+(2, \mathbb{R})) \simeq \mathbb{Z}$ induced by $S^1 = U(1) \subset \mathrm{Gl}_+(2, \mathbb{R})$.
For $k \in \mathbb{Z}$ can lift $\exp(i\pi\phi) \in U(1)$ to

$$\{\mathcal{P}(t)\} \longmapsto \{\mathcal{P}'(t) = \mathcal{P}(t + \phi + 2k)\}.$$

Example $\exp(i\pi)$ lifts to the action $F \longmapsto F[2k+1]$.

$\mathrm{Stab}(X) := \mathrm{Stab}(\mathrm{D^b}(X))$, $K \subset H^*(X)$.

Question $\mathrm{Stab}(X) \neq \emptyset$ for X a CY-threefold, abelian variety, hyperkähler manifold?

- Curves: OK (Bridgeland, Macri, Okada)
- Abelian and K3 surfaces: OK (Bridgeland)
- $\mathbb{P}^n$, del Pezzo surfaces (Macri)
- Special open CY-threefolds (Bridgeland), generic complex tori (Meinhardt)
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Examples: Curves

C curve, $g(C) \geq 1$, $K(C) = K(\mathrm{D}^{\mathrm{b}}(C)) = \mathbb{Z} \oplus \mathrm{Pic}(C)$. Choose $K(C) \twoheadrightarrow K := \mathbb{Z} \oplus \mathrm{NS}(C) = \mathbb{Z} \oplus \mathbb{Z}$.

Example: $Z : K \longrightarrow \mathbb{C}$, $(r, d) \longmapsto -d + i \cdot r$, $\mathcal{A} = \mathrm{Coh}(C)$. This yields stability condition with $\mathcal{A} = \mathrm{Coh}(C)$ and $\mathcal{P}(\phi) =$ set of semi-stable vector bundles E with $\cot(\pi\phi) = -\mu(E) = -d/r$ (for $\phi \in (0, 1)$).

Theorem $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(C)) \simeq \widetilde{\mathrm{Gl}}_+(2, \mathbb{R})$ and $\mathrm{Stab} \longrightarrow K_{\mathbb{C}}^*$ is universal covering $\widetilde{\mathrm{Gl}}_+(2, \mathbb{R}) \longrightarrow \mathrm{Gl}_+(2, \mathbb{R})$.

Curve like

For X surface let $\mathrm{Coh}(X) \twoheadrightarrow \mathrm{Coh}'(X)$ quotient by subcategory of sheaves concentrated in dim < 1. Let $\mathcal{T} := \mathrm{D^b}(\mathrm{Coh}'(X))$. Then $K(\mathcal{T}) = \mathbb{Z} \oplus \mathrm{Pic}(X)$. Consider $K(\mathcal{T}) \twoheadrightarrow K := \mathbb{Z} \oplus \mathrm{NS}(X)$.

Example For ω ample class $Z(E) := -(\mathrm{c}_1(E).\omega) + i \cdot \mathrm{rk}(E)$ defines a stability function on $\mathrm{Coh}'(X)$ with HN-property. So $\mathrm{Stab}(\mathcal{T}) \neq \emptyset$.

Remark If $\mathrm{Pic}^0(X) \neq 0$, then $\widetilde{\mathrm{Gl}}_+(2, \mathbb{R})$-orbit is (simply-) connected component. For $\rho(X) > 1$ get infinitely many connected components!

Examples: K3 surfaces

X K3 surface. Fix $K(X) \twoheadrightarrow N(X) \subset H^*(X, \mathbb{Z})$ algebraic classes and $\omega, B \in \mathrm{NS}(X)$ with ω ample, $\omega^2 > 2$. Consider full subcategory

$$\mathcal{A}(B + i\omega) \subset \mathrm{D}^{\mathrm{b}}(X) \quad \text{of complexes} \quad F^{-1} \longrightarrow F^0$$

s.t. ker torsion free, $\mu_{\max} \leq (B.\omega)$ and $\mu_{\min}(\mathrm{coker}/\mathit{Tors}) > (B.\omega)$.

Then $\mathcal{A}(B + i\omega)$ is heart of bounded t-structure and

$$Z : \mathcal{A} \longrightarrow \mathbb{C}, \quad Z(E) = \langle \exp(B + i\omega), v(E) \rangle$$

is a stability function with HN-property (Bridgeland).

Recall $\langle\ ,\ \rangle$ = Mukai pairing and $v(E) = \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(X)}$

Derived equivalence of K3 surfaces

Theorem (Mukai, Orlov) For X and X' K3 surfaces:
$\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(X') \Leftrightarrow \widetilde{H}(X, \mathbb{Z}) \simeq \widetilde{H}(X', \mathbb{Z})$.

Recall $\widetilde{H}(X) := H^*(X)$ with Mukai pairing and orthogonal weight-two Hodge structure given by $\widetilde{H}^{2,0}(X) = H^{2,0}(X)$.

Theorem $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(X')$

- $\Leftrightarrow$ $X \simeq X'$ or $X \simeq$ moduli space of μ-stable bundles on X'.
- $\Leftrightarrow$ There exist rational $B + i\omega$ and $B' + i\omega'$ such that $\mathcal{A}_X(B + i\omega) \simeq \mathcal{A}_{X'}(B' + i\omega')$ (as $\mathbb{C}$-linear abelian categories).

Action on cohomology

Theorem (Mukai) $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X') \rightsquigarrow$ Hodge isometry $\Phi_{\mathcal{E}}^H : \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X',\mathbb{Z})$, $\alpha \longmapsto p_*(q^*\alpha.v(\mathcal{E}))$.

$\rightsquigarrow$ representation: $\rho : \mathrm{Aut}(\mathrm{D^b}(X)) \longrightarrow \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))$.

Question Can one describe image and kernel of ρ ?

Theorem (Mukai, Orlov, Hosono et al, Ploog):

$$\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})) \subset \mathrm{Im}(\rho).$$

$\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})) =$ subgroup of Hodge isometries preserving the orientation of the positive four-space $\langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1+\omega^2/2, \omega \rangle$.

Examples of autoequivalences

- Automorphism $f : X \xrightarrow{\sim} X$
  $\rightsquigarrow \Phi_{\mathcal{E}} := f_* : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$, $\mathcal{E} = \mathcal{O}_{\mathrm{Graph}(f)}$
  $\rightsquigarrow \Phi_{\mathcal{E}}^H = f_*$.
- Shift: $\Phi_{\mathcal{E}} : F \longmapsto F[1]$, $\mathcal{E} = \mathcal{O}_\Delta[1]$
  $\rightsquigarrow \Phi_{\mathcal{E}}^H = -\mathrm{id}$.
- Line bundle twist: $L \in \mathrm{Pic}(X)$
  $\rightsquigarrow \Phi_{\mathcal{E}} : F \longmapsto F \otimes L$, $\mathcal{E} = \Delta_* L$.
  $\rightsquigarrow \Phi_{\mathcal{E}}^H = \mathrm{ch}(L) \cdot \quad .$
- Spherical twist: $E \in \mathrm{D^b}(X)$ with $\mathrm{Ext}^*(E, E) = H^*(S^2, \mathbb{C})$
  $\rightsquigarrow \Phi_{\mathcal{E}} = T_E$, $\mathcal{E} = \mathrm{Cone}(E^\vee \boxtimes E \longrightarrow \mathcal{O}_\Delta)$.
  $\rightsquigarrow \Phi_{\mathcal{E}}^H = s_{v(E)} =$ reflection in hyperplane $v(E)^\perp$.
- Universal family $\mathcal{E} \in \mathrm{Coh}(X \times M)$ of stable sheaves with dim $M = 2$ and M projective. Sometimes $X \simeq M$.

Bridgeland's conjecture

Let $\Sigma \subset \mathrm{Stab}(X)$ be component containing $\mathcal{A}(B + i\omega)$.

Theorem (Bridgeland) $\Sigma \twoheadrightarrow \mathcal{P}_0^+(X)$, $\sigma \longmapsto Z$ is a covering map.

• $\mathcal{P}^+(X) \subset \{\varphi \in N(X)_{\mathbb{C}} \mid \langle \mathrm{Re}(\varphi), \mathrm{Im}(\varphi)\rangle$ positive plane$\}$ connected component containing $\exp(B + i\omega)$.
• $\mathcal{P}_0^+(X) := \mathcal{P}^+(X) \setminus \bigcup_{\delta\in\Delta} \delta^\perp$, where $\Delta = \{\delta \in N(X) \mid \delta^2 = -2\}$.

Conjecture (strong form) (Bridgeland) $\Sigma \subset \mathrm{Stab}(X)$ is the only connected component and Σ is simply connected.

Consequence There is a short exact sequence

$$0 \longrightarrow \pi_1(\mathcal{P}_0^+(X)) \longrightarrow \mathrm{Aut}(\mathrm{D^b}(X)) \longrightarrow \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})) \longrightarrow 0.$$

Theorem (H., Macri, Stellari) Suppose $\mathrm{Pic}(X) = 0$. Then

- Σ is simply-connected.
- $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) = \mathrm{Aut}(X) \oplus \mathbb{Z}[1] \oplus \mathbb{Z}T_{\mathcal{O}}$.

Theorem (H., Macri, Stellari) If X is projective, then

$$\mathrm{Im}\left(\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))\right) = \mathrm{O}_{+}(\widetilde{H}(X,\mathbb{Z})).$$

Szendroi: This is the 'mirror dual' of

Theorem (Donaldson) If X is differentiable K3, then

$$\mathrm{Im}\left(\mathrm{Diff}(X) \longrightarrow \mathrm{O}(H^2(X,\mathbb{Z}))\right) = \mathrm{O}_{+}(H^2(X,\mathbb{Z})).$$

Consider smooth family $\mathbb{X} \longrightarrow \mathbb{P}^1$ (not projective!) with special fibre $X = \mathbb{X}_0$ and formal neighbourhood $\mathcal{X} \longrightarrow \mathrm{Spf}(\mathbb{C}[[t]])$.
Encode general fibre $\mathcal{X}_K \longrightarrow \mathrm{Spec}(\mathbb{C}((t)))$ by abelian category

$$\mathrm{Coh}(\mathcal{X}_K) := \mathrm{Coh}(\mathcal{X})/\mathrm{Coh}(\mathcal{X})_{t-\mathrm{tors}}.$$

Then $\mathrm{Coh}(\mathcal{X}_K)$ behaves like K3 surface over $\mathbb{C}((t))$. If $\mathbb{X} \longrightarrow \mathbb{P}^1$ generic twistor space, then $\mathcal{O}_{\mathcal{X}_K} \in \mathrm{Coh}(\mathcal{X}_K)$ is the only spherical object.

Consider $\Phi_{\mathcal{E}} \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ with $\Phi^H_{\mathcal{E}_0} = \pm\mathrm{id}_{H^2} \oplus \mathrm{id}_{H^0 \oplus H^4}$. Using deformation theory for complexes:

Theorem (Sketch) $\mathcal{E}_0$ deforms to $\mathcal{E}_K \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_K \times \mathcal{X}_K)$ with $\Phi_{\mathcal{E}_K}$ equivalence. Hence $\Phi_{\mathcal{E}_K} = T^n_{\mathcal{O}_K} \circ [m]$.

Corollary $\Phi_{\mathcal{E}_0}$ and $T^n_{\mathcal{O}} \circ [m]$ behave identically on H^* and CH^*.

Deformation theory of complexes: details

Consider extension of $\mathcal{X}_n \longrightarrow \mathrm{Spec}(R_n := \mathbb{C}[t]/t^{n+1})$ to $\mathcal{X}_{n+1} \longrightarrow \mathrm{Spec}(R_{n+1})$ and perfect complex $\mathcal{E}_n \in \mathrm{D^b}(\mathcal{X}_n)$. Let $i_n : \mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$.

Theorem (Lieblich, Loewen) There is an 'obstruction class'

$$o(\mathcal{E}_n) \in \mathrm{Ext}^2_X(\mathcal{E}_0, \mathcal{E}_0),$$

such that $\mathcal{E}_{n+1} \in \mathrm{D^b}(\mathcal{X}_{n+1})$ perfect with $Li_n^*(\mathcal{E}_{n+1}) \simeq \mathcal{E}_n$ exists if and only if $o(\mathcal{E}_n) = 0$.

Theorem (H., Macri, Stellari) $o(\mathcal{E}_n) = A(\mathcal{E}_n) \cdot \kappa_n$.

Use Atiyah class $A(\mathcal{E}_n) \in \mathrm{Ext}^1(\mathcal{E}_n, \mathcal{E}_n \otimes \Omega_{\mathcal{X}_n})$ and Kodaira–Spencer class $\kappa_n \in \mathrm{Ext}^1(\Omega_{\mathcal{X}_n}, \mathcal{O}_X)$.

Chow ring under derived equivalence (in progress)

Consider $A(X) := \mathbb{Z}[X] \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}[x] \subset \mathrm{CH}(X)_{\mathbb{Q}}$, where $x \in C$ rational curve in X.

Theorem (Beauville, Voisin) $A(X) \subset CH(X)$ is a subring.

Corollary If $\Phi_{\mathcal{E}}^{H} = \mathrm{id}$, then

$$\Phi_{\mathcal{E}}^{\mathrm{CH}} : A(X) \xrightarrow{\sim} A(X).$$

Open Consider $\mathcal{E}$ universal family of stable bundles.
Is then $\Phi_{\mathcal{E}}^{\mathrm{CH}}(A(X)) = A(M)$?
($\Leftrightarrow \mathrm{ch}(E) \in A(X)$ for any spherical object $E \in \mathrm{D^b}(X)$.)