Hyperkähler Geometry

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ITS Colloquium

May 30, 2022

European Research Council
Established by the European Commission

HYPERK

UNIVERSITÄT BONN

HAUSDORFF CENTER FOR MATHEMATICS

Goals:

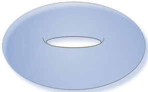
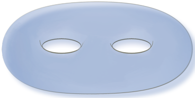
- Place of HK in the landscape of (complex) geometry
- Main features of HKs
- Classsification & Examples
- What would we like to know?
- Recent results (joint work with Debarre, Macrì, and Voisin)

Warning:

Here HK are compact! Non-compact HK is a different world (with some links)

Reminder on the classification of compact Riemann surfaces

Topology:

S^2	$S^1 \times S^1$	
$g = 0$	$g = 1$	$g \geq 2$

Equations: $F(x_0, x_1, x_2) = 0 \subset \mathbb{P}^2_{\mathbb{C}}$

$\deg(F) = 1, 2$	$\deg(F) = 3$	$\deg(F) \geq 4$

Parametrization:

$\mathbb{P}^1_{\mathbb{C}}$	$\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$	$\mathbb{H}/\Gamma$

Curvature: $\mathrm{c}_1 = [\omega] \in H^2(C, \mathbb{R})$, locally $\omega = \sqrt{-1}\, h\, dz \wedge d\bar{z}$

$h > 0$	$h = 0$	$h < 0$

Classification of compact complex(!) surfaces

Topology: ??

Equations: $F(x_0, x_1, x_2, x_3) = 0 \subset \mathbb{P}^3_{\mathbb{C}}$

$\deg(F) \leq 3$	$\deg(F) = 4$		$\deg(F) \geq 5$

Parametrization / Examples:

$\mathbb{P}^2_{\mathbb{C}}$	$\mathbb{C}^2/\Lambda$	...	$\mathbb{H}^2/\Gamma$

Curvature: $c_1 = [\omega] \in H^2(S, \mathbb{R})$, locally $\omega = \sqrt{-1} \sum h_{ij}\, dz_i \wedge d\bar{z}_j$

$(h_{ij}) > 0$	$(h_{ij}) = 0$	...	$(h_{ij}) < 0$

Holomorphic symplectic: $\exists$ 2-form σ: locally $f\, dz_1 \wedge dz_2$, $f \neq 0$, holomorphic

- $\mathbb{C}^2/\Lambda$ complex torus
- K3 surface $\approx S \subset \mathbb{P}^3$: $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$

Kummer, Kähler, Kodaira, et de la belle montagne de K2

Classification in higher dimensions

Curvature:

$c_1 > 0$ $c_1 = 0$ ... ... $c_1 < 0$

Beauville–Bogomolov–Yau 1983: $X \subset \mathbb{P}^N_{\mathbb{C}}$ compact, complex, manifold with $c_1 = 0$

$$\Rightarrow \quad \widetilde{X} \cong \mathbb{C}^N/\Lambda \times \prod_i \mathrm{HK}_i \times \prod_i \mathrm{CY}_i$$

Hyperkähler manifold (HK): $X =$ compact complex manifold

& $\exists\, \sigma$ holomorphic symplectic form: pointwise $\sigma = \sum dz_i \wedge dz_{n+i}$

Additional assumptions: $X \subset \mathbb{P}^N_{\mathbb{C}}$ (or Kähler) & $\pi_1(X) = \{1\}$ & σ unique

($\rightsquigarrow$ exclude tori $\mathbb{C}^n/\Lambda$ and products)

Examples: K3 surfaces, ??

Bogomolov 1978: HK do not exist in $\dim_{\mathbb{C}} > 2$, i.e. K3 surfaces are the only HK

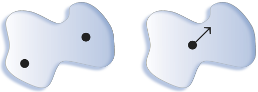
...and yet they exist

Beauville, Fujiki 1983:

- $S =$ K3 surface $\rightsquigarrow$ $S \times S \longrightarrow (S \times S)/\mathfrak{S}_2 \longleftarrow S^{[2]}$ = HK, $\dim_{\mathbb{C}} = 4$

  σ unique, $\pi_1 = \{1\}$

  but singular



Similar: $S^n \longrightarrow S^n/\mathfrak{S}_n \longleftarrow S^{[n]}$ = HK, $\dim_{\mathbb{C}} = 2n$

- $A = \mathbb{C}^2/\Lambda$ $\rightsquigarrow$ $A^n \longrightarrow A^n/\mathfrak{S}_n \longleftarrow A^{[n]} \supset K_{n-1}(A)$ = HK, $\dim_{\mathbb{C}} = 2(n-1)$

$\rightsquigarrow$ **Two series of examples of HK (topologically):**

'Hilbert schemes' $S^{[n]}$ and 'Kummer varieties' $K_n(A)$

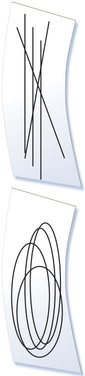
Recall: Topologically there exists only one torus in each dimension: $\mathbb{C}^n/\Lambda \approx (S^1)^{2n}$

$\rightsquigarrow$ The HK world is richer!

Remark: $\exists$ unexplained further complex structures, not coming from S or A

O'Grady 1999 & 2000: $\exists$ two further examples $\dim_{\mathbb{C}} = 6$ & $\dim_{\mathbb{C}} = 10$

Start again with K3 or torus, but get topologically new HK



Beauville–Donagi 1985:

Start with $Y \subset \mathbb{P}^5_{\mathbb{C}}$: $x_0^3 + \cdots + x_5^3 = 0$

$\rightsquigarrow X = F(Y) = \{\, \ell \subset Y \subset \mathbb{P}^5_{\mathbb{C}} \mid \text{line} \,\}$ is HK of $\dim_{\mathbb{C}} = 4$

But topologically not new $\approx S^{[2]}$

Lehn–Lehn–Sorger–van Straten 2017:

Start again with $Y \subset \mathbb{P}^5_{\mathbb{C}}$ $\rightsquigarrow$ $F'(Y)$ HK of $\dim_{\mathbb{C}} = 8$, topologically $\approx S^{[4]}$

General problem: There is a shortage of methods to construct interesting varieties

HK are not constructed neither by equations nor by parametrizations

Main features: BBF form

Principles:

- Strong restriction on topology
- HK behave like (K3) surfaces, i.e. topological fourfolds
- H^2 rules

Beauville–Bogomolov–Fujiki form: $X = \text{HK}$, $\dim_{\mathbb{C}} = 2n \quad (\Rightarrow \dim_{\mathbb{R}} = 4n)$

$$\Rightarrow \exists\, q\colon H^2(X,\mathbb{Z}) \to \mathbb{Z} \text{ non-deg. quadratic form } \ \& \ c \in \mathbb{Q}:$$

$$q(\alpha)^n = c \cdot \int_X \alpha^{2n} \quad \forall \alpha \in H^2(X,\mathbb{Z})$$

For $n = 1$: $q = (\ .\ )$ intersection form

Question: How restrictive is this for compact topological manifolds?

Verbitsky, Bogomolov 1996: Cohomology ring

$$S^*H^2(X) \twoheadrightarrow S^*H^2(X)/(H_{2n+2}) \hookrightarrow H^{2*}(X)$$

Main features: Using the BBF form

Surjectivity of period map 1999: X HK $\rightsquigarrow [\sigma] \in H^2(X, \mathbb{C})$: $q(\sigma) = 0$ & $q(\sigma, \bar{\sigma}) > 0$

If $[\sigma]' \in H^2(X, \mathbb{C})$ with ... $\Rightarrow$ comes from X'

Finiteness 2003: Fix $q\colon H^2(X, \mathbb{Z}) \to \mathbb{Z} \;\Rightarrow\; \exists$ at most finitely many topological HK X.

Global Torelli theorems:

(i) **K3 surfaces** [Pjateckiĭ-Šapiro–Šafarevič '71, Burns–Rapoport '75]:

$$H^2(S, \mathbb{Z}) \cong H^2(S', \mathbb{Z}) \;\&\; q \;\&\; \sigma \;\Rightarrow\; S \cong S'$$

(ii) **HK** [Verbitsky '13, Bourbaki '12, Looijenga '19]:

$$H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z}) \;\&\; q \;\&\; \sigma \;\&\; \text{monodromy} \Rightarrow\; X \sim X'$$

Central role of BBF form: What do we know? Not much.

For sure: $\mathrm{sing}(q) = (3, b_2 - 3)$ & $b_2 = 23$ or $b_2 \leq 8$ in $\dim_{\mathbb{C}} = 4$

Challenge: Exclude the case $b_2(X) = 3$, i.e. $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 3}$

Further features

Curves in K3 surfaces

$S =$ K3 surface, $C \subset S$ smooth curve

$$c_1(C) > 0 \iff q(C) < 0$$
$$c_1(C) = 0 \iff q(C) = 0$$
$$c_1(C) < 0 \iff q(C) > 0$$

Hypersurfaces in HK

$X =$ HK, $D \subset X$ smooth, $\dim_{\mathbb{C}} D = \dim_{\mathbb{C}} X - 1$

[Amerik–Campana, Abugaliev 2014-2021]

$$c_1(D) > 0 \iff q(D) < 0$$
$$c_1(D) = 0 \iff q(D) = 0$$
$$\vdots \quad \textit{do not occur!}$$
$$c_1(D) < 0 \iff q(D) > 0$$

Hirzebruch–Riemann–Roch

$S =$ K3 surface, $L \in \mathrm{Pic}(S)$

$$\Rightarrow \chi(S, L) = \tfrac{1}{2} \cdot q(c_1(L)) + 2$$

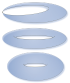
Riemann–Roch polynomials

$X =$ HK, $L \in \mathrm{Pic}(X)$

$$\Rightarrow \chi(X, L) = P(q(c_1(L)))$$
$$= \tfrac{c}{(2n)!} \cdot q(c_1(L))^n + \cdots + (n+1)$$

Lagrangian fibrations (integrable systems)

- $S \subset \mathbb{P}^3_{\mathbb{C}}$: $x_0^4 + \cdots + x_3^4 = 0$ Fermat surface: $[x_0 : \cdots : x_3] \longmapsto [x_0^2 + \xi x_1^2 : x_2^2 - \xi x_3^2]$

  $\rightsquigarrow \quad S =$  $\longrightarrow\!\!\!\!\!\rightarrow \mathbb{P}^1_{\mathbb{C}}$ with elliptic curves as generic fibres $S_t \cong \mathbb{C}/\Lambda$

  Special! But after small deformation every K3 surface $S \;\rightsquigarrow\; S' \longrightarrow\!\!\!\!\!\rightarrow \mathbb{P}^1_{\mathbb{C}}$

- Higher dimensions: Start with $S \longrightarrow \mathbb{P}^1_{\mathbb{C}} \;\rightsquigarrow\; S \times S \longrightarrow \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$

$\rightsquigarrow\; S^{[2]} \longrightarrow \mathbf{(S \times S)/\mathfrak{S}_2} \longrightarrow \mathbf{(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}})/\mathfrak{S}_2} \cong \mathbb{P}^2$ with fibres $S^{[2]}_{t,s} = S_t \times S_s \cong \mathbb{C}^2/\Lambda$

SYZ Conjecture: $X =$ HK, σ holomorphic two-form

$\Rightarrow$ after deformation $\exists \; X' \longrightarrow\!\!\!\!\!\rightarrow \mathbb{P}^n_{\mathbb{C}}$ with generic fibres $X'_t \cong \mathbb{C}^n/\Lambda$

Concrete SYZ Conjecture: If $\exists\, \alpha \in H^2(X, \mathbb{Z})$: $q(\alpha) = 0$ & $q(\alpha, \sigma) = 0$

$\Rightarrow \exists \; X \dashrightarrow\!\!\!\!\!\rightarrow \mathbb{P}^n_{\mathbb{C}}$ (Lagrangian fibration)

Going through the examples: True for all series of examples (non-trivial!)

Towards a classification in $\dim_{\mathbb{C}} = 4$

Theorem (with Debarre, Macrì, Voisin 2022):

- If X is HK, $\dim_{\mathbb{C}} = 4$ with $H^*(X,\mathbb{Z}) \cong H^*(S^{[2]},\mathbb{Z})$, then X is member of the $S^{[2]}$-clan
- Enough: $\exists\, \alpha,\beta \in H^2(X,\mathbb{Z})$ with $\int \alpha^4 = \int \beta^4 = 0$ and $\int \alpha^2 \cdot \beta^2 = 2$ (minimal)
- SYZ conjecture holds in this case

Meaning: $X \to \mathbb{P}^2$ is Lagrangian fibration:

(i) $\alpha =$ pullback of hyperplane class $\in H^2(\mathbb{P}^2,\mathbb{Z}) \;\; \Rightarrow \;\; \int \alpha^4 = 0$

(ii) $\mathbb{Z}\cdot\beta = \;\mathrm{Im}(H^2(X,\mathbb{Z}) \longrightarrow H^2(A,\mathbb{Z}))$ for smooth fibre $A = X_t \cong \mathbb{C}^2/\Lambda$

(iii) $\int \alpha^2 \cdot \beta^2 = 2$ is the case of principally polarized abelian surfaces, e.g. $A \cong E_1 \times E_2$

$\rightsquigarrow$ Maybe that's it ?

Expectation (today 5:10pm): One more example in dimension four

Open questions & current trends

- Cohomological description of higher-dimensional $S^{[n]}$
- Cohomological description of second series $K_2(A)$
- $A =$ polarized abelian variety of $\dim_{\mathbb{C}} = n$. When $\exists\, A \subset X =$ HK of $\dim_{\mathbb{C}} = 2n$?

  No restriction in small dimensions

- Test standard conjectures: Hodge, Tate, Grothendieck, ...
- Group actions on HK
- Geometric Langlands, $P = W$ conjecture
- Generalize existing theory to mildly singular HK. More examples!
- Dimension reduction (via derived categories)