Seshadri Constants on Algebraic Surfaces

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1 Introduction

This Bachelor thesis deals with the so-called Seshadri constant of a polarized variety (X, L) (i.e., a variety X with a line bundle L) at a closed point $x \in X$. Roughly speaking, it measures the positivity of L with respect to the intersection pairing at x. In particular, the Nakai–Moishezon criterion correctly suggests that it is positive for ample line bundles.
Originally defined by Demailly in 1990, this invariant has sparked interest because some theorems indicate that it might encode a lot of geometric information, like the existence of certain fibrations, and because the famous Nagata conjecture can be formulated via (multi-point) Seshadri constants. However, Seshadri constants are very difficult to calculate, so that one often has to make do with estimates.
This thesis is structured as follows: After recalling some basic definitions and facts from intersection theory we will define and discuss Seshadri constants. In the next three chapters we restrict ourselves to the surface case and present various theorems about Seshadri constants: We will provide a proof of a result by Ein and Lazarsfeld ([EL93]) concerning a lower bound for the Seshadri constant at 'very general' points. Then we will explain the main results of [Bau99], where the author discusses Seshadri constants of the canonical bundle on minimal surfaces of general type. Finally we will sketch the proof of a result that computes Seshadri constants for a large class of abelian surfaces.

Notations and Conventions. *Variety* always refers to a proper integral scheme X over a fixed algebraically closed field k of arbitrary characteristic. If we want to drop the condition of being integral or proper we say *nonintegral* or *nonproper* varieties. Still we want nonproper varieties to be separated and of finite type snd we want nonintegral varieties to be equidimensional. A *subvariety* of X is always assumed to be a *closed* subvariety, unless stated otherwise. A *proper subvariety* is a subvariety that does not coincide with X. A point $x \in X$ is always assumed to be a closed point. A *curve* (*surface*) is a variety of dimension 1 (resp. 2). In particular curves and surfaces are always integral. Given a morphism $f : X \to T$ of varieties and a point $t \in T$ we denote its fibre by X_t. If $\pi : \tilde{X} \to X$ is a blow up and $V \subset X$ a subvariety, its strict transform is denoted by $\tilde{V}$.
Given a coherent sheaf F on the variety X we use the short cut $h^i(F) := h^i(X, F) := \dim_k H^i(X, F)$ for $i \geq 0$.
Given two sequences a, b with $a_n, b_n \geq 0$ for $n \gg 0$ write $a = O(b)$ if $\exists C > 0 : \forall n \gg 0 : a_n \leq C \cdot b_n$. If $a_n, b_n > 0$ for $n \gg 0$ we write $a \sim b$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

2 Intersection theory

In this chapter we recall basic notions and facts from intersection theory. Fix a variety X over an algebraically closed field k (actually most facts in this chapter still work if we drop the integrality assumption). Its Picard group $\operatorname{Pic} X$ is isomorphic to the group of Cartier divisors modulo linear equivalence. So we can and will make no distinction between line bundles and Cartier divisors. Very often we will write $L + M$ instead of $L \otimes M$ for line bundles L, M. By abuse of notation we will not distinguish between Cartier divisors (like curves on a surface) and Cartier divisors modulo linear equivalence in many cases. If X has a canonical bundle, denote it by K_X. We have the

Theorem 2.1 (Snapper)**.** *Let F be a coherent $\mathcal{O}_X$-module and let $L_1, \ldots, L_t$ be line bundles on X. Then*

$$\varphi(n_1, \ldots, n_t) \coloneqq \chi\left(F \otimes L_1^{n_1} \otimes \ldots \otimes L_t^{n_t}\right)$$

is a numerical polynomial in $n_1, \ldots, n_t$ of degree $\leq s = \dim(\operatorname{Supp}(F))$.

For nonsingular X this follows from the Hirzebruch–Riemann–Roch theorem. But for general X there is a much easier proof in [Kle66]. If $s \leq t$, this theorem enables us to give the

Definition 2.1. The *intersection number* or *intersection multiplicity of $L_1, \ldots, L_t$ with F*, denoted by $(L_1 \ldots L_t \cdot F)$, is the coefficient of the monomial $n_1 \cdot \ldots \cdot n_t$ in φ. When talking about the intersection number as a function of $L_1, \ldots, L_t$, we also say *intersection form* or *intersection pairing*, if $t = 2$. If $L \coloneqq L_1 = \ldots = L_t$, we use the short cut $\left(L^{\cdot t} \cdot F\right)$ for the intersection number.
If $F = \mathcal{O}_Y$ with Y a closed subscheme of X, write $(L_1 \ldots L_t \cdot Y)$ instead of $(L_1 \ldots L_t \cdot \mathcal{O}_Y)$. If $Y = X$ we write $(L_1 \ldots L_t)$ for short.

By general properties of numerical polynomials the intersection number is always an integer.
Some facts stated here for future use are:

Lemma 2.2. $(L_1 \ldots L_t \cdot F)$ *is a symmetric multilinear form in $L_1, \ldots, L_t$.*

Lemma 2.3. *If $t = \dim X$, the leading coefficient of the polynomial $\chi\left(L^{\otimes n}\right)$ is $\left(L^{\cdot t}\right)/t!$.*

Lemma 2.4. *Let $f : Y \to X$ be a proper morphism between varieties and assume $t \geq \dim X, \dim Y$. Then for all line bundles $L_1, \ldots, L_t$ on X we have*

$$(f^* L_1 \ldots f^* L_t) = \deg(f) \cdot (L_1 \ldots L_t).$$

Note that we are using the convention $\deg(f) = 0$, if f is not generically finite.

Lemma 2.5. *If $Y \subset Z \subset X$ are subvarieties with $\dim Y = s$, then for all line bundles $L_1, \ldots, L_s$ on X we have*

$$\left(i^* L_1 \ldots i^* L_s \cdot Y\right) = \left(L_1 \ldots L_s \cdot Y\right),$$

where $i : Z \hookrightarrow X$ is the canonical embedding.

(Quick) proofs can be found in [Kle66]. Note that Lemma 2.3 and [Har77, Prop. III.5.3] imply

Corollary 2.6. *If L is ample, then $h^0\left(X, L^{\otimes n}\right) \sim n^t \left(L^{\cdot t}\right)/t!$.*

The next issue is positivity. Let us recall the well-known

Theorem 2.7 (Nakai–Moishezon Criterion)**.** *A line bundle L on X is ample if and only if $\left(L^{\cdot t} \cdot V\right) > 0$ for all subvarieties $V \subset X$ of positive dimension t.*

This allows us to think of ample line bundles as 'positive' line bundles. By analogy 'semi-positive' and 'zero' line bundles are given by the

Definition 2.2. A line bundle L on X is called *nef* (*numerically trivial*), if $\left(L^{\cdot t} \cdot V\right)$ is nonnegative (trivial) for all subvarieties $V \subset X$ of dimension $t > 0$. If $L - M$ is numerically trivial for line bundles L, M, we say that those line bundles are *numerically equivalent* and write $L \equiv M$.

Definition 2.3. A nef line bundle L on X is called *big and nef*, if its top-intersection $\left(L^{\cdot \dim X}\right)$ is positive.

We have:

Lemma 2.8. *If $f : Y \to X$ is a proper morphism between varieties and if L is a nef line bundle on X, then $f^* L$ is also nef.*

Proof. This is immediate from Lemma 2.4. Note that the proof breaks down, if we replace 'nef' by 'ample', because f might contract subvarieties of Y. □

Theorem 2.9 (Kleiman)**.** *L is nef (numerically trivial), if and only if $(L \cdot C)$ is nonnegative (resp. trivial) for all curves $C \subset X$.*

Proof. [Kle66, Thm. II.2.1, III.2.1] □

One is tempted to think the theorem still holds when 'nef' is substituted by 'ample' and '$\geq$' by '> 0', but this is false. Instead the correct analogon is Seshadri's criterion, which will be proved in the next chapter.
The quotient $N^1(X) := \mathrm{Pic}(X)/\equiv$ is called the *Néron-Severi group* of X. We have the

Theorem 2.10 (Theorem of the base)**.** *The abelian group $N^1(X)$ is free of finite rank $\rho(X)$. This rank is also called the* Picard number *of X.*

Proof. [LN59]. For k algebraically closed there is a quick proof in [Har77, Appendix B.5]. More information about available proofs is given in [Kah06, Chap. 1]. $\square$

A basic notion that will play a role in the next chapters is the multiplcity of points in varieties:

Definition 2.4. Let X be an arbitrary variety and let $x \in X$. The Hilbert polynomial P_x of the local ring $\mathcal{O}_{X,x}$ has degree $n = \dim X$. We define the *multiplicity* $\operatorname{mult}_x X$ to be $n!$ times the leading coefficient of P_x.
Using the Chinese remainder theorem one can show that $\operatorname{mult}_x(-)$ is an additive function of effective d-cycles for every $d \geq 0$.

Remark 2.5. If X is a variety that is smooth at $x \in X$, and if $V \subset X$ is a subvariety of codimension 1, then there is an alternative definition for $\operatorname{mult}_x V$: It is the largest integer m such that $\mathcal{I}_{V,x} \subset \mathfrak{m}^m$, where $\mathcal{I}_V \subset \mathcal{O}_X$ is the ideal sheaf of V and $\mathfrak{m}$ the maximal ideal of $A := \mathcal{O}_{X,x}$.
Indeed, by the regularity of A the prime ideal $\mathcal{I}_{V,x}$ is principal. It is generated by, say, $f \in \mathfrak{m}^m \setminus \mathfrak{m}^{m+1}$. Consider the exact sequence

$$0 \longrightarrow \left(\mathfrak{m}^i : (f)\right)/\mathfrak{m}^i \longrightarrow A/\mathfrak{m}^i \xrightarrow{f} A/\mathfrak{m}^i \longrightarrow A/\mathfrak{m}^i + (f) \longrightarrow 0$$

for $i \geq m$. Using the integrality of the associated graded ring of A we get $\left(\mathfrak{m}^i : (f)\right) = \mathfrak{m}^{i-m}$. Therefore

$$\dim A/\mathfrak{m}^i + (f) = \dim \mathfrak{m}^{i-m}/\mathfrak{m}^i = \dim A/\mathfrak{m}^i - \dim A/\mathfrak{m}^{i-m}.$$

An elementary calculation shows that this is asymptotically a polynomial with leading coefficient $m/(\dim X - 1)!$.

It turns out that this multiplicity can be described via intersection numbers:

Proposition 2.11. *Let X be an arbitrary variety of dimension n and let $\pi : \tilde{X} \to X$ be the blow up of a point $x \in X$. Then*

$$\operatorname{mult}_x X = (-1)^{1+n} \cdot (E^{\cdot n}),$$

where E is the exceptional divisor.

Proof. Let $\mathfrak{m} \subset \mathcal{O}_X$ be the ideal sheaf of x. We have $\tilde{X} = \operatorname{Proj} \bigoplus_{i \geq 0} \mathfrak{m}^i$ and $E = \operatorname{Proj} \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. Its ideal sheaf is

$$\mathcal{I}_E = \mathcal{O}_{\tilde{X}}(1) = \widetilde{\bigoplus_{i \geq 0} \mathfrak{m}^{i+1}}.$$

For $m \geq 0$ let $E^{(m)}$ be given by the ideal sheaf $\mathcal{I}_E^m = \mathcal{O}_{\tilde{X}}(m)$. Then we have

$$h^0\left(\tilde{X}, \mathcal{O}_{E^{(m)}}\right) = h^0\left(\tilde{X}, \bigoplus_{i\geq 0} \widetilde{\mathfrak{m}^i/\mathfrak{m}^{i+m}}\right) = \dim \mathcal{O}_{X,x}/\mathfrak{m}_x^m \sim P_x(m).$$

We want to compare this expression with $\chi\left(\mathcal{O}_{E^{(m)}}\right)$. This will be done by studying the short exact sequence

$$0 \longrightarrow \mathcal{I}_E^m \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{E^{(m)}} \longrightarrow 0.$$

Applying the direct image functor π_* and using the additivity of the Euler characteristic gives

$$\begin{aligned} \sum_{q=0}^{n} (-1)^q \cdot \chi\left(R^q\pi_*\mathcal{O}_{\tilde{X}}\right) &= \sum_{q=0}^{n} (-1)^q \cdot \chi\left(R^q\pi_*\mathcal{I}_E^m\right) \\ &\quad + \sum_{q=0}^{n} (-1)^q \cdot \chi\left(R^q\pi_*\mathcal{O}_{E^{(m)}}\right) =: \Sigma_1 + \Sigma_2. \end{aligned}$$

By a standard argument using the Leray spectral sequence we can show $\Sigma_1 = \chi\left(\mathcal{I}_E^m\right)$ (cf. [Băd01, proof of Lem. 1.18]). Also note that the line bundle $\mathcal{O}_{\tilde{X}}(1)$ is relatively ample, so $R^q\pi_*\mathcal{I}_E^m = 0$ for $m \gg 0$ and $q > 0$ by [Gro61, Prop. 2.6.1]. Hence $R^q\pi_*\mathcal{O}_{E^{(m)}} \cong R^q\pi_*\mathcal{O}_{\tilde{X}}$ for $m \gg 0$ and $q > 0$. In particular this expression is independent of m if m is sufficiently large. Combining all this we calculate

$$\begin{aligned} \operatorname{mult}_x X &= \lim_{m\to\infty} \frac{n! \cdot P_x(m)}{m^n} = \lim_{m\to\infty} \frac{n! \cdot \Sigma_2}{m^n} \\ &= \lim_{m\to\infty} -\frac{n! \cdot \Sigma_1}{m^n} = -\left((-E)^{\cdot n}\right) = (-1)^{1+n} \cdot \left(E^{\cdot n}\right). \end{aligned}$$

□

Corollary 2.12. *Let H be an effective divisor and let C be a curve on a projective variety X such that C intersects H in finitely many points. Choose s points $x_1, \ldots, x_s \in H \cap C$ and assume that none of them is a singularity of X. Then*

$$(H \cdot C) \geq \sum_{i=1}^{s} \operatorname{mult}_{x_i} H \cdot \operatorname{mult}_{x_i} C.$$

Proof. Assume $s \neq 0$ (otherwise this is trivial). Let $\pi : \tilde{X} \to X$ be the blow up of $x_1, \ldots, x_s$. Denote the exceptional divisors by $E_1, \ldots, E_s$ (in the corresponding order).
Then we have $\pi^*H = \tilde{H} + \sum \mu_i \cdot E_i$ for some integers μ_i. By the projectivity assumption $\operatorname{Pic} X$ contains ample line bundles. Using similar arguments as

in [Har77, Exc. II.7.5] we can write $H = L_1 - L_2$, where L_1, L_2 are very ample. Hence H is linearly equivalent to a divisor that does not contain $x_1, \ldots, x_s$, because we can find divisors in $|L_i|$, $i = 1, 2$, which do not go through those points. Therefore

$$\begin{aligned} 0 = \left(E_i^{\cdot n-1} \cdot \pi^* H\right) &= \left(E_i^{\cdot n-1} \cdot \tilde{H}\right) + \mu_i \cdot \left(E_i^{\cdot n}\right) \\ &= (-1)^n \cdot \operatorname{mult}_x H + \mu_i \cdot (-1)^{1+n} \\ \Rightarrow \mu_i &= \operatorname{mult}_x H. \end{aligned}$$

for all i by Proposition 2.11. Note that we used smoothness of x_i here. This, $\left(E_i \cdot \tilde{C}\right) = \operatorname{mult}_x C$ and Lemma 2.4 imply

$$(H \cdot C) = \left(\pi^* H \cdot \tilde{C}\right) = \left(\tilde{H} \cdot \tilde{C}\right) + \sum \operatorname{mult}_{x_i} H \cdot \operatorname{mult}_{x_i} C.$$

This finishes the proof. □

Next, some cohomological results:

Theorem 2.13 (Fujita's vanishing theorem). *Let X be a projective k-scheme and let H be an ample line bundle. Given any coherent sheaf F, there exists an integer $m(F, H)$, such that for all nef divisors L we have*

$$\forall m \geq m(F, H) : \forall i > 0 : H^i(X, F \otimes (mH + L)) = 0.$$

Proof. [Fuj83, 5.1]. Note that the theorem holds in every characteristic. □

Corollary 2.14. *Let L be a nef line bundle in the situation as in Fujita's theorem and let n be the dimension of X. Then*

$$h^i(X, F \otimes (mL)) = O\left(m^{n-i}\right).$$

Proof of the corollary. We will just reproduce the corresponding proof of [Laz04, Thm. 1.4.40]. We proceed by induction on n. For $n = 0$ this is trivial. Assume the statement holds for $n - 1$. Choose an ample line bundle H. By the previous theorem all the higher cohomologies of $F \otimes (mL + \mu H)$ vanish for $\mu \geq m(F, H)$ and $m > 0$. Without loss of generality we have $m(F, H) = 1$. We may assume that there exists a Cartier divisor $D \in |H|$ that does not contain any of the associated points of F. This gives us a short exact sequence (use [Liu02, Exc. 7.1.12])

$$0 \longrightarrow F \otimes (mL) \longrightarrow F \otimes (mL + H) \longrightarrow F \otimes (mL + H)|_D \longrightarrow 0$$

and we obtain

$$h^i(X, F \otimes (mL)) \leq h^{i-1}(D, F \otimes (mL + H)|_D) = O\left(m^{(n-1)-(i-1)}\right)$$

for $i > 0$. This and $\chi(F \otimes (mL)) = O(m^n)$ also imply the result for $i = 0$. □

Corollary 2.15. *In the situation as in Fujita's theorem assume* $(L^{\cdot n}) > 0$ *and choose a positive real number* α*, such that* $\alpha^n < (L^{\cdot n})$*. Then for* $m \gg 0$ *the following holds: For all smooth* $x \in X$ *there exists a divisor* $D_x \in |mL|$ *such that* $\mathrm{mult}_x(D_x) \geq m \cdot \alpha$*.*

Proof. The sections of $|mL|$ with multiplicity $\geq c$ at an arbitrary smooth point $x \in X$ correspond to the kernel of the map

$$H^0(X, mL) \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x^c$$

corresponding to an isomorphism $(mL)_x \xrightarrow{\sim} \mathcal{O}_{X,x}$ by Remark 2.5. Hence we can guarantee the existence of such a section, if the dimension d_1 of the left vector space is larger than the dimension d_2 of the right vector space. We compute $d_1 = h^0(X, mL) \sim \frac{m^n(L^{\cdot n})}{n!}$ and $d_2 = \binom{n+c-1}{n} \sim \frac{c^n}{n!}$ using the regularity of the local ring $\mathcal{O}_{X,x}$. So for $m \gg 0$ and $m \cdot \alpha < c < m \cdot (L^{\cdot n})^{1/n}$ we are done. □

Theorem 2.16. *If* X *is a smooth projective variety and* $\mathrm{char}\, k = 0$ *and* L *a big and nef line bundle, then*

$$H^i(X, K_X + L) = 0$$

for $i > 0$*.*

Proof. [Laz04, Thm. 4.3.1]. Note that this is a generalization of the Kodaira vanishing theorem. □

The next thing we are going to discuss are $\mathbb{R}$-divisors, i.e., elements of $\mathrm{Pic}_{\mathbb{R}}(X) = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ or $N^1_{\mathbb{R}}(X) = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. The intersection form on $\mathrm{Pic}(X)$ naturally induces an intersection form on $\mathrm{Pic}_{\mathbb{R}}(X)$. The definitions of amplitude, nefness and numerical equivalence in terms of intersection multiplicities on $\mathrm{Pic}(X)$ can be transferred to $\mathbb{R}$-divisors verbatim. One can show that $N^1_{\mathbb{R}}(X)$ is the quotient of $\mathrm{Pic}_{\mathbb{R}}(X)$ modulo numerical equivalence. Kleiman's criterion for $\mathbb{R}$-divisors still holds. Both facts are not obvious, but proofs can be found in [Laz04, Chap. 1.3, 1.4]
One advantage of working with $\mathbb{R}$-divisors is the following

Proposition 2.17. *For projective* X *amplitude (nefness) is an open (resp. closed) property, i.e., the set of ample (nef) divisors* $\mathrm{Amp}(X), \mathrm{Nef}(X) \subset N^1_{\mathbb{R}}(X) \cong \mathbb{R}^{\rho(X)}$ *is open (resp. closed). Furthermore the first set is the interior of the second one and the second set is the closure of the first one. If* X *is just proper, nefness is still a closed property.*

Clearly both sets are convex and invariant under multiplication with positive real numbers, so those sets are sometimes called the *ample* and the *nef cone.* Finally we state the

Theorem 2.18 (Hodge index theorem)**.** *If X is a projective nonsingular surface, then the intersection pairing has signature $(1, \rho(X) - 1)$.*

Corollary 2.19. *In the same situation as above we have have*

$$(L \cdot D)^2 \geq \left(L^{\cdot 2}\right) \cdot \left(D^{\cdot 2}\right)$$

for any two divisors, such that $\left(L^{\cdot 2}\right) > 0$.

Proof of the corollary. If $(L \cdot D) = 0$ then $\left(D^{\cdot 2}\right) \leq 0$ by the previous theorem and the inequality is true. If $(L \cdot D) \neq 0$, let $\lambda := \frac{-\left(L^{\cdot 2}\right)}{(L \cdot D)}$. Then $(L \cdot (L + \lambda \cdot D)) = 0$ and the same argument implies

$$0 \geq (L + \lambda \cdot D)^2 = \left(L^{\cdot 2}\right) \cdot \left(-1 + \frac{\left(L^{\cdot 2}\right) \cdot \left(D^{\cdot 2}\right)}{(L \cdot D)^2}\right).$$

This is equivalent to the inequality we want to show. □

Usually we will refer to the corollary by the name 'Hodge index theorem'. Let us close this chapter with a lemma that will be of some use in chapter 4. Note that this is Exc. V.1.11(a) in [Har77].

Lemma 2.20. *Given a smooth surface X, an ample line bundle L and $d \geq 0$, the set*

$$\left\{c \in N^1(X) \mid 0 \leq (c \cdot L) \leq d, \exists M \in c : |M| \neq \emptyset\right\}$$

is finite. Remember that c is just a numerical equivalence class by definition.

Proof. Let c be an element of this set and let $C \subset X$ be a nonintegral curve whose numerical equivalence class is equal to c. By the Hodge index theorem we have $\left(C^{\cdot 2}\right) \leq \frac{(L \cdot C)^2}{\left(L^{\cdot 2}\right)}$, so $\left(c^{\cdot 2}\right)$ is bounded from above. By the openness of amplitude $L - \frac{1}{k} K_X$ is ample for some sufficiently large integer $k > 0$. Hence $kL - K_X$ is ample. Nakai–Moishezon implies $((kL - K_X) \cdot C) > 0 \Rightarrow kd \geq k(L \cdot C) > (K_X \cdot C)$ and by the adjunction formula we have

$$\left(C^{\cdot 2}\right) = 2p_a(C) - 2 - (K_X \cdot C) > -2 - kd.$$

So $\left(c^{\cdot 2}\right)$ can take on only finitely many values. Say, $\left(c^{\cdot 2}\right) = \nu$ and $(c \cdot L) = \lambda$. Let $M_1, \ldots, M_r$ be an orthonormal basis of $L^{\perp}$ (in $N^1_{\mathbb{R}}(X)$). We have $c = \lambda L + \sum_{i=1}^{r} \mu_i M_i$ with real numbers $\mu_1, \ldots, \mu_r$. Because of $\left(c^{\cdot 2}\right) = \nu^2$ we have $\sum_{i=1}^{r} \mu_i^2 = \text{const}$. Hence c lies in a ball with center λL. Since $c \in N^1_{\mathbb{R}}(X)$ is also contained in the lattice $N^1(X)$, there are only finitely many possibilities. □

Remark 2.6. Once we know that $\left(c^{\cdot 2}\right)$ takes on only finitely many values, it suffices to assume that $\left(L^{\cdot 2}\right) > 0$.

3 Basic properties of Seshadri constants

For this section let X denote a variety over an algebraically closed field k with $\dim X \geq 2$. We will closely follow [Laz04, Ch. 5.1].

Definition 3.1. For a (nef) line bundle L on X and a point $x \in X$ define the *(local) Seshadri constant*

$$\epsilon(L; x) \coloneqq \inf \left\{ \frac{(L \cdot C)}{\operatorname{mult}_x C} \mid C \subset X \text{ is a curve through } x \right\}.$$

Also define the *global Seshadri constant* by

$$\epsilon(L) \coloneqq \inf_{x \in X} \epsilon(L; x).$$

For future reference define

$$\epsilon(L; 1) \coloneqq \sup_{x \in X} \epsilon(L; x);$$

later we will call it the *Seshadri constant at a very general point.*

Remark 3.2. Note that if we considered nonintegral curves C, this would not make a difference, because for any two nonintegral curves C, $D \ni x$ we have

$$\frac{(L \cdot (C+D))}{\operatorname{mult}_x (C+D)} = \frac{(L \cdot C) + (L \cdot D)}{\operatorname{mult}_x C + \operatorname{mult}_x D} \geq \min\left(\frac{(L \cdot C)}{\operatorname{mult}_x C}, \frac{(L \cdot D)}{\operatorname{mult}_x D} \right).$$

Remark 3.3. Of course we could make the same definition if L is more generally an $\mathbb{R}$-divisor. Fixing $x \in X$ it is clear that $\epsilon(-; x)$ is invariant under numerical equivalence, so it induces a map $\operatorname{Nef}(X) \to \mathbb{R}_{\geq 0}$. One easily sees that $\epsilon(-; x)$ is a convex function and in particular continuous everywhere and differentiable almost everywhere. Also note that the Seshadri constant is positively homogeneous, i.e., $\epsilon(\lambda L; x) = \lambda \cdot \epsilon(L; x)$ for all $\lambda \geq 0$. So the Seshadri constant is well-behaved in the first variable. Nevertheless it might behave wildly in the second variable.

Remark 3.4. In general local and global Seshadri constants of a line bundle L differ. But they clearly coincide if X is homogeneous, i.e., if $\operatorname{Aut} X$ acts transitively on X. In particular this holds for abelian varieties X.

Remark 3.5 (Nagata conjecture)**.** One can also define the *r-point Seshadri constant of L at points $x_1, \ldots, x_r$*

$$\epsilon(L; x_1, \ldots, x_r) \coloneqq \inf \left\{ \frac{(L \cdot C)}{\sum_{\rho=1}^{r} \operatorname{mult}_{x_\rho} C} \mid C \cap \{x_1, \ldots, x_r\} \neq \emptyset \text{ is a curve} \right\}.$$

The Nagata conjecture says that for a very general (cf. next chapter) set of points $x_1, \ldots, x_r \in \mathbb{P}^2$ with $r \geq 9$ we have

$$\epsilon\left(\mathcal{O}_{\mathbb{P}^2}(1); x_1 \ldots x_r\right) = \frac{1}{\sqrt{r}}.$$

In 1959 Nagata found a short proof for this when r is a perfect square and used this to construct a counterexample for Hilbert's 14th problem ([Nag59]). Although the Nagata conjecture deals with the most basic example of calculating r-point Seshadri constants one can think of, it is still unsolved in all the other cases. Note that the previous remark is of no use here, because $\operatorname{Aut} \mathbb{P}^2$ does not even act 3-transitively on $\mathbb{P}^2$.
As 1-point Seshadri constants are already difficult to calculate, we will restrict ourselves to the case $r = 1$ in the remainder of this thesis. Sporadically though, we will point out when easy generalizations of some proofs are possible.

Remark 3.6. Let L be very ample. So think of X as a closed subvariety of some $\mathbb{P}^n$ and assume $L = \mathcal{O}_X(1)$. Then for all curves $C \subset X$ that pass through x we have

$$(L \cdot C) = (\mathcal{O}_{\mathbb{P}^n}(1) \cdot C) = \deg C \geq \operatorname{mult}_x C$$

by Lemma 2.5 and Corollary 2.12 (x might not be a smooth point of X, but it is a smooth point of $\mathbb{P}^n$), hence $\epsilon(L) \geq 1$.
If L is just ample, then

$$\epsilon(L) \geq \frac{1}{m}$$

by Remark 3.3, where $m > 0$ is chosen in such a way that mL is very ample. Thus global Seshadri constants of ample line bundles are positive. We will show below that the conversion also holds, such that we obtain the *Seshadri criterion of amplitude*. Actually the name 'Seshadri constant' comes from this criterion.

Example 3.7. Let V be a vector space of dimension n, let $r \leq n$ and let $\operatorname{Gr}(r, n)$ be the corresponding Grassmannian. Assume $\dim \operatorname{Gr}(r, n) = r(n - r) \geq 2$. Let L be the line bundle associated to the Plücker embedding. Let $x \in X$. We already know $\epsilon(L; x) \geq 1$ by the previous remark. In fact equality holds. In order to see this, assume $X \subset \mathbb{P}(\bigwedge^r V)$. If $(v_1, \ldots, v_n)$ is a basis of V, we may assume $x = [v_1 \wedge \ldots \wedge v_r]$. Then the line C that passes through x and $[v_1 \wedge \ldots \wedge v_{r-1} \wedge v_{r+1}]$ lies in X. We obtain

$$\epsilon(L; x) = 1$$

because of $(L \cdot C) = \deg C = 1 = \operatorname{mult}_x C$. Note that we obtain

$$\epsilon(\mathcal{O}_{\mathbb{P}^n}(1); x) = 1$$

for all points $x \in \mathbb{P}^n$ $(n \geq 2)$ as a special case.

Example 3.8. Now let $X = \mathbb{P}^1 \times \mathbb{P}^1$. We have $\operatorname{Pic} X = \mathbb{Z}^2$ with $L_1 = \mathbb{P}^1 \times \{0\}$, $L_2 = \{0\} \times \mathbb{P}^1$ as generators. Both generators are inverse images of the nef divisor $\{0\} \subset \mathbb{P}^1$ under the canonical projections $\pi_1, \pi_2 : X \to \mathbb{P}^1$ and so nef themselves by Lemma 2.8. One easily shows $\left(L_1^{\cdot 2}\right) = \left(L_2^{\cdot 2}\right) = 0$ and $(L_1 \cdot L_2) = 1$. Using this one can show that the divisors $mL_1 + nL_2$ for nonnegative integers m, n are the only nef divisors on X. We want to calculate the Seshadri constants of $mL_1 + nL_2$. The surface X is homogeneous, so local and global Seshadri constants coincide. We claim $\epsilon\left(mL_1 + mL_2\right) = \min\left(m, n\right)$. Assume $m \leq n$. The Segre embedding $X \to \mathbb{P}^3$ is given by $L_1 + L_2$, so this line bundle is very ample and $1 \leq \epsilon\left(L_1 + L_2\right)$. Thus $m \leq \epsilon\left(mL_1 + mL_2\right) \leq \epsilon\left(mL_1 + nL_2\right)$ by convexity. Equality holds because of $((mL_1 + nL_2) \cdot L_2) = m$.

Remark 3.9. Unfortunately Seshadri constants are very difficult to calculate. Even for surfaces nontrivial examples with known values are hard to come by. So one should aim for good bounds instead. Upper bounds are easy to obtain by the first definition. If one wants to estimate, say, $\epsilon\left(L; x\right)$ one can calculate $\frac{(L \cdot C)}{\operatorname{mult}_x C}$ for an arbitrary curve $C \ni x$ and this is automatically an upper bound. In Remark 3.3 we will get to know a basic, but important upper bound. However, lower bounds are more difficult to find. Indeed, establishing a lower bound is one of the most challenging parts of the proofs in the last three chapters.

Remark 3.10. All known values of the Seshadri constant are rational. As we will see later, one can even show that the Seshadri constants of a nef line bundle L on a (smooth) surface are rational or equal to $\sqrt{(L^2)}$ respectively.

There is an alternative definition: Fix a point $x \in X$ (possibly singular) and consider its blow up $\pi : \tilde{X} \to X$. Let $E = \pi^{-1}\left(x\right)$ be the exceptional divisor. Then we have:

Proposition 3.1. $\epsilon\left(L; x\right) = \max\left\{\epsilon \geq 0 \mid \pi^* L - \epsilon E \textit{ is nef}\right\}$

Proof. Let $\epsilon \geq 0$. It suffices to show the equivalence

$$\epsilon \leq \epsilon\left(L; x\right) \Longleftrightarrow \pi^* L - \epsilon E \text{ is nef.}$$

That the maximum on the right hand side is really a maximum follows from the closedness of nefness.
Let $C \subset X$ be an arbitrary curve. Then we have

$$\left(L \cdot C\right) = \left(\pi^* L \cdot \tilde{C}\right)$$

and $\operatorname{mult}_x C = \left(E \cdot \tilde{C}\right)$ by Proposition 2.11. So $\epsilon \leq \epsilon\left(L; x\right)$ is equivalent to

$$\left(L \cdot C\right) - \epsilon \cdot \operatorname{mult}_x C \geq 0 \Leftrightarrow \left(\left(\pi^* L - \epsilon E\right) \cdot \tilde{C}\right) \geq 0$$

for all such curves. Now every curve $C' \subset \tilde{X}$ that is not a subvariety of E satisfies $\widetilde{\pi_* C'} = C'$, so this is equivalent to $((\pi^* L - \epsilon E) \cdot C') \geq 0$ for all C'. If C' is a subvariety of E, then this inequality is still correct, because $\pi^* L$ is nef by Lemma 2.8 and $(E \cdot C') < 0$. Hence this is equivalent to $\pi^* L - \epsilon E$ being nef. $\square$

Remark 3.11. Given r points $x_1, \dots, x_r$, we still have

$$\epsilon\left(L; x_1, \dots, x_r\right) = \max\left\{\epsilon \geq 0 \mid \pi^* L - \epsilon E \text{ is nef}\right\}.$$

But now $\pi : \tilde{X} \to X$ is the blow up of $x_1, \dots, x_r$ and E is the sum of the r exceptional divisors.

Remark 3.12. Note that for $0 \leq \epsilon < \epsilon\left(L; x\right)$ the divisor $\pi^* L - \epsilon E$ might still lie on the boundary of the nef cone, nevertheless it is at least big and nef, because

$$\left((\pi^* L - \epsilon E)^{\cdot \dim X}\right) > \left((\pi^* L - \epsilon\left(L; x\right) \cdot E)^{\cdot \dim X}\right) \geq 0.$$

If L is ample, one can say that $\pi^* L - \epsilon E$ is ample as well by Nakai–Moishezon: Let $V' \subset \tilde{X}$ be a subvariety and let $V = \pi\left(V'\right)$. Then we have

$$\left((\pi^* L - \epsilon E)^{\cdot \dim V'} \cdot V'\right) > \left((\pi^* L - \epsilon\left(L; x\right) \cdot E)^{\cdot \dim V'} \cdot V'\right) \geq 0,$$

if $V' \cap E \neq \emptyset$. If $V' \cap E = \emptyset$, the left-hand side is equal to $\left(L^{\cdot \dim V} \cdot V\right) > 0$.

Theorem 3.2 (Seshadri's Criterion)**.** *A line bundle L on X is ample if and only if the global Seshadri constant $\epsilon\left(L\right)$ is defined and positive.*

Proof. One direction has already been proven in Remark 3.6. So assume $\epsilon \coloneqq \epsilon\left(L\right) > 0$. Then we have $\epsilon\left(L|_V\right) > 0$ for all closed subvarieties $V \subset X$. By induction on $n = \dim X$ we may assume that $L|_V$ is ample for all closed subvarieties $V \subset X$ of smaller dimension. In particular $\left(L^{\cdot \dim V} \cdot V\right) > 0$, so Nakai–Moishezon tells us that it suffices to show $(L^{\cdot n}) > 0$.
Indeed, let $\pi : \tilde{X} \to X$ be the blow up with respect to some point $x \in X$ and let E be the exceptional divisor. We may assume that x is no base-point of $|L|$. This implies $\left(\pi^* L^{\cdot m} \cdot E^{\cdot (n-m)}\right) = 0$ for all $0 < m < n$. Now the nefness of $\pi^* L - \epsilon E$ and Kleiman's criterion imply

$$(L^{\cdot n}) - \epsilon^n \cdot \operatorname{mult}_x X = ((\pi^* L - \epsilon E)^{\cdot n}) \geq 0.$$

$\square$

We will later see $\epsilon\left(L; 1\right) \geq 1$ for ample L in Theorem 4.5 under some extra assumptions. So one might wonder if there is a better universal bound in the theorem above. The next example shows that the answer is no.

Example 3.13 (Miranda)**.** Let $\delta > 0$. We want to construct an ample line bundle L on some surface X, such that

$$\epsilon(L;x) < \delta$$

for some $x \in X$.
Choose an integer $m > \frac{1}{\delta}$ and a curve $\Gamma \subset \mathbb{P}^2$ with a point $p \in \Gamma$ of multiplicity m. We may assume that Γ has degree $d > 1$. We can choose a nonsingular curve Γ' in $\mathbb{P}^2$ of degree d that meets Γ transversally in ordinary points (points of multiplicity 1) of Γ, such that the linear system generated by Γ and Γ' only contains curves, i.e., not nonintegral curves.
This follows from the following argument: Using the standard identification $|\mathcal{O}_{\mathbb{P}^2}(e)| = \mathbb{P}^{\binom{e+1}{2}-1}$ for $e > 0$ the set of nonintegral degree d curves corresponds to the union of the images of the Segre embeddings

$$\sigma_{e_1,e_2} : \mathbb{P}^{\binom{e_1+1}{2}-1} \times \mathbb{P}^{\binom{e_2+1}{2}-1} \to \mathbb{P}^{\binom{d+1}{2}-1}$$

for all positive integers e_1, e_2 adding up to d. Setting $e \coloneqq e_1$ the dimension of $\operatorname{im}\sigma_{e_1,e_2}$ is equal to

$$\begin{aligned}\binom{e+1}{2} + \binom{d-e+1}{2} - 2 &= \frac{e(e+1)}{2} + \frac{(d-e)(d-e+1)}{2} - 2 \\ &\leq \frac{1(1+1)}{2} + \frac{(d-1)d}{2} - 2 = \frac{d^2-d-2}{2}.\end{aligned}$$

Thus the nonintegral curves in $|\mathcal{O}_{\mathbb{P}^2}(d)|$ form an algebraic set B of codimension at least $d \geq 2$. So the 'cone' $\overline{\Gamma, B}$ has codimension at least 1. By Bertini almost all nonintegral degree d curves Γ' intersect Γ transversally. Also almost all of them do not contain a multiple point of Γ and again by Bertini almost all of them are nonsingular. Choose one outside of $\overline{\Gamma, B}$ and we are done.
Having guaranteed the existence of Γ' we can define the blow up

$$\pi : X \to \mathbb{P}^2$$

at the base points $\Gamma \cap \Gamma'$ of $\langle \Gamma, \Gamma' \rangle$. Let C, C' be the strict transforms of Γ, Γ'. The morphism π induces a map $\pi^* : \langle \Gamma, \Gamma' \rangle \to \langle C, C' \rangle$ given by strict transforms. Each $\Gamma'' \in \langle \Gamma, \Gamma' \rangle$ is birational to $\pi^*(\Gamma'')$, so all elements of $\langle C, C' \rangle$ are integral. Furthermore Γ and Γ' are isomorphic to their respective strict transform. In particular C has a point x of multiplicity m. By construction $\langle C, C' \rangle$ is basepointfree and it induces a morphism $f : X \to \mathbb{P}^1$. Let $E \subset X$ be an exceptional divisor.
Define $L \coloneqq aC + E$ for some integer $a \geq 2$. We claim that this is the example we are looking for. The first thing we need to check is amplitude. We know that all elements of $\langle C, C' \rangle$ are numerically equivalent, so $\left(C^{\cdot 2}\right) = (C \cdot C') = 0$. The equalities $(C \cdot E) = 1$ and $\left(E^{\cdot 2}\right) = -1$ imply

$$\left(L^{\cdot 2}\right) = 2a - 1 > 0, (L \cdot E) = a - 1 > 0, (L \cdot C) = 1 > 0.$$

If $D \subset X$ is an arbitrary curve, there are two cases: D might dominate $\mathbb{P}^1$. Then $D = E$ or $(E \cdot D) \geq 0 \Rightarrow (L \cdot D) \geq (aC \cdot D) > 0$. Or D is contained in a fiber of f, i.e., in an element of $\langle C, C' \rangle$. But all elements of this linear system are integral, so D lies in this linear system and is numerically equivalent to C. Therefore L is ample by Nakai–Moishezon.
Apart from that we have

$$\epsilon\,(L;x) \leq \frac{(L \cdot C)}{\mathrm{mult}_x\,(C)} = \frac{1}{m} < \delta,$$

as stated.

However, we have a good upper bound:

Proposition 3.3. *Let $V \subset X$ be a subvariety of dimension $d > 0$ passing through x. Then*

$$\epsilon\,(L;x) \leq \left(\frac{\left(L^{\cdot d} \cdot V\right)}{\mathrm{mult}_x\,(V)}\right)^{\frac{1}{d}}$$

with equality for some V.

Proof. We adopt the notation of the proof of Seshadri's criterion. Write $\epsilon := \epsilon\,(L;x)$. The divisor $\pi^* L - \epsilon E$ is nef, so

$$\left((\pi^* L - \epsilon E)^{\cdot d} \cdot \tilde{V}\right) \geq 0.$$

Now Lemma 2.4 and Proposition 2.11 imply the inequality. As $\pi^* L - \epsilon E$ is not ample, $\pi^* L - \epsilon E$ has trivial intersection with some subvariety V'. Then we have equality with $V = \pi\,(V')$. ☐

Remark 3.14. Given $x \in X$ and L it is very difficult to say which dimension V might have if equality holds for V. This corresponds to the fact, that all known values of the Seshadri constants are rational.

Remark 3.15. If $d = \dim X$, this implies that

$$\epsilon\,(L;x) \leq \sqrt[d]{(L^{\cdot d})}.$$

If X is a surface and $x \in X$ a smooth point, we also see that 'inf' can be replaced by 'min' in our first definition if equality does not hold, or more concretely: There exists a curve $C \ni x$ with

$$\epsilon\,(L;x) = \frac{(L \cdot C)}{\mathrm{mult}_x\, C}.$$

In particular $\epsilon\,(L;x)$ is rational unless it is equal to $\sqrt{(L^{\cdot 2})}$.
This proposition also tells us that Seshadri constants are uninteresting when L is not big and nef.

Remark 3.16. The proof can be easily transferred to r-point Seshadri constants. Of course V has to pass through some of the points $x_1, \ldots, x_r$ and $\operatorname{mult}_x V$ has to be substituted by $\sum \operatorname{mult}_{x_i} V$. Consequently we get

$$\epsilon(L; x_1, \ldots, x_r) \leq \sqrt{\frac{(L^{\cdot 2})}{r}}$$

in the surface case. Note that this implies 'one half' of the Nagata conjecture.

Remark 3.17. In the case when X is a projective surface, one can prove Remark 3.15 differently using:

Lemma 3.4. *Let L be a big and nef line bundle on a projective surface X. Given a real number $\xi > 0$, a smooth point $x \in X$ and $k > 0$ such that there exists a $D \in |kL|$ with*

$$\frac{(L \cdot D)}{\operatorname{mult}_x D} \leq \xi \sqrt{(L^{\cdot 2})},$$

the following holds: Every curve $C \subset X$ passing through x with

$$\frac{(L \cdot C)}{\operatorname{mult}_x C} < \frac{1}{\xi} \sqrt{(L^{\cdot 2})}$$

is a component of D.

Proof. Assume by way of contradiction $C \not\subset D$. Then $C \cap D$ is finite (by our conventions curves are always integral) and by Corollary 2.12 we have

$$k(L \cdot C) = (D \cdot C) \geq \operatorname{mult}_x D \cdot \operatorname{mult}_x C > \frac{(L \cdot D)}{\xi \sqrt{(L^{\cdot 2})}} \cdot \frac{\xi (L \cdot C)}{\sqrt{(L^{\cdot 2})}} = k(L \cdot C),$$

contradiction. $\square$

Assume now $\epsilon(L; x) < \sqrt{(L^{\cdot 2})}$. Fix a real number $1 < \xi < \sqrt{(L^{\cdot 2})}/\epsilon(L; x)$. Choose a sequence of curves $C_n \ni x$ with $\epsilon(L; x) = \lim_{n \to \infty} \frac{(L \cdot C_n)}{\operatorname{mult}_x C_n}$. Then we have $\frac{(L \cdot C_n)}{\operatorname{mult}_x C_n} < \frac{1}{\xi} \sqrt{(L^{\cdot 2})}$ for $n \gg 0$. By Corollary 2.15 with $\alpha = \frac{1}{\xi}$ there exists $D \in |kL|$ with $\frac{(L \cdot D)}{\operatorname{mult}_x D} \leq \xi \sqrt{(L^{\cdot 2})}$ for some $k > 0$. Lemma 3.4 implies that C_n is a component of D for $n \gg 0$. In particular there are only finitely many C_n and the result follows.

If L is ample and x smooth there is also a third definition for the local Seshadri constant. Therefore we first give a

Definition 3.18. Given a line bundle L on X, a point $x \in X$ with ideal sheaf $\mathfrak{m}_x \subset \mathcal{O}_X$ and $s \geq -1$, we say that $|L|$ *separates s-jets at x* if the canonical map

$$H^0(X, L) \to H^0\left(X, L \otimes \mathcal{O}_X/\mathfrak{m}_x^{s+1}\right) = L_x/\mathfrak{m}_x^{s+1} L_x$$

is surjective.
Denote by $s(L; x)$ the maximum over all s such that $|L|$ separates s-jets at x.

Remark 3.19. First, note that the maximum does exist because of $\dim L_x = \dim \mathcal{O}_{X,x} = \infty$. Second, note that $s(L;x) \geq 0$ if and only if x is no base point of the linear system. Third, note that if $|L|$ separates s-jets at x, then it also separates s'-jets for $s' \leq s$. Fourth, also note that this is equivalent to the injectivity of

$$H^1\left(X, L \otimes \mathfrak{m}_x^{s+1}\right) \to H^1(X, L).$$

This is for instance the case when the left cohomology group vanishes.

Now we have the

Theorem 3.5. *Let L be an ample line bundle on X and $x \in X$ be a smooth point. Then*

$$\epsilon(L;x) = \lim_{k\to\infty} \frac{s(kL;x)}{k}.$$

Proof. Set $\epsilon \coloneqq \epsilon(L;x)$ and $s_k \coloneqq s(kL;x)$. We will show

$$\epsilon \geq \limsup_{k\to\infty} \frac{s_k}{k} \geq \liminf_{k\to\infty} \frac{s_k}{k} \geq \epsilon.$$

So our proof consists of two parts.
First we will show $\epsilon \geq \frac{s_k}{k}$ for $k > 0$. Despite Remark 3.9 this will be the elementary half. Fix a curve $C \ni x$. Denote its ideal sheaf localized at x by I. In $\mathcal{O}_{X,x}$ this is a prime ideal of height 1. We have $I \cap \mathfrak{m}_x^{s_k}/\mathfrak{m}_x^{s_k+1} \neq \mathfrak{m}_x^{s_k}/\mathfrak{m}_x^{s_k+1}$, because otherwise we would have $\sqrt{I} = \mathfrak{m}_x$. Choose an element $\alpha \in \mathfrak{m}_x^{s_k} \setminus \left(I \cup \mathfrak{m}_x^{s_k+1}\right)$ and choose an isomorphism $\varphi : (kL)_x \to \mathcal{O}_{X,x}$. We have $\varphi^{-1}(\alpha) \notin \mathfrak{m}_x^{s_k+1}(kL)_x$. Now let D be a divisor given by some preimage of the nontrivial element $\overline{\varphi^{-1}(\alpha)}$ under the surjective map $H^0(X, kL) \to (kL)_x/\mathfrak{m}_x^{s_k+1}(kL)_x$. Then we have $\operatorname{mult}_x D \geq s_k$ and $C \not\subset D$ by construction, hence

$$\begin{aligned} k \cdot (L \cdot C) = (kL \cdot C) = (D \cdot C) \geq \operatorname{mult}_x D \cdot \operatorname{mult}_x C \geq s_k \cdot \operatorname{mult}_x C \\ \Rightarrow \epsilon(L;x) \geq \frac{s_k}{k}. \end{aligned}$$

Part two will build upon Fujita's vanishing theorem and the following fact, which we will prove afterwards:

Lemma 3.6. *Let X be a proper k-scheme and let $x \in X$ be a smooth point with ideal sheaf $\mathfrak{m}_x \subset \mathcal{O}_X$. Let $\pi : \tilde{X} \to X$ be its blow up with exceptional divisor E. Then we have*

$$H^i\left(\tilde{X}, \pi^*L - aE\right) \cong H^i\left(X, L \otimes \mathfrak{m}_x^a\right)$$

for all line bundles L on X and for all integers $i, a \geq 0$.

The assumption that $x \in X$ is smooth is crucial here. Otherwise the lemma would be false.
Fix integers p_0, $q_0 \gg 0$, such that $0 < \epsilon - \frac{p_0}{q_0} \ll 1$. Denote the blow up at x by $\pi : \tilde{X} \to X$ and denote the exceptional divisor by E. By Remark 3.12 $\pi^* (q_0 L) - p_0 E$ is ample. By Fujita's vanishing theorem there exists m_0, such that we have

$$H^1 \left(\tilde{X}, m \left(\pi^* \left(q_0 L\right) - p_0 E\right) + D\right) = 0$$

for every $m \geq m_0$ and every nef line bundle D. Now let $k > m_0 q_0$ be arbitrary. Write $k = m q_0 + q_1$ with $0 \leq q_1 < q_0$. Setting $D = \pi^* (q_1 L)$ we obtain in particular

$$H^1 \left(X, kL \otimes \mathfrak{m}_x^{mp_0}\right) \cong H^1 \left(\tilde{X}, \pi^* (kL) - mp_0 E\right) = 0$$

by Lemma 3.6, so $|kL|$ separates $(mp_0 - 1)$-jets. Thus

$$\frac{s_k}{k} \geq \frac{mp_0 - 1}{k} \geq \frac{mp_0 - 1}{(m+1)\,k} = \left(\frac{m}{m+1}\right) \frac{p_0}{q_0} - \frac{1}{(m+1)\,q_0}.$$

By easy convergence arguments we obtain $\liminf \frac{s_k}{k} \geq \epsilon$. □

Proof of Lemma 3.6. Let $a \geq 0$. First, note that for all $a \geq 0$ we have $\pi_* (-aE)\,|_E = \mathfrak{m}_x^a / \mathfrak{m}_x^{a+1}$. Second, by the smoothness of x we have $E = \mathbb{P}\left(\mathfrak{m}_x / \mathfrak{m}_x^2\right) \cong \mathbb{P}^{\dim X - 1}$. Hence $R^i \pi_* (-aE) = R^i \pi_* \left(\mathcal{O}_E (a)\right) = 0$ for $i > 0$. Besides, we have a short exact sequence

$$0 \longrightarrow -(a+1)\,E \longrightarrow -aE \longrightarrow (-aE)\,|_E \longrightarrow 0.$$

By induction on a we obtain $\pi_* (-aE) = \mathfrak{m}_x^a$ and $R^i \pi_* (-aE) = 0$ for $i > 0$. Indeed, by [Har77, Prop. V.3.4] (the proof given there also works for $\dim X \neq 2$) the assertion holds for $a = 0$. If the induction hypothesis works for a, then we get the result for $a + 1$ by applying π_* to the short exact sequence from above and by using the two facts we stated at the beginning. Now the proposition easily follows from [Har77, Exc. III.8.1] and [Har77, Exc. III.8.3]. □

Unfortunately this theorem is hardly suitable for applications. But if we stick to a special case there is a better result:

Theorem 3.7. *Let X be smooth of dimension n and $\operatorname{char} k = 0$. If $\epsilon (L; x) > 0$ and k, s are nonnegative integers, such that $\epsilon (kL; x) > s + n$, then $|K_X + kL|$ separates s-jets.*
If conversely there exist positive real numbers c, ϵ (like $n/\epsilon (L; x)$), such that for all nonnegative integers k, s with $k\epsilon > s + c\epsilon$ the linear system $|K_X + kL|$ separates s-jets, then $\epsilon (L; x) \geq \epsilon$.

Proof. We may assume $k = 1$. Adopt the notation from the previous proof. We have $K_{\tilde{X}} = \pi^* K_X + (n-1) E$. Since $\epsilon(L; x) > s + n$, the divisor $\pi^* L - (n+s) E$ is big and nef by Remark 3.12. Hence

$$K_{\tilde{X}} + \pi^* L - (n+s) E = \pi^* (K_X + L) - (s+1) E$$

and Lemma 3.6 and Theorem 2.16 imply

$$H^1 \left(X, (K_X + L) \otimes \mathfrak{m}_x^{s+1} \right) = H^1 \left(\tilde{X}, \pi^* (K_X + L) - (s+1) E \right) = 0$$

and we are done. Note that Theorem 2.16 requires the assumption $\operatorname{char} k = 0$.

For the second part let $C \ni x$ be a curve. Write $m = \operatorname{mult}_x C$. Let $s > 0$ be arbitrary and $k := \left\lfloor \frac{s}{\epsilon} + c \right\rfloor + 1$. By assumption $|K_X + kL|$ separates s-jets. By the same argument as in the proof of the previous theorem this implies the existence of a global section $D \in |K_X + kL|$ with $\operatorname{mult}_x D \geq s$ and $C \not\subset D$. Hence

$$\begin{aligned} ((K_X + kL) \cdot C) &\geq \operatorname{mult}_x (D) \cdot \operatorname{mult}_x (C) \geq s \cdot m \\ \Rightarrow \frac{(L \cdot C)}{m} &\geq \frac{s}{k} - \frac{(K_X \cdot C)}{k \cdot m}. \end{aligned}$$

Letting $s \to \infty$ gives the result. □

4 Seshadri constants at very general points

In Miranda's example we have seen an ample line bundle L on some surface X, such that $\epsilon(L; x)$ is very small for a specific point $x \in X$. In this chapter we will see that this is an 'exception', i.e., for most of the points $x \in X$ the Seshadri constant $\epsilon(L; x)$ is much larger. The surface X was even birational to $\mathbb{P}^2$, and in $\mathbb{P}^2$ such behaviour does not occur. This leads us to consider the value of the Seshadri constant at a 'generic' point (not to be confused with the scheme theoretical generic point). The following definition will be useful for future use:

Definition 4.1. Let X be a variety. We say that a property for closed points holds at a *general (very general point)* $x \in X$, if it holds for all x off a proper subvariety (respectively countable union of proper subvarieties) of X.

From now on we assume that k is uncountable. Otherwise the definition of very general points would be inane. Also assume $\operatorname{char} k = 0$. We have:

Proposition 4.1. *Let $f : X \to T$ be a proper morphism of noetherian schemes and L be a line bundle on X. Write $L_t = L|X_t$ for all fibres X_t. If L_{t_0} is ample for some t_0, then $L|U$ is ample for some open neighbourhood $U \ni t_0$.*

Proof. Proofs can be found in [Laz04, Thm. 1.2.17] or in [Gro61, III.4.7.1]. □

Corollary 4.2 ([Laz04, Prop. 1.4.14])**.** *Let $f : X \to T$ be a projective morphism of varieties and let L be a line bundle on X. If L_{t_0} is nef for at least one point $t_0 \in T$, then L_t is nef at a very general point.*

Proof. Let A be a line bundle of X which is ample relative to T. In particular A_t is ample for all t. If L_t is nef, $L_t + \frac{1}{m}A_t$ is ample for all integers $m > 0$ by Nakai–Moishezon. The conversion holds by the closedness of nefness. Fix $m > 0$. Then $mL_{t_0} + A_{t_0}$ is ample and by Proposition 4.1 $mL_t + A_t$, and therefore also $L_t + \frac{1}{m}A_t$ is ample for all t off a proper subvariety $S_m \subset T$. Thus L_t is nef for all $t \notin \bigcup S_m$. □

Remark 4.2. If L is a $\mathbb{Q}$-divisor, i.e., if kL is a line bundle for some integer $k > 0$, then for all $t \in T$ the divisor kL_t is nef if and only if L_t is nef. Hence the result is also true for $\mathbb{Q}$-divisors.

Corollary 4.3. *Let $f : X \to T$ be a surjective projective morphism with a section $x : T \to X$, $t \mapsto x_t$ and let L be a line bundle on X. Then there exists $\epsilon > 0$ such that $\epsilon\,(L_t; x_t) = \epsilon$ for very general t.*

Proof. Let
$$\epsilon := \sup_{t \in T} \epsilon\,(L_t; x_t)$$
By Remark 3.15 $\epsilon < \infty$. Let $\pi : \tilde{X} \to X$ be the blow up along the image of x. Denote the exceptional divisor by E. For all $t \in T$ this morphism induces the blow up $\pi_t : \tilde{X}_t \to X_t$ at x_t. Denote the exceptional divisor by $E_t \subset \tilde{X}_t$. Now let δ be a *rational* number such that $\epsilon\,(L_{t_0}; x_{t_0}) \geq \delta$ for some point $t_0 \in T$. Then $(\pi^*L - \delta E)_{t_0}$ is nef. Remark 4.2 implies that $(\pi^*L - \delta E)_t$ is nef for very general t, i.e., $\epsilon\,(L_t; x_t) \geq \delta$ for very general t. Apply this to a sequence $(\delta_m)_m$ of rational numbers $\delta_m \uparrow \epsilon$ and we obtain $\epsilon \geq \epsilon\,(L_t; x_t) \geq \epsilon$ for very general t. □

Now let X be a variety, $p_1, p_2 : X \times X \to X$ the projection onto the first, second factor and $\Delta : X \to X \times X$ the diagonal morphism. Let L be a nef line bundle on X and let $\mathcal{L} := p_1^*(L)$. Setting $f = p_2$, $x = \Delta$ and $L = \mathcal{L}$ in the corollary above gives us the

Corollary 4.4. *For all nef line bundles L on a variety X we have*
$$\epsilon\,(L; 1) = \epsilon\,(L; x)$$
for very general points $x \in X$.

After this discussion we now come to the main theorem of this chapter, shown by Ein and Lazarsfeld in [EL93]:

Theorem 4.5. *Let e be a positive integer and let L be line bundle on the smooth surface X with $\left(L^{\cdot 2}\right) \geq 2e^2 - 2e + 1$ and $(L \cdot C) \geq e$ for every irreducible curve $C \subset X$. Then $\epsilon(L; 1) \geq e$. If $e > 1$ or more generally $\left(L^{\cdot 2}\right) > 1$, we even have $\epsilon(L; x) \geq e$ for* general *$x \in X$.*

Note that L is ample by Nakai–Moishezon. For $e = 1$ the assumptions just say that L shall be ample.

Before giving a proof we recall the notion of an algebraic family. We say that a flat morphism $f : \mathcal{X} \to T$ of arbitrary nonproper varieties is an *algebraic family*. Another way to think of that is to say f determines a 'continuous' deformation $(X_t)_{t \in T}$. If f is projective, flatness of f is equivalent to saying that the Hilbert polynomial P_t of X_t is independent of t by [Har77, Thm. III.9.9]. In particular all fibres have the same dimension. In the proof we will mainly encounter the special case, where T is a smooth nonproper curve and $\mathcal{X}$ is a closed subscheme of $X \times T$ of codimension 1, i.e., an effective Cartier divisor. In that case we write $C_t = \mathcal{X}_t$ and we have a family of curves.

An important tool for studying such a family at a point $t \in T$ is the *Kodaira–Spencer map*

$$\rho_t : T_t T \to H^0(C_t, \mathcal{N}_t),$$

where $T_t T$ is the tangent space of T at t and $\mathcal{N}_t$ is the normal bundle to C_t in X (note that $\mathcal{N}_t = C_t|_{C_t}$ is invertible also for singular C_t). We will investigate this map later. For the proof we just need to know two things: First, it is intuitively the derivative of $t \mapsto C_t$. So one expects it to be trivial for all t if and only if C_t does not move, i.e., $C_{t_1} = C_{t_2}$ for all $t_1, t_2 \in T$.

Second, if we also have a family $(x_t)_{t \in T}$ of points on the curves C_t with high multiplicity, then C_t has a high self-intersection number if ρ_t is nontrivial. More precisely: If $\mathrm{mult}_{x_t}(C_t) \geq m$ for all t, then $\left(C_t^{\cdot 2}\right) \geq m(m-1)$ if $\rho_t \neq 0$.

Now we start with the proof:

Proof. In order to show $\epsilon(L, 1) \geq e$, we need to inspect the 'bad' points x and want to show there are not too many of them, i.e., we consider the points x with

$$\epsilon(L; x) < e \Leftrightarrow \exists \text{ curve } C \subset X : \mathrm{mult}_x(C) > \frac{(L \cdot C)}{e}.$$

Therefore we consider the set

$$S = \left\{ (C, x) \mid x \in C \subset X \text{ curve}, \mathrm{mult}_x(C) > \frac{(L \cdot C)}{e} \right\}.$$

It is the union of the sets

$$S_d = \left\{ (C, x) \mid x \in C \subset X \text{ curve}, \mathrm{mult}_x(C) > \frac{(L \cdot C)}{e},\ (L \cdot C) \leq d \right\}.$$

for $d = 0, 1, \ldots$. Let us say a subset $M \subset S$ is *parametrized* by a family $\mathcal{C} \subset X \times X \times T \to T$, if the fibres of that family make up M. Each S_d can be parametrized by a bounded family, so that S can be parametrized by countably many families (*). We will come to a proof of this statement later.
If we can show that each family $(C_t \ni x_t)_{t \in T}$ is trivial, we get the first statement of Theorem 4.5. Suppose this family is nontrivial. Without loss of generality T is a nonproper curve (otherwise substitute T by a curve $C \subset T$, such that the family is still nontrivial) and smooth (after substitution of T by an open subscheme). Each C_t is reduced and hence only finitely many points on it have multiplicity greater 1 (curves are always integral by our conventions). Thus $m_t \coloneqq \mathrm{mult}_{x_t}(C_t) > \frac{(L \cdot C_t)}{e} \geq 1$ for all t implies that $(C_t)_{t \in T}$ is already nontrivial. We may assume that $(m_t)_t$ is constantly m (cf. (*)). By a previous remark we have $\rho_{t^*} \neq 0$ for some point t^*. As we mentioned before, this implies $\left(C^{\cdot 2}\right) \geq m(m-1)$, where $C \coloneqq C_{t^*}$. On the other hand $\left(L^{\cdot 2}\right)\left(C^{\cdot 2}\right) \leq (L \cdot C)^2$ by the Hodge index theorem. Hence

$$\left(2e^2 - 2e + 1\right) m(m-1) \leq \left(L^{\cdot 2}\right)\left(C^{\cdot 2}\right) \leq (L \cdot C)^2 \leq (em-1)^2,$$

and elementary arguments show that this is impossible for $m > 1$. This finishes the proof of the first statement.
Now suppose $\left(L^{\cdot 2}\right) \geq 2$. We are done if we can show $S = S_d$ for some integer d. An easy calculation shows $\alpha \coloneqq e/\sqrt{(L^{\cdot 2})} < 1$. Observe that by Corollary 2.15 there exists a positive integer k such that for all $y \in X$ we can find a divisor $D_y \in |k \cdot L|$ with $\mathrm{mult}_y(D_y) \geq k \cdot \alpha$. We claim that we can choose $d \coloneqq k \cdot \left(L^{\cdot 2}\right)$. Indeed, let $(C, x) \in S$. We have

$$\frac{(L \cdot C)}{\mathrm{mult}_x C} < e = \alpha \sqrt{(L^{\cdot 2})}$$

and

$$\frac{(L \cdot D_x)}{\mathrm{mult}_x D_x} \leq \frac{1}{\alpha}\sqrt{(L^{\cdot 2})}.$$

By Lemma 3.4 the curve C is a component of D_x and we obtain

$$(L \cdot C) \leq (L \cdot D_x) = d,$$

i.e., $(C, x) \in S_d$. □

Filling the gaps. (*) Here, Hilbert schemes will prove themselves helpful. So let us first review those objects. Denote the category of locally noetherian k-schemes by Sch_k. Fix a projective variety X and an embedding in some $\mathbb{P}^n$. Define the *Hilbert functor*

$$\begin{aligned} \mathrm{Hilb}_X : \mathrm{Sch}_k &\longrightarrow \mathrm{Sets} \\ T &\longmapsto \{\text{closed subschemes } \mathcal{C} \subset X \times T \text{ flat over } T\}. \end{aligned}$$

It is a highly nontrivial fact that this functor is represented by a scheme $\mathrm{Hilb}\, X$, the so-called *Hilbert scheme of* X. It is separated, but unfortunately not noetherian in general.
Nevertheless for each numerical polynomial $P \in \mathbb{Q}[x]$ one can define a subfunctor $\mathrm{Hilb}_X^P \subset \mathrm{Hilb}_X$ by imposing the following extra condition on the families $\mathcal{C} \to T$: All fibres $\mathcal{C}_t \subset \mathbb{P}^n$ have the same Hilbert polynomial P. Each of those functors is representable by a closed subscheme $\mathrm{Hilb}^P X \subset \mathrm{Hilb}\, X$. One can show that

$$\mathrm{Hilb}\, X = \coprod_{P \text{ num. pol.}} \mathrm{Hilb}^P X$$

and that each $\mathrm{Hilb}^P X$ is projective over k.
Finally, note that $\mathrm{Hom}(\mathrm{Spec}\, k, \mathrm{Hilb}\, X)$ corresponds to all closed subschemes of X and that $\mathrm{id} \in \mathrm{Hom}(\mathrm{Hilb}\, X, \mathrm{Hilb}\, X)$ corresponds to a family $\mathcal{C} \to \mathrm{Hilb}\, X$ that parametrizes all closed subschemes of X. Analogous statements hold when $\mathrm{Hilb}\, X$ is substituted by $\mathrm{Hilb}^P X$.
Now let us focus on the special case when X is a smooth surface. For k sufficiently large, kL is very ample. Fix the embedding $X \subset \mathbb{P}^n$ induced by kL. Fix $d \geq 0$. The set in Lemma 2.20 is finite; let $c_1, \ldots, c_s \in N^1(X)$ be its elements. Let C_i be effective representatives of c_i for $i = 1, \ldots, s$ respectively. If P_i denotes the Hilbert polynomial of C_i, it does not depend on C_i. Indeed, $\deg P_i = \dim C_i = 1$, $P_i(0) = 1 - p_a(C_i) = -\frac{1}{2} \cdot ((c_i + K_X) \cdot c_i)$ and the leading coefficient of P_i is $\deg C_i = (c_i \cdot kL)$. Conversely every nonintegral curve $C \subset X$ with Hilbert polynomial P_i satisfies $(L \cdot C) = \frac{1}{k} \cdot \deg C_i \leq d$. Therefore the set

$$S_d''' = \{C \subset X \mid C \text{ is a nonintegral curve with } (L \cdot C) \leq d\}$$

is parametrized by a subset $T \subset \coprod_{i=1}^s \mathrm{Hilb}^{P_i} X$. One can show that T is an open subvariety. Hence the set

$$S_d'' = S_d''' \times X$$

is parametrized by $T \times X$. Let S_d' be defined like S_d, but without the assumption that the curves C are integral. We are done if we can show that $S_d' \subset S_d''$ corresponds to a closed subscheme Y of $T \times X$ and that $S_d \subset S_d'$ corresponds to an open subscheme Z of Y. By tedious arguments using the notions of 'open' and 'closed' subfunctors (cf. [EH00, VI.1.1]) one sees that this boils down to the following two facts:
If T is a nonproper variety and $\mathcal{C} \subset X \times T \to T$ a family of curves, then $\{t \in T \mid \mathcal{C}_t \text{ is integral}\}$ is an open subset of $T(k)$. If additionally $x \subset X \times T$ is a family of points, then $\{t \in T \mid \mathrm{mult}_{x_t} \mathcal{C}_t > \mu\}$ is a closed subset of $T(k)$. We omit the proofs.
(**) The second thing that needs to be settled is to make the arguments

with the Kodaira–Spencer map more precise.
Fix families $f : \mathcal{C} \to T$, $x : \mathcal{D} \to T$ of curves $C_t \subset X$, points $x_t \in C_t$ respectively over a nonproper curve T. Fix a point $0 \in T$. We want to define ρ_0 now. An element of $T_0 T$ is essentially the same thing as a morphism $\tau : \operatorname{Spec} k[t]/t^2 \to T$ with image 0. We have $X = \operatorname{Proj} A$ for some graded regular ring A. The curve $C_0 \subset X$ is given by a homogeneous ideal $I \subset A$. After base change of f via τ we obtain an infinitesimal deformation $C \to \operatorname{Spec} k[t]/t^2$ of C_0. This corresponds to a flat ring homomorphism $k[t]/t^2 \to A[t]/J$ with $t \mapsto t$, such that $A/I \cong A[t]/(J,t)$. One easily shows that $A[t] \to A, t \mapsto 0$ sends J to I and that the kernel of $A \to A[t]/J \xrightarrow{t} A[t]/J$ is contained in J. Now define $\rho_0(\tau) \in \operatorname{Hom}_{A/I}\left(I/I^2, A/I\right)$ as follows: If $a \in I$ there exists $b \in A$ such that $a + bt \in J$. Set $\rho_0(\tau)(a) = b$. Tedious checks show that ρ_0 is k-linear and well-defined.
Now we want to show:

Lemma 4.6. *If f is nontrivial, the set $\{t \in T \mid \rho_t \neq 0\}$ is dense in T.*

Lemma 4.7. *If $\operatorname{mult}_{x_t} C_t \geq m$ for all $t \in T$, then ρ_t vanishes to order $\geq m-1$ at x_t, i.e., $\operatorname{im} \rho_t \subset H^0\left(C_t, \mathcal{N}_t \otimes \mathfrak{m}_{x_t}^{m-1}\right)$, where $\mathfrak{m}_{x_t} \subset \mathcal{O}_X$ is the ideal sheaf of $\{x_t\}$.*

We will just prove those lemmas for $k = \mathbb{C}$. By standard arguments this can be extended to the general case.
Without loss of generality T is connected and f nontrivial. Then f stays nontrivial if we substitute T by an open subset. Hence it suffices to prove $\{t \in T \mid \rho_t \neq 0\} \neq \emptyset$ in Lemma 4.6. Because of $k = \mathbb{C}$ the nonproper curves $\mathcal{D}$, T only have finitely many singularities. After shrinking T we may assume that $\mathcal{D}$ and T are nonsingular. Now after application of [Har77, Cor. III.10.7] we may even assume that x is étale.
Choose a point $0 \in T$ and a tangent $\tau : \operatorname{Spec} k[t]/t^2 \to T$ at 0. We have a factorization

$$\operatorname{Spec} k[t]/t^2 \xrightarrow{\iota} \operatorname{Spec} k[\![t]\!] \xrightarrow{\varphi} \operatorname{Spec} \widehat{\mathcal{O}}_{T,0} \xrightarrow{\kappa} \operatorname{Spec} \mathcal{O}_{T,0}$$

The morphism φ is an isomorphism by regularity of T, but it is *not* unique. If $u \in \mathcal{O}_{T,0}$ is a uniformizing parameter though, we have $(\kappa \circ \varphi)^*(u) = \lambda_\tau t + r$ with a uniquely determined scalar λ_τ and some $r \in t^2 k[\![t]\!]$. Let $p : \mathcal{C}[\![t]\!] := \mathcal{C} \times_T \operatorname{Spec} k[\![t]\!] \to \operatorname{Spec} k[\![t]\!]$ be the canonical projection. An easy calculation shows

$$\rho_0(\tau) = \frac{\partial}{\partial t} p^*(\lambda_\tau t + r)\,|_{t=0} \in H^0(C_0, \mathcal{N}_0).$$

Now we will proceed by analytical methods: From now on we think of $\mathcal{C}$, $\mathcal{D}$, T as complex analytical spaces. As x is étale, it is a local biholomorphism. The morphism $\kappa \circ \varphi$ corresponds to a chart $t : U \to V \subset T$ with $U \subset \mathbb{C}$

open. We may assume $U = V = T$ and $t = \mathrm{id}$. After shrinking X and T we may assume $X \subset \mathbb{A}^n$ and $x : \mathcal{D} \tilde{\to} T$. Write $x^{-1}(t) = (a_1(t), \ldots, a_n(t))$ for $t \in T$. Use the shortcut $F_t(x_1, \ldots, x_n) = p^*(\lambda_\tau t + r)$. Then C_t is given by the homogeneous polynomial F_t for all $t \in T$. By assumption F_t has multiplicity $\geq m$ at $(0, \ldots, 0)$ for all t. Writing $F_t = \sum_i F_i t^i$, this implies $F_i \in (x_1, \ldots, x_n)^m := (x_1, \ldots, x_n)^m k[\![x_1, \ldots, x_n, t]\!]$ for all i. Hence $\frac{\partial}{\partial t} F_t \mid_{t=0} \in (x_1, \ldots, x_n)^{m-1}$ and Lemma 4.7 follows.
Lemma 4.6 follows from the fact, that F_t is a holomorphic function.
In applications we will not directly use Lemma 4.7, but the following

Corollary 4.8. *If in the situation of the lemma $\rho_0 \neq 0$ for some $0 \in T$, then $\left(C_0^{\cdot 2}\right) \geq m(m-1)$.*

Proof. We write $C = C_0$ and $x = x_0$. Let $\pi : \tilde{X} \to X$ be the blow up of x with exceptional divisor E. By the proof of Corollary 2.12 we have $\pi^* C = \tilde{C} + \mu E$, where $\mu := \mathrm{mult}_x C$. It follows from our assumption $\rho_0 \neq 0$ and from Lemma 4.7 that $H^0\left(C, \mathcal{O}_C(C) \otimes \mathfrak{m}_x^{m-1}\right)$ is nontrivial. By Lemma 3.6 this is equivalent to $\left|\left(f^*\left(\mathcal{O}_C(C)\right) + (1-m)E\right)\mid_{\tilde{C}}\right| \neq \emptyset$. Thus

$$\left(f^*\left(\mathcal{O}_C(C)\right) \cdot \tilde{C}\right) \geq \left((m-1)E \cdot \tilde{C}\right) = \mu(m-1)$$

$$\Rightarrow \left(C^{\cdot 2}\right) = \left(\tilde{C}^{\cdot 2}\right) = \left(f^*\left(\mathcal{O}_C(C)\right) \cdot \tilde{C}\right) \geq \mu(m-1) \geq m(m-1).$$

□

Remark 4.3. Write $C = C_0$. By similar, but more elaborate arguments, Xu shows $\left(C^{\cdot 2}\right) \geq m(m-1)+1$ for $m > 1$ in [Xu95, Lem. 1]. The main trick lies in showing, for instance, that the nontrivial section of $H^0\left(C, \mathcal{O}_C(C) \otimes \mathfrak{m}_x^{m-1}\right)$ also vanishes at another point, if $x \in C$ has at two least tangent directions. With this result one can improve Theorem 4.5: We just substitute $2e^2-2e+1$ by $2e^2 - 2e$.

Remark 4.4. Unfortunately one cannot directly apply the Lefschetz principle to Theorem 4.5 because of the notion of very general points. But if we just stick to statements about properties for general points, we get analogous results for arbitrary algebraically closed fields k. Let us give one example: If L is ample, we have $\left(M^{\cdot 2}\right) \geq 2e^2 - 2e$ and $(M \cdot C) \geq e$ for all curves $C \subset X$ with $e = 2$ and $M = 2L$. Then we have

$$\epsilon(L; x) = \frac{1}{2} \cdot \epsilon(M; x) \geq \frac{1}{2}$$

for general $x \in X$.

Now let us close this chapter with another application of the previous lemma. We present the following proposition shown by Steffens in [Ste98]:

Proposition 4.9. *If the smooth surface X has Picard number 1, then $\epsilon(L;1) \geq \left\lfloor \sqrt{(L^{\cdot 2})} \right\rfloor$ for all nef line bundles L.*

Proof. Without loss of generality L is a generator of $N^1(X)$. Let $\alpha := \left\lfloor \sqrt{(L^{\cdot 2})} \right\rfloor$. Assume the statement is false. By the arguments in Theorem 4.5 there exists a nontrivial family $(C_t, x_t)_{t\in T}$ with $\alpha \cdot \operatorname{mult}_{x_t} C_t > (L \cdot C_t)$ for all t. By Lemma 4.6 and Corollary 4.8 this gives us a curve C with $(L \cdot C) < \alpha \cdot \operatorname{mult}_x C =: \alpha m$ and $\left(C^{\cdot 2}\right) \geq m(m-1)$. By the assumption $N^1(X) = \mathbb{Z}$ we have $C \equiv dL$ for some $d \geq 0$. Hence

$$(L \cdot C) = d\left(L^{\cdot 2}\right) \geq d\alpha^2 \Rightarrow d\alpha < m \Rightarrow d\alpha \leq m-1.$$

Therefore

$$m(m-1) \leq \left(C^{\cdot 2}\right) = d(L \cdot C) < d\alpha m \leq m(m-1),$$

contradiction. □

5 Example: Canonical bundle on certain surfaces

After our somewhat general discussion of the Seshadri constant in the previous two chapters we now turn to concrete examples. In this chapter we will discuss the main results of [BS08].
The setting is as follows: We fix a smooth surface X over an algebraically closed, uncountable field k of characteristic 0, which is minimal and of general type, i.e., X does not contain (-1)-curves and has Kodaira dimension 2 (keep in mind that X is automatically projective).

Remark 5.1. This is actually equivalent to K_X being big and nef:
Indeed, assume K_X is big and nef. That X has Kodaira dimension 2 is clear by Corollary 2.14. Choose $m > 0$, such that $|mK_X| \neq \emptyset$, and choose an element D. If C is a (-1)-curve, then $(K_X \cdot C) = -1$ by the adjunction formula. But K_X is nef, contradiction.
If X is minimal and of general type, again choose $D \in |mK_X|$ for $m \gg 0$. If K_X is not nef, there exists a curve C with $(K_X \cdot C) < 0 \Rightarrow (D \cdot C) < 0$. Hence C is a component of D and $\left(C^{\cdot 2}\right) < 0$. By the adjunction formula C is a (-1)-curve, contradiction. By the same arguments as above we also get the bigness of X.

So X is a surface for which

- it makes sense to consider the Seshadri constants of K_X (K_X is nef) and for which

- not all Seshadri constants of K_X are 0 (K_X is big and nef).

Although this is the most general setting in which it is sensible to study the Seshadri constants of the canonical bundle, one can show the following three results ([BS08], the first result also holds if we drop the extra assumptions on k):

Theorem 5.1. *Let $x \in X$.*

(a) $\epsilon(K_X; x) = 0$ *if and only if x lies on a (-2)-curve.*

(b) *If $0 < \epsilon(K_X; x) < 1$, we have*

$$\epsilon(K_X; x) = \frac{m-1}{m}$$

for some integer $m \geq 2$ and this Seshadri constant is computed by a curve C, such that $\mathrm{mult}_x(C) = m$.

(c) *If $0 < \epsilon(K_X; x) < 1$ and $\left(K_X^{\cdot 2}\right) \geq 2$, then*

(i) *$m = 2$ and C can be chosen in such a way that $p_a(C) = 1$, or*

(ii) *$m = 3$ and C can be chosen in such a way that $p_a(C) = 2$.*

(d) *If $0 < \epsilon(K_X; x) < 1$ and $\left(K_X^{\cdot 2}\right) \geq 3$ only the first case is possible.*

Theorem 5.2. *We have $\epsilon(K_X; 1) \geq 1$ where equality holds if and only if $\left(K_X^{\cdot 2}\right) = 1$.*

Theorem 5.3. *If $\left(K_X^{\cdot 2}\right) \geq 6$, we have $\epsilon(K_X; 1) \geq 2$ with equality if and only if X admits a genus 2 fibration over a smooth curve.*

Remark 5.2. Case (a) in Theorem 5.1 only happens for x in a proper subvariety of X. Indeed, if C is a (-2)-curve we have $(K_X \cdot C) = 0$ by the adjunction formula. By Remark 2.6 with $L = K_X$ and $d = 0$ there are only finitely many numerical equivalence classes c which contain a (-2)-curve. Since $\left(c^{\cdot 2}\right) = -2$, the class c contains only one curve and we are done.

Remark 5.3. It is unknown whether every value for m in Theorem 5.1(b) can occur. However, Bauer gives an example for $m = 2$.

Proof of Theorem 5.1. (a) Because of $\left(K_X^{\cdot 2}\right) \geq 1$ we have $\epsilon(K_X; x) = 0$ if and only if there exists a curve $C \ni x$ with $(K_X \cdot C) = 0$. This and the Hodge index theorem automatically imply $\left(C^{\cdot 2}\right) < 0$. So the adjunction formula implies that $(K_X \cdot C) = 0$ is equivalent to C being a (-2)-curve.
(b) Remark 3.15 implies $\epsilon(K_X; x) = \frac{d}{m}$ for some curve $C \ni x$ with $m :=$ $\mathrm{mult}_x(C)$ and $d := (K_X \cdot C)$. The inequality $0 < \epsilon(K_X; x) < 1$ implies $m \geq d + 1 \geq 2$. We want to show $m \leq d + 1$. If $m = 2$, this is trivial, so assume $m \geq 3$. By the Hodge index theorem we have

$$\left(C^{\cdot 2}\right) \leq \frac{d^2}{\left(K_X^{\cdot 2}\right)} =: \frac{d^2}{c}.$$

By the adjunction formula we have

$$p_a(C) = 1 + \frac{1}{2}((C + K_X) \cdot C) \leq 1 + \frac{d(d/c+1)}{2}.$$

But C has a point of multiplicity m, hence

$$\binom{m}{2} \leq p_a(C) - p_g(C) \leq 1 + \frac{d(d/c+1)}{2} \tag{1}$$

by [Har77, Ex. V.3.9.2]. This easily implies $m \leq d+1$ because of $m \geq 3$ and $c \geq 1$.
(c) Know we assume $c \geq 2$. Then (1) implies $m \leq 3$ because otherwise we would have $m < d+1$, and we are done.
(d) Same argument as before, now using $c \geq 3$. □

Theorem 5.2 and Theorem 5.3 require some more work. Indeed, their proofs crucially depend on the following lemma that requires the type of variational arguments we came across in the previous chapter:

Lemma 5.4. *Let L be a big and nef line bundle on a smooth surface X. If*

$$\epsilon(L;1) < \sqrt{\frac{3}{4}(L^{\cdot 2})},$$

then X is fibered by nonintegral curves computing $\epsilon(L;1)$, or more precisely: There is a flat family $f : X \to B$ ($F_x := f^{-1}(x)$ for $x \in B$) such that

$$\epsilon(L;1) = \epsilon(L;x) = \frac{(L \cdot F_x)}{\operatorname{mult}_y(F_x)}$$

for general $y \in F_x$.
Note that $\epsilon(L;1)$ is an integer in this case, because for general $y \in F_x$ we have $\operatorname{mult}_y F_x = 1$.

Proof. Let

$$\Sigma := \left\{(C,x) \mid C \ni x \text{ is a curve}, \frac{(L \cdot C)}{\operatorname{mult}_x C} \leq \epsilon(L;1)\right\}.$$

By the arguments we came across in the last chapter this set is parametrized by countably many families. As k is uncountable and $\epsilon(L;1) = \epsilon(L;x)$ for very general $x \in X$, there exists a nontrivial family $(C_t \in x_t)_{t \in T}$ of elements of Σ such that $(C_t)_{t \in T}$ is also a nontrivial family. We may also assume $\epsilon(L;1) = \epsilon(L;x_t)$ for very general $t \in T$. Call those points $t \in T$ *good.* Without loss of generality this family and T are smooth and $m := \operatorname{mult}_{x_t} C_t$ is constant by (*).
By the Hodge index theorem we have

$$\left(L^{\cdot 2}\right) \cdot \left(C_t^{\cdot 2}\right) \leq (L \cdot C_t)^2 < \frac{3}{4} m^2 \left(L^{\cdot 2}\right)$$

for all $t \in T$. This and Remark 4.3 imply

$$m^2 - m + 1 < \frac{3}{4} m^2 \Leftrightarrow (m-2)^2 < 0$$

for $m > 1$, contradiction. Hence $m = 1$ and $\left(C_t^{\cdot 2}\right) < \frac{3}{4} \Rightarrow \left(C_t^{\cdot 2}\right) \leq 0$. Corollary 4.8 implies equality.
Fix a good point $0 \in T$. Embed T into a smooth proper curve $\bar{T}$. For $k \gg 0$ the linear system $|k0|$ (in $\bar{T}$) is very ample. By Bertini $k0$ is linearly equivalent to some nonsingular divisor $t_1 + \ldots + t_k$ with support in $T \setminus \{0\}$. In other words, the points t_i are pairwise different. The family $(C_t)_{t \in T}$ is given by a flat morphism $f : \mathcal{C} \to T$ and [Liu02, Lem. 7.1.33] implies $C_{t_1} + \ldots + C_{t_k} \in |kC_0|$. The base-point-free linear system $\langle kC_0, C_{t_1} + \ldots + C_{t_k} \rangle$ induces a fibration $g : X \to \mathbb{P}^1$. Unfortunately this is not the fibration we are looking for if $k > 1$. But there is the Stein factorization

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[d, "h"] \\ & \mathbb{P}^1 \end{tikzcd}$$

over a smooth curve B. We have $g^{-1}(1) = C_{t_1} + \ldots + C_{t_k}$ for some point $1 \in \mathbb{P}^1$. This curve has exactly k connected components, because the curves C_{t_κ} are connected (by integrality). Hence $h^{-1}(1)$ consists of k points and we have $\deg h = k$. This implies that for instance C_{t_1} is a fiber of the fibration f. But $C_{t_1} \equiv C_0$, so f is the fibration with the right properties (because all fibres are numerically equivalent). $\square$

Remark 5.4. The constant $\frac{3}{4}$ is optimal. Indeed, let $X \subset \mathbb{P}^3$ be a smooth cubic and $L = \mathcal{O}_X(1)$. Then we have

$$\epsilon(L; 1) = \frac{3}{2} = \sqrt{\frac{3}{4}\left(L^{\cdot 2}\right)}.$$

In particular, this Seshadri constant is no integer anymore.
Let us prove this. Fix a point $x \in X$. By inspection of the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

we obtain $h^0(X, L) = h^0\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)\right) - h^0\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)\right) = 4$, because the higher cohomologies of $\mathcal{O}_{\mathbb{P}^3}(-2)$ vanish. Since this is larger than $\binom{2+2-1}{2} = 3$, there exists a divisor $C \in |L|$ with $\operatorname{mult}_x C \geq 2$ by the proof of Corollary 2.15. By [Har77, Ex. II.8.20.3] we have $K_X = -L$ and by [Har77, Exc. V.1.5(a)] we have $\left(L^{\cdot 2}\right) = \left(K_X^{\cdot 2}\right) = 3$, hence

$$\epsilon(L; x) \leq \frac{(L \cdot C)}{\operatorname{mult}_x C} \leq \frac{3}{2} = \sqrt{\frac{3}{4}\left(L^{\cdot 2}\right)}.$$

Assume by way of contradiction that $\epsilon(L;1) < \frac{3}{2}$. Then Lemma 5.4 implies $\epsilon(L;1) = 1$ and the existence of a fibration $f : X \to B$ of nonintegral curves F with $(L \cdot F) = 1$. It follows from the proof of the lemma that at least one fiber is integral. By one of the two facts stated at the end of (*) in the previous chapter this implies that all fibers except for finitely many are integral. But if F is one of those fibers, we have $\left(F^{\cdot 2}\right) \leq (L \cdot F) / \left(L^{\cdot 2}\right) = 1/3 \Rightarrow \left(F^{\cdot 2}\right) \leq 0$ by the Hodge index theorem and

$$\left(F^{\cdot 2}\right) = 2p_a(F) - 2 - (K_X \cdot F) = 2p_a(F) - 2 + (L \cdot F) = 2p_a(F) - 1,$$

hence $p_a(F) = 0$ and F is a (-1)-curve. So F is one of the 27 lines on X and this contradicts the infinitude of possibilities for F (alternative argument: F is the only effective divisor in its numerical equivalence class and by Lemma 2.20 there are only finitely many possibilities).

Proof of Theorem 5.2. If $\left(K_X^{\cdot 2}\right) = 1$, we have $\epsilon(K_X;1) \leq 1$ by Remark 3.15. Lemma 5.4 implies $\epsilon(K_X;1) = 1$.
Conversely assume now $\epsilon(K_X;1) \leq 1$ and $\left(K_X^{\cdot 2}\right) \geq 2$. Again Lemma 5.4 implies equality and the existence of infinitely many curves $C \subset X$ with $(K_X \cdot C) = 1$. But by the Hodge index theorem we have

$$\left(K_X^{\cdot 2}\right)\left(C^{\cdot 2}\right) \leq (K_X \cdot C)^2 = 1 \Rightarrow \left(C^{\cdot 2}\right) \leq 0.$$

By the adjunction formula $p_a(C) = 1 + \frac{1}{2}\left(\left(C^{\cdot 2}\right) + (K_X \cdot C)\right)$ the self-intersection number of C is odd, so we have $\left(C^{\cdot 2}\right) \leq -1$. Hence C is the only curve in its numerical equivalence class. By Lemma 2.20 there are only finitely many elements in $N^1(X)$ with degree 1 with respect to K_X. This gives us a contradiction. □

Proof of Theorem 5.3. Assume $\epsilon(K_X;1) \leq 2$. Theorem 5.2 and Lemma 5.4 tell us that equality holds. Let $f : X \to B$ be like in Lemma 5.4. Then all fibres C satisfy

$$p_a(C) = 1 + \frac{1}{2}\left(\left(C^{\cdot 2}\right) + (K_X \cdot C)\right) = 2.$$

If conversely $f : X \to B$ is a genus 2 fibration over a smooth curve, then for all $x \in X$ we have $(K_X \cdot F_x) = 2 \Rightarrow \epsilon(K_X;x) \leq 2$ by the adjunction formula, where $F_x := f^{-1}(f(x))$. Hence $\epsilon(K_X;1) \leq 2$ and, as already noted, this implies equality. □

Remark 5.5. Bauer uses a slightly stronger version of Lemma 5.4.

6 Example: Calculation of Seshadri constants on abelian surfaces with Picard number 1

Seshadri constants on abelian surfaces X are particularly well understood. We will demonstrate this fact by presenting a result shown by Bauer in

[Bau99] in the case $k = \mathbb{C}$ and $\rho(X) = 1$. This theorem is remarkable, because it computes Seshadri constants in terms of the minimal solution of some Pell equation. Of course, this is clear evidence for the supposition that Seshadri constants are difficult to compute in general.
The assumption $\rho(X) = 1$ just says $N^1(X) = \mathbb{Z}L$ for some line bundle L (one can show $\rho(X) \leq 4$ ([BL04, Exc. 2.5]) and [BL04, Exc. 10.7] suggests that the case $\rho(X) = 1$ is the 'normal' one). The surface X is projective, hence kL is very ample for some integer $k \neq 0$. Without loss of generality we have $k > 0$ and L is ample. Our goal is to compute its Seshadri constants. Note that $\epsilon(L; x) = \epsilon(L)$ for all $x \in X$ by Remark 3.4. Also note that by Riemann–Roch we have

$$\chi(L) = \frac{((L - K_X) \cdot L)}{2} + \chi(\mathcal{O}_X).$$

It is a general fact that on abelian surfaces the canonical bundle is trivial. Hence $\left(L^{\cdot 2}\right) = 2d$ for some positive integer d. We show

Theorem 6.1. *In the situation described in the paragraph above we have:*

(a) *If $\sqrt{2d}$ is rational, then $\epsilon(L) = \sqrt{2d}$.*

(b) *If $\sqrt{2d}$ is irrational, then*

$$\epsilon(L) = 2d \cdot \frac{k_0}{l_0},$$

where (k_0, l_0) is the primitive solution of the Pell equation $l^2 - 2dk^2 = 1$.

Before we can start with the proof we need to state some facts:

1. The 2-torsion group consists of exactly 16 points $e_1 = 0, \ldots, e_{16}$ on X. In other words, those are the fixed points of the natural involution $\iota : X \to X$ (cf. [Mum74, Appl. 3 in Ch. II.6]). Let $\pi : \tilde{X} \to X$ denote the blow up at those points. The morphism ι lifts to a fixed-point-free involution $\tilde{\iota} : \tilde{X} \to \tilde{X}$. This gives us a group action of $\mathbb{Z}/2\mathbb{Z}$ on $\tilde{X}$. Its quotient K is a smooth K3 surface, the so-called *Kummer surface* of X. There is a canonical morphism $\varphi : \tilde{X} \to K$.

2. The line bundle $2L$ is numerically equivalent to a *symmetric* line bundle, i.e., a line bundle L_{sym} with $\iota^* L_{\text{sym}} \cong L_{\text{sym}}$. This is the reason: The map $\iota^* : N^1(X) \to N^1(X)$ is a group isomorphism, so we either have $\iota^* L \equiv L$ or $\iota^* L \equiv -L$. But by Lemma 2.8 the line bundle $\iota^* L$ is nef, so we are left with the first case. Therefore we have $2L \equiv L + \iota^* L =: L_{\text{sym}}$. Now let k be a positive integer. Clearly kL_{sym} is still symmetric. Hence ι^* induces an involution on $|kL_{\text{sym}}| = \mathbb{P}\left(H^0(X, kL_{\text{sym}})\right)$. Let

$$H^0(X, kL_{\text{sym}}) = H^0(X, kL_{\text{sym}})^+ \oplus H^0(X, kL_{\text{sym}})^-$$

be the corresponding eigenspace decomposition, i.e., the decomposition into so-called symmetric and anti-symmetric divisors. Then one can show that the dimension of the first summand is $2+2dk^2$. Bauer shows this in [Bau94] by the following line of argument: He shows that $\varphi_*\pi^*L_{\mathrm{sym}}$ admits a decomposition into line bundles M^+, M^- with $h^0(K, M^\pm) = \dim H^0(X, L_{\mathrm{sym}})^\pm$ (this is elementary). Using the properties of K3 surfaces and Riemann–Roch he establishes $\chi(M^+) = 2+2dk^2$, and using more involved arguments he shows that the higher cohomologies of M^+ vanish.
Given that, we can prove the existence of a symmetric divisor $D \in |kL_{\mathrm{sym}}|$ with $\mathrm{mult}_{e_1}(D) \geq \left\lfloor 2\sqrt{2dk^2+1} \right\rfloor$ by using similar arguments as in Corollary 2.15. But instead of $H^0(X, kL_{\mathrm{sym}})$ we consider the vector space $H^0(X, kL_{\mathrm{sym}})^+$ and instead of $\mathcal{O}_{X,x}/\mathfrak{m}_{e_1}^c$ we consider the subspace of symmetric elements, i.e., elements that are invariant under ι. One can show that the symmetric elements are exactly the ones with odd degree. The dimension of the corresponding subspace is $\lceil c/2 \rceil^2$ and we want this number to be smaller than $2+2dk^2$. Choose $c = \left\lfloor 2\sqrt{2dk^2+1} \right\rfloor$ and we get our divisor D.

3. Let C be a symmetric effective divisor on X. We say that a 2-torsion point e is an even, odd 2-torsion point of C, if $\mathrm{mult}_e C$ is even, odd respectively. If o_C is the number of odd 2-torsion points of C, we have $o_C \in \{0, 4, 6, 10, 12, 16\}$. This is shown by analytical methods in [BL04, Ch. 4.7]. The main idea for the proof is as follows: The 2-torsion subgroup $X_2 \subset X$ is a 4-dimensional $\mathbb{F}_2$-vector space;
$$\begin{aligned} q : X_2 &\longrightarrow \mathbb{F}_2 \\ e &\longmapsto (\mathrm{mult}_e C - \mathrm{mult}_0 C) \bmod 2 \end{aligned}$$
is a quadratic form (this is the difficult part) and by basic arguments with quadratic forms one gets $\#q^{-1}(1) \in \{0, 4, 6, 10, 12, 16\}$.
If $o_C = 0$, one can show that C is divisible by 2 in $N^1(X)$.

Proof of Theorem 6.1. (a) This follows from Proposition 4.9.
(b) We will closely follow the proof of [Bau99, Thm. 6.1]. By one of the facts stated above there is a symmetric divisor in $|k_0 L_s|$ with
$$\mathrm{mult}_{e_1}(D) \geq \left\lfloor 2\sqrt{2dk^2+1} \right\rfloor = 2l_0.$$
Thus
$$\epsilon(L) \leq \frac{(L \cdot D)}{\mathrm{mult}_{e_1}(D)} \leq \frac{(L \cdot 2k_0 L)}{2l_0} = 2d \cdot \frac{k_0}{l_0}.$$
Now suppose this upper bound is not sharp, i.e., we have a curve $C \ni e_1$ with
$$\frac{(L \cdot C)}{\mathrm{mult}_{e_1}(C)} < 2d \cdot \frac{k_0}{l_0}.$$

As $\iota : X \to X$ is an automorphism, we have $\mathrm{mult}_{e_1}(\iota^* C) = \mathrm{mult}_{e_1}(C)$ and $(L \cdot \iota^* C) = (L \cdot C)$ due to $L \equiv \iota^* L$. By Lemma 3.4 we thus have $C = \iota^* C$, so C is symmetric, and by the same argument C is a component of D, in particular $C \equiv k_1 L$ with $k_1 \leq 2k_0$.
Write $m_i := \mathrm{mult}_{e_i}(C)$ for $i = 1, \ldots, 16$. Our next aim is to prove a Pell-like equation involving those multiplicites and k_1. Remember the blow up $\pi : \tilde{X} \to X$ and the morphism $\varphi : \tilde{X} \to K$. The proper transform of C under the blow up is

$$\tilde{C} = \pi^* C - \sum_{i=1}^{16} \mathrm{mult}_{e_i}(C) \cdot E_i,$$

where $E_i = \pi^{-1}(e_i)$, $i = 1, \ldots, 16$ are the exceptional divisors. The curve C and therefore also the curve $\tilde{C}$ are symmetric. Hence $\tilde{C} = \varphi^{-1}\varphi\left(\tilde{C}\right)$, so $\bar{C} := \varphi\left(\tilde{C}\right)$ is still an integral curve. Then $h^0\left(K, \bar{C}\right) = 1$, because otherwise there are infinitely many curves in $|C|$ with the same multiplicities at the 2-torsion points as C. But this contradicts Lemma 3.4.
This implies $\left(\bar{C}^{\cdot 2}\right) = -2$, because from the exact sequence

$$0 \longrightarrow -\bar{C} \longrightarrow \mathcal{O}_K \longrightarrow \mathcal{O}_{\bar{C}} \longrightarrow 0$$

we get $h^2\left(K, \bar{C}\right) = h^0\left(K, -\bar{C}\right) = 0$ and $h^1\left(K, \bar{C}\right) = h^1\left(K, -\bar{C}\right) = 0$ by Serre duality and $h^1(K, \mathcal{O}_K) = 0$ and finally by Riemann–Roch

$$1 = \chi\left(\bar{C}\right) = \frac{1}{2}\left(\bar{C}^{\cdot 2}\right) + \chi(\mathcal{O}_K) = \frac{1}{2}\left(\bar{C}^{\cdot 2}\right) + 2.$$

Now we arrive at the equation

$$\begin{aligned} k_1^2 \cdot 2d - \sum_{i=1}^{16} m_i^2 &= \left(C^{\cdot 2}\right) - \sum_{i=1}^{16} m_i^2 \\ &= \left(\tilde{C}^{\cdot 2}\right) = \left(\varphi^* \bar{C}^{\cdot 2}\right) = \deg \varphi \cdot \left(\bar{C}^{\cdot 2}\right) = -4. \end{aligned}$$

Hence

$$k_1^2 \cdot 2d - m_1^2 \geq -4.$$

Note that this expression is negative, because otherwise we would have

$$\frac{(L \cdot C)}{m_1} = \frac{k_1}{m_1} \cdot 2d \geq \sqrt{2d}.$$

So $N := k_1^2 \cdot 2d - m_1^2 \in \{-4, -3, -2, -1\}$. This gives us 4 cases:
Case 1. $N = -4$. We automatically get $2|m_1$ and $m_i = 0$ for $i > 1$. Hence $o_C = 0$ and as we noted before, this implies $2|k_1$. Now $(k_1/2, l_1/2)$ is a

solution of $l^2 - 2dk^2 = 1$ and by $k_1 \leq 2k_0$ and the minimality of (k_0, l_0) we obtain

$$k_1 = 2k_0 \text{ and } m_1 = 2l_0 \Rightarrow \frac{(L \cdot C)}{m_1} = 2d \cdot \frac{k_0}{l_0},$$

contradiction.
Case 2. $N = -3$. By elementary arguments we get that m_1 is odd. Exactly one of the numbers m_i, $i \geq 2$, is 1 and all the other numbers are 0 because of $\sum_{i=2}^{1} 6m_i^2 = 1$. So we have $o_C = 2$ and this is impossible.
Case 3. $N = -2$. Then m_1 is even and by similar arguments as before, we get $o_C = 3$ and a contradiction.
Case 4. $N = -1$. In other words: (k_1, m_1) solves the Pell equation $l^2 - 2dk^2 = 1$ and similarly as in the first case we get $k_1 = k_0$ and $m_1 = l_0$ and the same contradiction as in the first case. □

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