Stable pair invariants after Pandharipande–Thomas

Maximilian Wiesmann

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Betreuer: Prof. Dr. Daniel Huybrechts

Zweitgutachter: Prof. Dr. Georg Oberdieck

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Deutsche Zusammenfassung

Das Zählen von Kurven hat eine lange und reiche Geschichte: Bereits die algebraischen Geometer des 19. Jahrhunderts wussten, dass es 27 Geraden auf einer kubischen Fläche gibt, 2785 Geraden auf einer allgemeinen dreidimensionalen Quintik, 5819539783680 gedrehte Kubiken tangential zu 12 allgemeinen Quadriken in $\mathbb{P}^3$, ...

Das Thema bekam in den 1990er Jahren einen völlig neuen Schub, als das Zählen von Kurven durch Ideen aus der Stringtheorie Einzug in die Physik fand. In dieser Zeit wurden Gromov–Witten-Invarianten (im Folgenden GW-Invarianten) eingeführt, welche Einbettungen von (stabilen) Kurven in eine komplexe, dreidimensionale Mannigfaltigkeit „zählen" und es ermöglichen, Pfadintegrale in der Stringtheorie zu definieren. (Man stelle sich die Weltfläche vor, die ein geschlossener String erzeugt, der durch sechs kompakte Dimensionen propagiert – es entsteht eine Einbettung einer Riemannschen Fläche in einen Raum mit sechs reellen Dimensionen, d.h. eine Einbettung einer Kurve in einen komplexen, dreidimensionalen Raum; es scheint daher plausibel, dass es eine Verbindung zwischen dem Zählen von Kurven auf einer dreidimensionalen Mannigfaltigkeit und Pfadintegralen in der Stringtheorie gibt.) Allerdings ist der Ausdruck „zählen" an dieser Stelle etwas euphemistisch, denn a priori sind GW-Invarianten rationale Zahlen, da sie durch Integration gegen die (virtuelle) Fundamentalklasse des Modulraums stabiler Abbildungen $\overline{\mathcal{M}}(X)$ definiert sind, welcher im Allgemeinen kein Schema, sondern ein Deligne–Mumford-Stack ist. Dennoch fand man in vielen Fällen heraus, dass den GW-Invarianten tatsächlich ganze Zahlen zugrunde liegen. Eine geometrische Interpretation für diesen Umstand wäre wünschenswert, worauf das Konzept von Stabile-Paare-Invarianten abzielt.

Stabile-Paare-Invarianten wurden von R. Pandharipande und R. Thomas in 2009 eingeführt [30]. Auf einer glatten, projektiven, dreidimensionalen Mannigfaltigkeit bilden sie eine ganzzahlige und deformationsinvariante Theorie. In vielen Fällen konnte man inzwischen zeigen, dass sie äquivalent zu GW-Invarianten sind, was die oben beschriebene Frage nach der Ganzzahligkeit beantwortet. Auch spielten sie eine Hauptrolle im Beweis der Äquivalenz von GW- und Donaldson–Thomas-Theorie (ein weiterer Ansatz, um Kurven zu zählen, der aus algebraischer Sichtweise etwas natürlicher erscheint als GW-Invarianten) [29].

Das Ziel dieser Arbeit ist es, die Konstruktion von Stabile-Paare-Invarianten zu erklären. Im ersten Abschnitt führen wir den Begriff des stabilen Paares

$$\mathcal{O}_X \xrightarrow{s} F$$

ein, wobei F eine kohärente, eindimensionale Garbe und s ein nichttrivialer Schnitt ist. Wir benutzen auch die Notation (F, s) für ein solches Paar. Man kann sich vorstellen, dass ein stabiles Paar eine Einbettung einer Cohen–Macaulay-Kurve C mit zusätzlichen Punkten auf der Kurve nach X ist. Falls $\iota\colon C \hookrightarrow X$ eine solche Einbettung und D ein Divisor auf C ist, mit kanonischem Schnitt $s_D\colon \mathcal{O}_X \to \iota_*\mathcal{O}_C(D)$, dann ist

$$(\iota_*\mathcal{O}_C(D), s_D)$$

ein Beispiel für ein stabiles Paar.

Was bedeutet „stabil" in diesem Kontext? Um den Modulraum von stabilen Paaren zu konstruieren, muss man Automorphismen auf den Paaren eliminieren, was durch die Einführung von Stabilitätskriterien erreicht wird. Diese Kriterien sind zunächst von numerischer Gestalt und kommen aus der Geometrischen Invariantentheorie (GIT). Diese Theorie liefert die Konstruktion des Modulraums stabiler Paare $P(X)$ mittels eines GIT-Quotienten. Bemerkenswerterweise ist jede Komponente $P_n(X,\beta)$ von $P(X)$ eine projektive Varietät. Wie sich herausstellt, lassen sich die numerischen Kriterien in geometrische übersetzen und ein Paar (F,s) ist stabil, wenn F von echter Dimension eins und der Kokern von s nulldimensional ist.

Es ergibt sich aber das Problem, dass der Modulraum $P(X)$ nicht rein nulldimensional ist, d.h. es ist a priori nicht klar, wie man stabile Paare zählen kann. Üblicherweise, wie etwa auch im Falle der GW-Invarianten, schafft die Konstruktion einer virtuellen Fundamentalklasse hier Abhilfe. Leider liefert die natürliche Obstruktionstheorie von $P(X)$ keine solche Klasse. Um dieses Problem zu umgehen, fassen wir ein stabiles Paar als Zwei-Term-Komplex

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\}$$

in der derivierten Kategorie $D^b(X)$ auf; dies behandeln wir im zweiten Abschnitt.

Der nächste Schritt in der Konstruktion besteht darin, $P(X)$ in den größeren Modulraum von perfekten Komplexen in $D^b(X)$ mit trivialer Determinante einzubetten. Hierzu untersuchen wir die Deformationstheorie von stabilen Paaren und Komplexen mit trivialer Determinante und zeigen im dritten Abschnitt, dass diese übereinstimmen. Dies ist die Aussage des Hauptresultats aus §3:

Satz ([30, Thm. 2.7])**.** *Sei $\iota\colon B_0 \hookrightarrow B$ eine Aufdickung N-ter Ordnung von quasi-projektiven Schemata. Sei $I^\bullet$ eine Deformation über B eines Komplexes $I_0^\bullet$, der zu einem stabilen Paar (F_0,s_0) auf $X\times B_0$ assoziiert ist, mit* $\det(I^\bullet)\cong\mathcal{O}_{X\times B}$. *Dann ist $I^\bullet$ quasi-isomorph zu einem Komplex*

$$\{\mathcal{O}_{X\times B} \xrightarrow{s} F\},$$

wobei F eine flache Deformation von F_0 mit Schnitt s ist.

Die natürliche Obstruktionstheorie des Modulraums der perfekten Komplexe mit trivialer Determinante ermöglicht nun die Konstruktion einer virtuellen Klasse. Eine kurze Einführung in virtuelle Klassen wird im vierten Abschnitt gegeben.

Im fünften Abschnitt konstruieren wir dann eine virtuelle Klasse für $P(X)$. Wenn X eine dreidimensionale Calabi–Yau-Mannigfaltigkeit ist, hat $P(X)$ virtuelle Dimension null und wir können Stabile-Paare-Invarianten definieren als

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1.$$

Im sechsten und letzten Abschnitt untersuchen wir erste Beispiele und skizzieren Zusammenhänge zu anderen Kurven zählenden Invarianten.

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Contents

Introduction

Curve counting has a long and rich tradition: already the algebraic geometers of the 19th century knew that there are 27 lines on a smooth cubic surface, 2785 lines on a general quintic threefold, 5819539783680 twisted cubics tangent to 12 general quadrics in $\mathbb{P}^3$, ...

The subject got a completely new direction in the 1990s when ideas from string theory opened the door for curve counting in physics. Gromov–Witten invariants (in the following GW invariants for short) were introduced, "counting" embeddings of (stable) curves into a complex threefold X and providing a method to define path integrals in string theory. (Imagine the worldsheet of a closed string propagating through six compact dimensions—one gets a Riemann surface embedded into a space with six real dimensions, i.e. a curve embedded in a complex threefold; therefore, it should sound reasonable that curve counting on a complex threefold is related to path integrals in string theory.) But the term "counting" is quite euphemistic here: a priori, GW invariants are rational numbers. This is because they are defined via integration against a (virtual) fundamental class of the moduli space of stable maps $\overline{\mathcal{M}}(X)$ which is in general not a scheme but a Deligne–Mumford stack. However, one discovered that in a huge class of examples GW theory is governed by integers. It would be highly desirable to achieve a geometric interpretation of this fact and the concept of stable pair invariants aims at that direction.

Stable pair invariants were introduced by R. Pandharipande and R. Thomas in 2009 [30]. On a smooth, projective, complex threefold they provide an integer valued and deformation invariant theory. One could show that they are equivalent to GW invariants in a large number of cases which solved the integrality problem described above. Moreover, they played a key role in establishing an equivalence between GW and Donaldson–Thomas theory (yet another way of counting embedded curves, from an algebraic point of view more naturally defined than GW invariants) [29].

The goal of this thesis is to present the construction of stable pair invariants.
In the first section we introduce the notion of a stable pair

$$\mathcal{O}_X \xrightarrow{s} F,$$

where F is a coherent, one-dimensional sheaf and s a nontrivial section. We also write (F, s) to denote such a pair. One can think of a stable pair as an embedding of a Cohen–Macaulay curve C with additional points on the curve into X. Indeed, if $\iota\colon C \hookrightarrow X$ is such an embedding and D is a divisor on C coming with the canonical section $s_D\colon \mathcal{O}_X \to \iota_*\mathcal{O}_C(D)$, then

$$(\iota_*\mathcal{O}_C(D), s_D)$$

is an example of a stable pair.

What does "stable" mean? To construct the moduli space of stable pairs one has to eliminate automorphisms on the pairs which is done by imposing certain stability conditions. These conditions are firstly stated as numerical conditions on the pair resulting from Geometric Invariant Theory (GIT). This theory leads to the construction of the moduli space of stable pairs $P(X)$ via GIT quotients. The remarkable thing about stable pairs is that each component $P_n(X, \beta)$ of $P(X)$ is a projective variety. It turns out that the numerical stability conditions translate into geometric terms and a pair (F, s) is stable if F is a pure sheaf of dimension one and s has zero-dimensional cokernel.

The problem is that $P(X)$ is not purely zero-dimensional so, a priori, it is not obvious how to define a counting of stable pairs. The usual workaround for this issue is the construction of a virtual fundamental class, as for example in the case of GW invariants. Unfortunately, the natural obstruction theory for $P(X)$ does not admit such a class. To remedy this problem a stable pair is regarded as a two-term complex

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\}$$

in the derived category $D^b(X)$; we discuss this in the second section.

The next step of the construction is to show that we can regard $P(X)$ as a component of the bigger moduli space of perfect complexes in $D^b(X)$ with trivial determinant. For this purpose, we study the deformation theory of stable pairs and complexes with trivial determinant in the third section and show that they agree. This is captured in the main theorem of §3:

Theorem ([30, Thm. 2.7]). *Let $\iota\colon B_0 \hookrightarrow B$ be an N-th order thickening of quasi-projective schemes. Let $I^\bullet$ be a deformation over B of a complex $I_0^\bullet$ associated to a stable pair (F_0, s_0) on $X \times B_0$ with* $\det(I^\bullet) \cong \mathcal{O}_{X\times B}$*. Then $I^\bullet$ is quasi-isomorphic to a complex*

$$\{\mathcal{O}_{X\times B} \xrightarrow{s} F\},$$

where F is a flat deformation of F_0 with section s.

The moduli space of these complexes has the advantage that its obstruction theory *does* admit a virtual class. We will give a short introduction to virtual classes in the fourth section.

A further study of the obstruction theory of the complexes associated to stable pairs leads to the construction of a virtual class for $P(X)$ in the fifth section. If X is a Calabi–Yau threefold, the virtual dimension of $P(X)$ is zero and we can finally define stable pair invariants as

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1.$$

In the sixth and last section we will study first examples and give some relations to other curve counting invariants.

1. Definitions and preliminaries

Notation 1.1. Throughout this thesis X denotes a smooth projective variety of dimension three over $\mathbb{C}$. Fix a very ample line bundle $L = \mathcal{O}_X(1)$ on X. For a coherent sheaf F on X we set $F(k) = F \otimes L^k$. A subsheaf $G \subset F$ is *proper* if it is nonzero and not equal to F. All rings are assumed to be commutative with unity. We write $\mathbb{P}^n$ shorthand for $\mathbb{P}^n_{\mathbb{C}}$.

1.1. The moduli space of stable pairs.

The following notion plays a crucial role in the definition of a stable pair.

Definition 1.2 ([17, Def. 1.1.2])**.** A coherent sheaf F is called *pure* if all nonzero subsheaves of F have support of the same dimension. The dimension of F is defined as the dimension of the support of F.

We can think of pure sheaves as generalizations of torsion-free sheaves (see also Remark 3.13). Let us recall the following definition.

Definition 1.3 ([15, §III.5])**.** Let F be a coherent sheaf on X. The *Hilbert polynomial* of F is the unique polynomial with rational coefficients satisfying

$$k \mapsto \chi(F(k))$$

for all $k \in \mathbb{Z}$.

Ultimately, we would like to have the (fine) moduli space of stable pairs. For the present, we consider the moduli space $P^q_n(X, \beta)$. Here, $n \in \mathbb{Z}$ is some integer and $0 \neq \beta \in H_2(X, \mathbb{Z})$ some nonzero homology class; these serve as parameters for the moduli problem. Moreover, there is the parameter $q \in \mathbb{Q}[k]$ with positive leading coefficient which is used as a stability parameter. Consider the moduli functor

$$\begin{array}{rcl} \mathfrak{P}^q_n(X,\beta)\colon \begin{pmatrix} \text{quasi-projective} \\ \text{schemes over } \mathbb{C} \end{pmatrix} & \to & (\mathrm{Set}) \\ B & \mapsto & \left\{\text{families of } q\text{-semistable pairs } \mathcal{O}_{X\times B} \xrightarrow{s} F\right\} / \simeq \end{array}$$

where a family of q-semistable pairs is a pair

$$\mathcal{O}_{X\times B} \xrightarrow{s} F$$

such that F is flat over B and for all closed points $b \in B$, the restriction (F_b, s_b) to the fiber $X \times \{b\}$ is a q-semistable pair (the notion of a q-semistable pair will be introduced in Definition 1.4 below) with Hilbert polynomial

$$\chi(F_b(k)) = k \int_\beta c_1(L) + n.$$

The families are considered up to equivalence, where two families (F, s) and (F', s') are equivalent if there exists an isomorphism $\psi\colon F \to F'$ with $\psi \circ s = s'$.

The corresponding moduli space $P^q_n(X, \beta)$ representing the functor $\mathfrak{P}^q_n(X, \beta)$ can be constructed via GIT, see [27] for a general reference.

What are stability conditions good for? Roughly speaking, a stability condition is needed to eliminate certain homomorphisms of the pairs. Homomorphisms of the objects the moduli space parametrizes can be "bad" for several reasons (this is of course very generally speaking and not an entirely precise statement):

(i) Automorphisms prevent the moduli space from being a scheme.

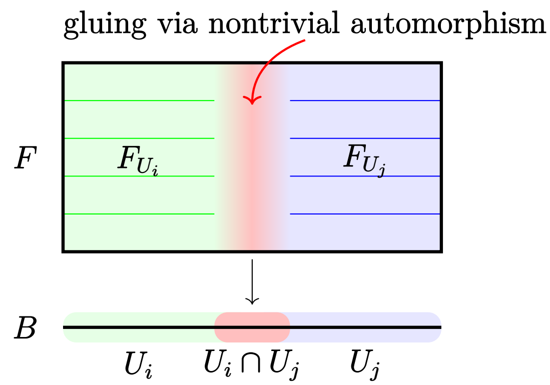

FIGURE 1.1. Construction of a nontrivial family F over a base scheme B via gluing along a nontrivial automorphism.

(ii) Homomorphisms between different objects could make the moduli space non-separated.

The first point can be illustrated quite intuitively. Suppose there exist nontrivial automorphisms on the families we want to parametrize. The idea is to find a base scheme B with a covering $B = \cup_i U_i$ and construct a nontrivial family F over B such that F_{U_i} is trivial for all i but the gluing

$$F_{U_i}|_{U_i \cap U_j} \cong F_{U_j}|_{U_i \cap U_j}$$

is done via some nontrivial automorphism, see Figure 1.1. Now suppose there is a scheme M representing the moduli functor and coming with a universal family $\mathbb{U}$. Then the map from B to M factors over the U_i and therefore the pullback of $\mathbb{U}$ is trivial, contradicting the nontriviality of F. An example where the second point becomes relevant can be found in [23, Prop. 6.6].

The (semi-)stability conditions for the moduli problem of stable pairs were calculated by Le Potier [24] and are summarized in the next paragraph, following [30, §1].

For the rest of this section let F denote a pure sheaf of dimension one on X and s a nonzero section of F. If G is a sheaf with $\dim(\mathrm{Supp}(G)) \leq 1$, then its Hilbert polynomial is given by

$$\chi(G(k)) = r(G)k + c$$

for some leading coefficient $r(G)$; we have $r(G) = 0$ if and only if $\dim(\mathrm{Supp}(G)) = 0$ (recall that the degree of the Hilbert polynomial equals the dimension of the support). In particular, by the purity of F any proper subsheaf $G \subset F$ has $r(G) > 0$.

Definition/Proposition 1.4. Let $q \in \mathbb{Q}[k]$ be a polynomial with positive leading coefficient. The pair (F, s) is *q-semistable* if for every proper subsheaf $G \subset F$

$$\frac{\chi(G(k))}{r(G)} \leq \frac{\chi(F(k)) + q(k)}{r(F)} \quad \text{for } k \gg 0 \tag{1}$$

and for every proper subsheaf G through which s factors

$$\frac{\chi(G(k)) + q(k)}{r(G)} \leq \frac{\chi(F(k)) + q(k)}{r(F)} \quad \text{for } k \gg 0 \tag{2}$$

hold. The pair is *q-stable* if the above inequalities are strict.

The moduli space $P_n^q(X,\beta)$ is realized as a subset of a product of a Quot-scheme and a Grassmannian and is a projective variety. We will not explain this construction, but provide the definition of a Quot-scheme.

Definition 1.5 ([10, §5.1])**.** Let S be a Noetherian scheme, $Y \to S$ a scheme of finite type over S and E a coherent sheaf on Y. For any scheme $B \to S$, a *family of quotients of E parametrized by B* is a pair (G,q) with

- G a coherent sheaf on $Y_B = Y \times_S B$ such that its schematic support is proper over B and G is flat over B;
- $q\colon E_B \twoheadrightarrow G$ a surjective $\mathcal{O}_{Y_B}$-linear homomorphism, where E_B is the pullback of E under the projection $Y_B \to Y$.

An equivalence relation is defined via $(G,q) \sim (G',q') \Leftrightarrow \ker(q) = \ker(q')$. Denote the corresponding equivalence class by $\langle G,q\rangle$.

Definition/Proposition 1.6 ([10, §5.1])**.** The *Quot-functor* is defined via

$$\begin{array}{rccl} \mathfrak{Quot}_{E/Y/S} & : & (\mathrm{Sch}_S)^{\mathrm{op}} & \to & (\mathrm{Set}) \\ & & B & \mapsto & \{\text{all } \langle G,q\rangle \text{ parametrized by } B\}. \end{array}$$

This functor is representable by a scheme $\mathrm{Quot}_{E/Y/S}$, the *Quot-scheme.*

Remark 1.7. Let G be a coherent sheaf on Y_B with the properties as in Definition 1.5. For each $b \in B$ let $\phi_b \in \mathbb{Q}[k]$ be the Hilbert polynomial of $G_b = G|_{Y_b}$. As G is flat over B the function $b \mapsto \phi_b$ is locally constant. Therefore, one obtains the stratification

$$\mathrm{Quot}_{E/Y/S} = \coprod_{\phi \in \mathbb{Q}[k]} \mathrm{Quot}^{\phi}_{E/Y/S} \ ,$$

where $\mathrm{Quot}^{\phi}_{E/Y/S}$ is the component parametrizing all quotients with fixed Hilbert polynomial ϕ.

Example 1.8. Let us consider the scheme $\mathrm{Quot}_{\mathcal{O}_Y/Y/S}$, parametrizing surjections $\mathcal{O}_Y \twoheadrightarrow G$. Such a surjection is characterized by its ideal sheaf. Therefore, $\mathrm{Quot}_{\mathcal{O}_Y/Y/S}$ is the moduli space parametrizing closed subschemes of Y. We also call this scheme the *Hilbert scheme* of Y over S and denote it by $\mathrm{Hilb}_{Y/S}$. Similarly to the Quot-scheme, the Hilbert scheme is a disjoint union of Hilbert schemes with fixed Hilbert polynomial, i.e.

$$\mathrm{Hilb}_{Y/S} = \coprod_{\phi \in \mathbb{Q}[k]} \mathrm{Hilb}^{\phi}_{Y/S} \ .$$

Hilbert schemes can be seen as generalizations of Grassmannians. Indeed, let

$$\phi_r = \binom{r+k}{r} \in \mathbb{Q}[k]$$

be the polynomial calculating the dimension of $H^0(\mathbb{P}^r_S, \mathcal{O}_{\mathbb{P}^r_S}(k))$. Then the Hilbert scheme $\mathrm{Hilb}^{\phi_r}_{\mathbb{P}^n_S/S}$ is isomorphic to the Grassmannian $\mathrm{Gr}_S(r+1,n+1)$.

The following example provides a computation of the moduli space of pairs $\mathcal{O}_X \to F$ where the sheaf F has support of dimension zero.

Example 1.9. The stability conditions from Definition 1.4 may be valid if F is not necessarily one-dimensional. Let F be a coherent sheaf with Hilbert polynomial $\chi(F(k)) = k\int_\beta c_1(L)+n$. If $\beta = 0$, then $\chi(F(k))$ is the constant n and we set $r(F)$ to be the leading coefficient n

in the Hilbert polynomial. Therefore, equation (1) is always satisfied. Consider a closed, zero-dimensional subscheme $Z \subset X$ satisfying $h^0(Z, \mathcal{O}_Z) = n$, i.e. a length n subscheme. The pair

$$\mathcal{O}_X \xrightarrow{1} \mathcal{O}_Z$$

is q-stable for any q as the section 1 already generates $\mathcal{O}_Z$, so case (2) is irrelevant.

Indeed, for $\beta = 0$, all stable pairs are of this form. Let $\mathcal{O}_X \xrightarrow{s} F$ be a q-stable pair and let Z be the zero-dimensional support of F, consisting of m points $z_1, \dots, z_m$. Then F can be written as a direct sum over its stalks, i.e. $F \cong F_{z_1} \oplus \cdots \oplus F_{z_m}$, satisfying

$$h^0(Z, F) = \sum_{i=1}^{m} h^0(Z, F_{z_i}) = n.$$

Suppose s factors through a proper subsheaf $G \subset F$. Then $r(G) < r(F)$ and equation (2) is violated. Therefore, s must generate F which implies that Z consists of n points and $F = \mathcal{O}_Z$.

Hence, q-stable pairs are in 1:1 correspondence with length n subschemes. Thus, $P_n^q(X, 0)$ is the Hilbert scheme $\mathrm{Hilb}^n_{X/\mathbb{C}}$.

To make the notion of stability independent of q we look at the limit case where q becomes large. The following definition states this precisely.

Definition 1.10. Let $(\mathbb{Q}[k], \leq)$ be the polynomial ring in one variable over the rationals together with lexicographic ordering, denoted by "$\leq$". A pair (F, s) is called *limit stable* if there exists some linear polynomial $q_0(k) = A_0 k + B_0 \in \mathbb{Q}[k]$ such that (F, s) is q-stable for any polynomial $q \in \mathbb{Q}[k]$ with $q_0 \leq q$.

The following lemma gives a geometric description of limit stability.

Lemma 1.11 ([30, Lem. 1.3]). *For q sufficiently large, stability and semistability coincide. A pair (F, s) is limit stable if and only if*

(i) *the sheaf F is pure of dimension one,*
(ii) *the section $\mathcal{O}_X \xrightarrow{s} F$ has zero-dimensional cokernel.*

Proof. For q sufficiently large, inequality (1) is always satisfied. Multiplying inequality (2) with $r(G)r(F)$ and using that $\chi((F/G)(k)) = \chi(F(k)) - \chi(G(k))$ one obtains

$$(r(F) - r(G))(\chi(F(k)) + q(k)) \leq r(F)\chi((F/G)(k)).$$

For q sufficiently large this implies

$$r(F) - r(G) = 0 \tag{3}$$

showing that any proper subsheaf G of F has one-dimensional support, i.e. F is pure. Furthermore, equality never holds as $\chi((F/G)(k))$ is a nonzero constant, so stability and semistability coincide in the limit case. Setting $G = \mathrm{im}(s)$ in (3) we see that s has zero-dimensional cokernel.

Conversely, if s has zero-dimensional cokernel we get $r(F) = r(\mathrm{im}(s))$. Take a proper subsheaf G of F through which s factors. By purity of F the support of G is one-dimensional and $r(\mathrm{im}(s)) \leq r(G) \leq r(F)$ implying that (3) holds. □

Definition 1.12. A *stable pair* (F, s) is a pair that is limit stable. The corresponding fine moduli space is denoted by $P_n(X, \beta)$. The fine moduli space of all stable pairs with no restriction on the Hilbert polynomial is denoted by $P(X)$.

Let us take a look at a first example of a stable pair which will provide much insight on how to think about stable pairs in general.

Example 1.13. Let C be a Cohen–Macaulay curve with a closed immersion $\iota\colon C \hookrightarrow X$ and let D be a Cartier divisor on C coming with the canonical section $s_D\colon \mathcal{O}_X \to \iota_*\mathcal{O}_C(D)$. Then

$$(\iota_*\mathcal{O}_C(D), s_D)$$

is a stable pair.

Clearly, the cokernel of s_D is zero-dimensional as its support is the divisor D. The assumption on C to be Cohen–Macaulay relates to the purity of $\iota_*\mathcal{O}_C(D)$ and is a little more subtle. As we will see in Lemma 1.19 the support of the sheaf belonging to a stable pair turns out to be always a Cohen–Macaulay curve.

1.2. **Stable pairs and Cohen–Macaulay curves.** To understand the relationship between Cohen–Macaulay curves and pure sheaves we first revise some definitions from [26, Ch. 6] and then discuss this relationship more detailed.

Definition 1.14. Let A be a ring and let M be an A-module. A finite sequence $(a_1, \dots, a_n)$ of elements $a_i \in A$ is called *regular* for M (or *M-regular*) if for all $i = 1, \dots, n$ the multiplication with a_i on the A-module $M/(a_1M + \cdots + a_{i-1}M)$ is injective and $M/(a_1M + \cdots + a_nM) \neq 0$.

Definition 1.15. Let A be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated A-module. Then $\operatorname{depth}_A(M)$ is the length of a maximal M-regular sequence with elements $a_i \in \mathfrak{m}$.

Definition 1.16. Let A be a Noetherian ring. It is called *Cohen–Macaulay* if $\operatorname{depth}(A_\mathfrak{p}) = \dim(A_\mathfrak{p})$ for all prime ideals $\mathfrak{p}$. A locally Noetherian scheme X is called Cohen–Macaulay if all local rings $\mathcal{O}_{X,x}$ are Cohen–Macaulay.

Proposition 1.17. *Let C be a locally Noetherian scheme of dimension one. Then the following are equivalent:*

(i) *C has no embedded points.*
(ii) *C is Cohen–Macaulay.*

Proof. Embedded points correspond to associated prime ideals that are not minimal. Suppose that C is not Cohen–Macaulay, so there exists some closed point $x \in C$ such that $\mathcal{O}_{C,x}$ has depth zero. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{C,x}$. As the multiplication with any element of $\mathfrak{m}$ cannot be injective, there must exist some $a \in \mathcal{O}_C$ with $a\mathfrak{m} = 0$ which implies that $\mathfrak{m} \in \operatorname{Ass}(\mathcal{O}_{C,x})$. Since $\mathfrak{m}$ is maximal, it is an embedded prime and therefore corresponds to an embedded point. The converse follows analogously. □

Example 1.18. Every smooth curve is Cohen–Macaulay. The cuspidal curve $\operatorname{Spec}(\mathbb{C}[x,y]/(y^3 - x^2))$ is an example for a singular Cohen–Macaulay curve. The scheme $\operatorname{Spec}(\mathbb{C}[x,y]/(x^2, xy))$ is not Cohen–Macaulay as it has an embedded point at the origin.

The following lemma summarizes the previous discussion.

Lemma 1.19 ([30, §1])**.** *Let (F, s) be a stable pair and let $C_F := \operatorname{Supp}(F) \subset X$ be the schematic support of F. Then F is isomorphic to the structure sheaf of C_F away from finitely many points. Moreover, $C_F = \operatorname{Supp}(\operatorname{im}(s))$ and C_F is a Cohen–Macaulay curve.*

Proof. The first claim follows immediately from condition (ii) of Lemma 1.11. The second claim can be proved locally. Recall that $\mathrm{Supp}(F)$ and $\mathrm{Supp}(\mathrm{im}(s))$ are defined by $V(\mathrm{Ann}(F))$ and $V(\mathrm{Ann}(\mathrm{im}(s)))$, respectively. We have the obvious inclusion $\mathrm{Ann}(F) \subseteq \mathrm{Ann}(\mathrm{im}(s))$. To prove the opposite inclusion assume that there exists some $a \in \mathrm{Ann}(\mathrm{im}(s)) \setminus \mathrm{Ann}(F)$ and let $f \in F$ be a section such that $af \neq 0 \in F$. Then the submodule of F generated by af has support of dimension zero as it sits inside the zero-dimensional cokernel of s. This is a contradiction to F being pure.

For the last statement notice that as a quotient of $\mathcal{O}_X$, $\mathrm{im}(s)$ is a structure sheaf, so by the part above $\mathrm{im}(s) \cong \mathcal{O}_{C_F}$. Since $\mathrm{im}(s)$ is a subsheaf of F, it is pure. Assume there exists some point $x \in C_F$ such that $\mathcal{O}_{C_F,x}$ has depth zero. Let $\mathfrak{m}$ be its maximal ideal; then there exists an element $a \in \mathcal{O}_{C_F}$ with $a\mathfrak{m} = 0$, so $\mathfrak{m}$ is an embedded prime in $\mathrm{Ass}(\mathcal{O}_{C_F,x})$. Therefore, x is an embedded point of C_F, contradicting the purity of $\mathcal{O}_{C_F}$. It follows that C_F is a Cohen–Macaulay curve. □

Therefore, we can think of a stable pair $s\colon \mathcal{O}_X \to F$ as an embedding of a Cohen–Macaulay curve (the support of F) into X with some additional points on the curve corresponding to the cokernel of s. The invariants we will define in the end then count such embeddings. The parameter $\beta \in H_2(X, \mathbb{Z})$ is the class of the curve inside X.

What happens if we loose the first stability condition, i.e. do not assume that the sheaf F is pure?

Example 1.20. Let C' be a smooth curve with an embedding $\iota\colon C' \hookrightarrow \mathbb{P}^3$ and let $p \in C'$ be some point on it. We define a (non-Cohen–Macaulay) curve C by fattening the point p. As an example, we could take C to be the projectivization of $\mathrm{Spec}(\mathbb{C}[x,y]/(x)\cdot(x,y)^3)$, i.e. a line with a fat point at the origin. Let D be the effective divisor corresponding to the fat point. We can find some nontrivial automorphism ϕ on the fat point (see Figure 1.2); in the example above we could take

$$\begin{array}{rcl} \phi\colon \mathbb{C}[x,y]/(x)\cdot(x,y)^3 & \to & \mathbb{C}[x,y]/(x)\cdot(x,y)^3 \\ x & \mapsto & x \\ y & \mapsto & y+y^2. \end{array}$$

This induces an automorphism ϕ on $\iota_*\mathcal{O}_C(D)$ making the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{s_D} & \iota_*\mathcal{O}_C(D) \\ \Vert & & \downarrow{\scriptstyle\phi} \\ \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{s_D} & \iota_*\mathcal{O}_C(D) \end{array}$$

commute, since the support of the cokernel of s_D is D, so s_D cannot "detect" the automorphism. Hence, we have constructed a nontrivial automorphism of the pair and according to the argument presented earlier, the moduli space of such pairs would not be a scheme.

For the further discussion it will be useful to denote the kernel of a stable pair $s\colon \mathcal{O}_X \to F$ by $\mathscr{I}_{C_F}$ and its cokernel by Q so that there is the following exact sequence

$$0 \to \mathscr{I}_{C_F} \to \mathcal{O}_X \xrightarrow{s} F \to Q \to 0. \tag{4}$$

There is the following characterization of stable pairs which we will not prove.

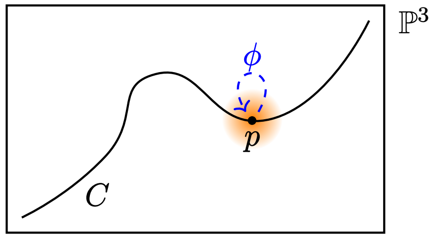

FIGURE 1.2. A curve C with a fattened point p admitting a nontrivial automorphism ϕ.

Proposition 1.21 ([30, Prop. 1.8]). *Let $C \subset X$ be a Cohen–Macaulay curve and $\mathfrak{m} \subset \mathcal{O}_C$ the ideal of a finite union of closed points. A stable pair (F, s) with support C satisfying* $\operatorname{Supp}^{red}(Q) \subset \operatorname{Supp}(\mathcal{O}_C/\mathfrak{m})$ *is equivalent to a subsheaf of* $\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C,\ r \gg 0$.

At the end of this section we look at an example for the moduli space $P_n(X, \beta)$ which can be found in [30, §4.1].

Example 1.22. Let X be the scheme given by the total space of the vector bundle

$$p\colon \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to \mathbb{P}^1,$$

i.e. $X = \mathbf{Spec}(\mathrm{Sym}((\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})^\vee))$. This is a scheme of relative dimension two over $\mathbb{P}^1$, hence a three-dimensional scheme over $\mathbb{C}$. The zero section

$$z\colon \mathbb{P}^1 \to X,$$

corresponding to the zero element inside $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$, is a closed immersion, i.e. its image is a class $[\mathbb{P}^1]$ inside X. In fact, the image of the zero section is the only curve in X with class $[\mathbb{P}^1]$. To see this, embed X in the projective bundle $\overline{X} = \mathbf{Proj}(\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1})^\vee)$. The image of z corresponds to $\mathbf{Proj}(\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}))$ inside $\overline{X}$. By the universal property of the projective bundle, every projectivization of a line bundle inside $\overline{X}$ comes from a section and vice versa, see [8, Prop. 9.14]. The only section of $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ is the zero section and the claim follows.

We can now investigate what the moduli space $P_n(X, [\mathbb{P}^1])$ is. Recall that any coherent sheaf on $\mathbb{P}^1$ is a finite direct sum of line bundles and zero-dimensional sheaves. Let (F, s) be a stable pair inside $P_n(X, [\mathbb{P}^1])$. Then F is a pure sheaf with support $[\mathbb{P}^1]$, hence a direct sum of line bundles. The Hilbert polynomial of F is given by

$$P_F(k) = k \int_{[\mathbb{P}^1]} c_1(\mathcal{O}_X(1)) + n = k + n.$$

This implies that the rank of F is one as it is the leading coefficient in the Hilbert polynomial. Hence, F is a line bundle with nonnegative degree and $h^0(\mathbb{P}^1, F) = n$, so we conclude F must be $\mathcal{O}_{\mathbb{P}^1}(n-1)$. Therefore, $P_n(X, [\mathbb{P}^1])$ parametrizes nonzero sections of $\mathcal{O}_{\mathbb{P}^1}(n-1)$ modulo scalar multiples, implying that

$$P_n(X, [\mathbb{P}^1]) \cong \mathrm{Sym}^{n-1}(\mathbb{P}^1) \cong \mathbb{P}^{n-1}.$$

Note that the space X is *local Calabi–Yau*, i.e. its canonical bundle is trivial but X is not proper. Using the tangent exact sequence and noticing that the tangent sheaf of X over $\mathbb{P}^1$ is just $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ we obtain the exact sequence

$$0 \to p^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \to T_X \to p^*(T_{\mathbb{P}^1}) \to 0.$$

Taking determinants yields

$$K_X \cong (\det(T_X))^\vee \cong (\mathcal{O}_X(-2) \otimes \mathcal{O}_X(2))^\vee \cong \mathcal{O}_X.$$

2. Stable pairs in the derived category

Unfortunately, the natural obstruction theory of the moduli space of stable pairs $P_n(X,\beta)$ does not admit a virtual fundamental class so there is no possibility to obtain a numerical invariant (how a (perfect) obstruction theory gives rise to a virtual class is discussed in Section 4). To circumvent this problem one considers a stable pair as a complex in the derived category.

Let $D^b(X)$ be the bounded derived category of coherent sheaves on X. To a stable pair (F, s) we associate the two-term complex

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X),$$

where $\mathcal{O}_X$ sits in degree zero. The purpose of this section is to prove the following statement.

Proposition 2.1 ([30, Prop. 1.21]). *The stable pairs (F, s) and (F', s') are isomorphic if and only if their associated complexes $\{\mathcal{O}_X \xrightarrow{s} F\}$ and $\{\mathcal{O}_X \xrightarrow{s'} F'\}$ are isomorphic in $D^b(X)$.*

Lemma 2.2. *The following sequences are exact triangles in $D^b(X)$:*

$$F[-1] \to I^\bullet \to \mathcal{O}_X \xrightarrow{s} F \tag{5}$$

$$\mathscr{I}_{C_F} \to I^\bullet \to Q[-1] \to \mathscr{I}_{C_F}[1]. \tag{6}$$

Proof. For the first part we show that (5) is a rotated version of the exact triangle

$$\mathcal{O}_X \xrightarrow{s} F \to C(s) \to \mathcal{O}_X[1]$$

corresponding to the short exact sequence of complexes

$$0 \to F \to C(s) \to \mathcal{O}_X[1] \to 0,$$

where $C(s)$ is the mapping cone of s. Indeed,

$$C(s) = \mathcal{O}_X[1] \oplus F \simeq \{\mathcal{O}_X \xrightarrow{s} F\}$$

with $\mathcal{O}_X$ in degree -1, which is just $I^\bullet[1]$.

For the second part, notice that the exact sequence (4) corresponds to a short exact sequence of complexes

$$0 \to \mathscr{I}_{C_F} \to I^\bullet \to Q[-1] \to 0$$

which gives the desired exact triangle. □

Let us recall the local-to-global spectral sequence.

Lemma 2.3 (Local-to-global spectral sequence, [16, (3.16)]). *Let $(Y, \mathcal{O}_Y)$ be a ringed space and let M and N be two $\mathcal{O}_Y$-modules. There is the following spectral sequence:*

$$E_2^{p,q} = H^p(Y, \mathscr{E}xt^q_{\mathcal{O}_Y}(M, N)) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}_Y}(M, N).$$

Lemma 2.4. *Let F be a coherent sheaf on X and $i < \dim(X) - \dim(\operatorname{Supp}(F))$ be an integer. Then $\mathscr{E}xt^i(F, \mathcal{O}_X) = 0$.*

Proof. By Serre's vanishing theorem [15, Thm. III.5.2] there exists an integer n such that for all $j > 0$, $H^j(\mathscr{E}xt^i(F, \mathcal{O}_X) \otimes L^n) = 0$ and $\mathscr{E}xt^i(F, \mathcal{O}_X) \otimes L^n$ is globally generated. Applying the local-to-global spectral sequence we obtain

$$H^0(\mathscr{E}xt^i(F, \mathcal{O}_X) \otimes L^n) \cong H^0(\mathscr{E}xt^i(F, L^n)) \cong \operatorname{Ext}^i(F, L^n) \tag{7}$$

as the E_2-page degenerates. By Serre duality,

$$\operatorname{Ext}^i(F, L^n) \cong \operatorname{Ext}^{\dim(X)-i}(L^n, F \otimes K_X)^\vee \cong H^{\dim(X)-i}(F \otimes K_X \otimes L^{-n})^\vee.$$

The term on the right-hand side vanishes for $i < \dim(X) - \dim(\operatorname{Supp}(F))$ by Grothendieck's vanishing theorem. As $\mathscr{E}xt^i(F, \mathcal{O}_X) \otimes L^n$ is globally generated, the statement then follows from (7). □

Lemma 2.5 ([30, Lem. 1.15])**.** *$\mathscr{E}xt^{\leq -1}(I^\bullet, I^\bullet) = 0$ and $\mathscr{H}om(I^\bullet, I^\bullet) \cong \mathcal{O}_X$.*

Proof. Applying $\mathscr{H}om(-, \mathcal{O}_X)$ to the exact triangle (5) yields

$$\mathscr{H}om(F, \mathcal{O}_X) \to \mathscr{H}om(\mathcal{O}_X, \mathcal{O}_X) \to \mathscr{H}om(I^\bullet, \mathcal{O}_X) \to \mathscr{H}om(F[-1], \mathcal{O}_X) \cong \mathscr{E}xt^1(F, \mathcal{O}_X).$$

As X has dimension three and F is pure of dimension one, the first and last term of this sequence vanish by Lemma 2.4. The second term is generated by the identity map which maps in the third term to the canonical map from (5) which is therefore a generator for

$$\mathscr{H}om(I^\bullet, \mathcal{O}_X) \cong \mathcal{O}_X. \tag{8}$$

We also get this canonical map as the image of the identity in the exact sequence

$$\mathscr{E}xt^{-1}(I^\bullet, F) \to \mathscr{H}om(I^\bullet, I^\bullet) \to \mathscr{H}om(I^\bullet, \mathcal{O}_X) \to 0 \tag{9}$$

which is again obtained from (5) by applying $\mathscr{H}om(I^\bullet, -)$. Applying $\mathscr{H}om(-, F)$ to the other exact triangle (6) yields

$$\mathscr{E}xt^{-2}(\mathscr{I}_{C_F}, F) \to \mathscr{H}om(Q, F) \to \mathscr{E}xt^{-1}(I^\bullet, F) \to \mathscr{E}xt^{-1}(\mathscr{I}_{C_F}, F).$$

The first and last term vanish for degree reasons. Moreover, $\mathscr{H}om(Q, F) = 0$ as F is pure and Q has only zero-dimensional support. Therefore, $\mathscr{E}xt^{-1}(I^\bullet, F) = 0$ and from (9) we get an isomorphism

$$\mathscr{H}om(I^\bullet, I^\bullet) \cong \mathscr{H}om(I^\bullet, \mathcal{O}_X) \overset{(8)}{\cong} \mathcal{O}_X.$$

By shifting all the sequences above to lower degrees and noticing that $\mathscr{E}xt^{\leq -1}(\mathcal{O}_X, \mathcal{O}_X) = 0 = \mathscr{E}xt^{\leq -1}(I^\bullet, \mathcal{O}_X)$, by the same argument as above we obtain the vanishing of $\mathscr{E}xt^{\leq -1}(I^\bullet, I^\bullet)$. □

Corollary 2.6 ([30, Eqn. 1.19])**.** *$\mathscr{E}xt^{\leq -1}(I^\bullet, I^\bullet \otimes K_X) = 0$ and $\mathscr{H}om(I^\bullet, I^\bullet \otimes K_X) \cong K_X$.*

Proof. This is obtained by tensoring the results of Lemma 2.5 with K_X, using that the functor $- \otimes K_X$ is exact as X is smooth, so K_X is a vector bundle. □

Corollary 2.7 ([30, Lem. 1.20])**.** *$\mathscr{E}xt^{\leq -1}(I^\bullet, \mathcal{O}_X) = 0$ and $\mathscr{H}om(I^\bullet, \mathcal{O}_X) \cong \mathcal{O}_X$.*

Proof. Both statements appeared before in the proof of Lemma 2.5. □

Now we are sufficiently prepared to give a

Proof of Proposition 2.1 [30, §1]. Let $I^\bullet \in D^b(X)$ be a complex quasi-isomorphic to the complex

$$I^\bullet \simeq \{\mathcal{O}_X \xrightarrow{s} F\}$$

associated to a stable pair (F, s). If we can show how to uniquely recover the pair (F, s) from $I^\bullet$ the proof is done.

By Corollary 2.7, the E_2-page of the local-to-global spectral sequence with

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(I^\bullet, \mathcal{O}_X))$$

has nonzero entries only in the first quadrant. Therefore, $E_2^{0,0} = E_\infty^{0,0}$ which lets us conclude

$$\operatorname{Hom}(I^\bullet, \mathcal{O}_X) \cong H^0(X, \mathcal{H}om(I^\bullet, \mathcal{O}_X)).$$

Using the second part of Corollary 2.7 and the fact that X is projective over $\mathbb{C}$ we obtain

$$\operatorname{Hom}(I^\bullet, \mathcal{O}_X) \cong \mathbb{C}.$$

Hence, there is a unique (up to scalars) nonzero element z inside $\operatorname{Hom}(I^\bullet, \mathcal{O}_X)$ which gives rise to an exact triangle

$$I^\bullet \xrightarrow{z} \mathcal{O}_X \to C(z) \to I^\bullet[1],$$

where $C(z)$ denotes the mapping cone. If

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\}$$

then we know from the exact triangle (5) that this mapping cone is F. Now for $I^\bullet$ quasi-isomorphic to $\{\mathcal{O}_X \xrightarrow{s} F\}$ we get a commutative diagram

$$\begin{array}{ccccccc} I^\bullet & \xrightarrow{z} & \mathcal{O}_X & \longrightarrow & C(z) & \longrightarrow & I^\bullet[1] \\ \downarrow{\scriptstyle\sim} & & \| & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \{\mathcal{O}_X \xrightarrow{s} F\} & \longrightarrow & \mathcal{O}_X & \xrightarrow{s} & F & \longrightarrow & \{\mathcal{O}_X \xrightarrow{s} F\}[1], \end{array} \tag{10}$$

implying that the mapping cones $C(z)$ and F are quasi-isomorphic. There is the canonical map

$$\phi \colon \mathcal{O}_X \to C(z)$$

which induces a map in cohomology

$$h^0(\phi) \colon \mathcal{O}_X \to F.$$

Comparing with (10), the map $h^0(\phi)$ is just the section s and we have recovered the stable pair (F, s) from the complex $I^\bullet$. Hence, no information is lost by considering the pair as a complex in the derived category. □

In the next section we are going to show that deformations of stable pairs and of complexes in the derived category agree. For this to be true, we will need to restrict the complexes to those with trivial determinant.

Lemma 2.8. *Let F be a coherent sheaf supported on a subscheme of codimension at least two and let s be a section. Then the associated complex $I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\}$ has trivial determinant.*

Proof. As $\det(I^\bullet) = \det(\mathcal{O}_X) \otimes \det(F)^\vee$ it suffices to show that $\det(F) = \mathcal{O}_X$. Consider a locally free resolution

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to F \to 0$$

of F. Outside the support of F the alternating tensor product over the determinants $\det(F_i)$ becomes trivial. Therefore, $\det(F)$ is trivial away from a subscheme of codimension at least two, hence it is trivial. $\square$

3. Deformation theory

The next step towards the construction of stable pair invariants is to embed the moduli space $P_n(X, \beta)$ into the moduli space of perfect complexes in $D^b(X)$ with trivial determinant $\mathcal{O}_X$. In the last section we have already seen that there is an injective map at the level of sets. To make this an embedding of spaces we have to show that their local geometries agree. This is done by establishing an equivalence of their deformation theories. The calculation of the deformation theory of stable pairs was done by Le Potier [24]. As before, we will not explain this calculation but present the deformation theory in the simpler case of Quot-schemes instead, following [17, §2.2]. The rest of this section is dedicated to the analysis of the relation between deformations of stable pairs and deformations of the associated complexes. Ultimately, we will see that they are the same.

3.1. **Deformation theory of Quot-schemes.** Let Q denote the Quot-scheme $\mathrm{Quot}_{E/Y/S}$, where E is an S-flat coherent $\mathcal{O}_Y$-module, and let (Artin/k) be the category of local Artinian k-algebras with residue field k. Let $\sigma\colon A' \to A$ be a surjective morphism in (Artin/k) such that there exists the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{q} & Q \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathrm{Spec}(A') & \xrightarrow{\psi} & S \end{array}$$

The morphism q corresponds by definition of the Quot-scheme to a short exact sequence $0 \to K \to E_A \to F \to 0$ of coherent sheaves on $Y_A = \mathrm{Spec}(A) \times_S Y$, where $E_A = A \otimes_{\mathcal{O}_S} E$. An extension q' of q with respect to A' is a morphism $q'\colon \mathrm{Spec}(A') \to Q$ making this diagram commute:

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{q} & Q \\ {\scriptstyle\sigma}\downarrow & \overset{q'}{\nearrow} & \downarrow{\scriptstyle\pi} \\ \mathrm{Spec}(A') & \xrightarrow{\psi} & S \end{array}$$

In other words, q' corresponds to a short exact sequence $0 \to K' \to E_{A'} \to F' \to 0$ and when restricted to Y_A, $A \otimes_{A'} F'$ and F agree. Let I be the kernel of σ and $\mathfrak{m}_{A'}$ the unique maximal ideal of A'. There exists some integer n such that $\mathfrak{m}_{A'}^n I = 0$. One restricts to the case $n = 1$ as otherwise one can break up the extension problem into n smaller ones each satisfying $\mathfrak{m}_{A'} I = 0$. Let us denote $F_0 = A/\mathfrak{m}_A \otimes_A F$ and likewise $K_0 = A/\mathfrak{m}_A \otimes_A K$. Using that $I \otimes_{A'} F' \cong I \otimes_k F_0$ and $I \otimes_{A'} K' \cong I \otimes_k K_0$ one obtains the following diagram, where both rows and columns are exact by flatness of F' and $E_{A'}$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I \otimes_k K_0 & \xrightarrow{1\otimes i_0} & I \otimes_k E & \xrightarrow{1\otimes q_0} & I \otimes_k F_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow j & & \downarrow & & \\
0 & \longrightarrow & K' & \xrightarrow{i'} & E_{A'} & \xrightarrow{q'} & F' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \sigma & & \downarrow & & \\
0 & \longrightarrow & K & \xrightarrow{i} & E_A & \xrightarrow{q} & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By lifting the morphism i along σ we get a morphism from K to $E_{A'}$, well-defined up to an element in the kernel $(j \circ (1 \otimes i_0))(I \otimes_k K_0)$, i.e. we get an induced morphism

$$\hat{i}\colon K \to E_{A'}/(j \circ (1 \otimes i_0))(I \otimes_k K_0).$$

Notice that F' is the cokernel of $\hat{i}$. Vice versa, any $\mathcal{O}_{Y_{A'}}$-morphism $\hat{i}$ with the property $\sigma \circ \hat{i} = i$ gives rise to an extension F' of F. In fact, any such F' is flat, see [17, Lem. 2.1.3]. Therefore, the existence of an extension F' of F and the existence of a morphism $\hat{i}$ with $\sigma \circ \hat{i} = i$ are equivalent. Consider the complex

$$0 \longrightarrow I \otimes_k K_0 \xrightarrow{j\circ(1\otimes i_0)} E_{A'} \xrightarrow{q\circ\sigma} F \longrightarrow 0$$

and let $B = \ker(q \circ \sigma)/\operatorname{im}(j \circ (1 \otimes i_0))$ be the middle homology of this complex. As $IB = 0$, B can be considered as an $\mathcal{O}_{Y_A}$-module. Then the existence of $\hat{i}$ is equivalent to the splitting of the exact sequence

$$0 \to I \otimes_k F_0 \to B \to K \to 0.$$

Therefore, the extension class

$$\mathfrak{o}(\sigma, q, \psi) \in \operatorname{Ext}^1_{Y_A}(K, I \otimes_k F_0)$$

is trivial if and only if an extension q' of q exists. Since K is A-flat and $I \otimes_k F_0$ is a vector space annihilated by $\mathfrak{m}_A$, so $I \otimes_k F_0$ can be considered as an $A/\mathfrak{m}_A$-module, there is a natural isomorphism

$$\operatorname{Ext}^1_{Y_A}(K, I \otimes_k F_0) = \operatorname{Ext}^1_{Y_s}(K_0, F_0) \otimes_k I.$$

We call $\mathfrak{o}(\sigma, q, \psi)$ the *obstruction* to extend q to q' and $\operatorname{Ext}^1_{Y_S}(K_0, F_0)$ the *obstruction space.* Note that if one splitting $\hat{i}$ exists, all other splittings differ by a morphism $K \to I \otimes_k F_0$. Therefore, our findings can be summarized as follows.

Lemma 3.1 ([17, Lem. 2.2.6])**.** *An extension q' of q exists if and only if $\mathfrak{o}(\sigma, q, \psi)$ vanishes. If this is the case, the possible extensions form a torsor under* $\operatorname{Hom}_{Y_s}(K_0, F_0) \otimes_k I$.

3.2. **Deformations of stable pairs and complexes in $D^b(X)$.** Returning to stable pairs, the results of [24] show that the tangent space of $P_n(X, \beta)$ is given by $\operatorname{Hom}(I^\bullet, F)$ and the obstruction space is $\operatorname{Ext}^1(I^\bullet, F)$.

Definition 3.2 ([17, p. 113])**.** Let E be a locally free sheaf on a scheme Y, then the trace map $\operatorname{tr}\colon \mathscr{E}nd(E) \to \mathcal{O}_Y$ induces maps $\operatorname{tr}^i\colon \operatorname{Ext}^i(E,E) = H^i(\mathscr{E}nd(E)) \to H^i(\mathcal{O}_Y)$. We define $\operatorname{Ext}^i(E,E)_0$ to be the kernel of tr^i and call it *trace-free Ext.*

Remark 3.3 ([17, §10.1.2]). The construction of a natural trace map is also possible for coherent sheaves E that are not necessarily locally free: let $E^\bullet$ be a finite complex of locally free sheaves. We define a map

$$\mathrm{tr}_{E^\bullet}\colon \mathscr{H}om^\bullet(E^\bullet,E^\bullet)\to \mathcal{O}_Y;\ \ \mathrm{tr}_{E^\bullet}|_{\mathscr{H}om(E^i,E^j)} = \begin{cases} (-1)^i\mathrm{tr}_{E^i} & \text{for } i=j;\\ 0 & \text{else.}\end{cases}$$

Here, the complex $\mathscr{H}om^\bullet(E^\bullet,E^\bullet)$ is given by

$$\mathscr{H}om^n(E^\bullet,E^\bullet)=\bigoplus_i \mathscr{H}om(E^i,E^{i+n}),$$

so the trace is only nonzero on $\mathscr{H}om^0(E^\bullet,E^\bullet)$. One can check that this map is really a chain morphism and thus induces morphisms

$$\mathrm{tr}^i\colon \mathrm{Ext}^i(E^\bullet,E^\bullet)\to H^i(\mathcal{O}_Y).$$

Now for a coherent sheaf E admitting a finite locally free resolution $E^\bullet\to E$, $\mathrm{Ext}^i(E^\bullet,E^\bullet)\cong \mathrm{Ext}^i(E,E)$ and we can define the trace-free $\mathrm{Ext}^i(E,E)_0$ as the kernel of $\mathrm{tr}^i\colon \mathrm{Ext}^i(E^\bullet,E^\bullet)\to H^i(\mathcal{O}_Y)$. For a perfect complex $I^\bullet\in D^b(X)$ the construction is done in the same way.

There is another map

$$i_{E^\bullet}\colon \mathcal{O}_Y\to \mathscr{H}om^0(E^\bullet,E^\bullet),$$

defined by sending

$$1\mapsto \sum_i \mathrm{id}_{E^i}.$$

The identity

$$\mathrm{tr}_{E^\bullet}(i_{E^\bullet}(1))=\sum_i(-1)^i\mathrm{rk}(E^i)=:\mathrm{rk}(E^\bullet)$$

is satisfied, yielding

$$\mathrm{tr}_{E^\bullet}\circ i_{E^\bullet}=\mathrm{rk}(E^\bullet)\cdot \mathrm{id}_{E^\bullet}.$$

In fact, the map $i_{E^\bullet}$ can also be shown to be a chain map and the identity above also holds if cohomology is taken.

The deformation theory of complexes $I^\bullet\in D^b(X)$ with fixed determinant $\mathcal{O}_X$ is governed by $\mathrm{Ext}^1(I^\bullet,I^\bullet)_0$ and $\mathrm{Ext}^2(I^\bullet,I^\bullet)_0$ as is shown in [20] and [25]. By applying $\mathrm{Hom}(I^\bullet,-)$ to the map $F[-1]\to I^\bullet$ of (5) we obtain a map

$$\mathrm{Ext}^i(I^\bullet,F)\to \mathrm{Ext}^{i+1}(I^\bullet,I^\bullet).$$

Claim 3.4. *The image of the above map is contained in the trace-free part.*

Proof. Looking again at the complex obtained by applying $\mathrm{Hom}(I^\bullet,-)$ to (5),

$$\cdots\to \mathrm{Ext}^i(I^\bullet,F)\to \mathrm{Ext}^{i+1}(I^\bullet,I^\bullet)\to \mathrm{Ext}^{i+1}(I^\bullet,\mathcal{O}_X)\to\cdots,$$

we see that $\mathrm{Ext}^i(I^\bullet,F)$ maps as zero into $\mathrm{Ext}^{i+1}(I^\bullet,\mathcal{O}_X)$. We will show that the trace map factors over $\mathrm{Ext}^i(I^\bullet,\mathcal{O}_X)$. To do so, it suffices to show this for the sheaf version

$$\cdots\to \mathscr{E}xt^i(I^\bullet,F)\to \mathscr{E}xt^{i+1}(I^\bullet,I^\bullet)\to \mathscr{E}xt^{i+1}(I^\bullet,\mathcal{O}_X)\to\cdots.$$

Let $A^\bullet$ be a finite complex of locally free sheaves quasi-isomorphic to $I^\bullet$. As $\mathrm{rk}(A^\bullet)=\mathrm{rk}(I^\bullet)=1$ we have

$$\mathrm{tr}_{A^\bullet}\circ i_{A^\bullet}=\mathrm{id}_{A^\bullet}=i_{A^\bullet}\circ \mathrm{tr}_{A^\bullet};$$

this induces a splitting

$$R\mathscr{H}om(A^\bullet,A^\bullet)\cong (A^\bullet)^\vee\otimes A^\bullet\cong \mathcal{O}_X\oplus((A^\bullet)^\vee\otimes A^\bullet)_0, \tag{11}$$

where the first summand is the image of $i_{A^\bullet}$ and the second summand is the kernel of the trace map. Note that the map $\mathscr{E}xt^{i+1}(I^\bullet, I^\bullet) \to \mathscr{E}xt^{i+1}(I^\bullet, \mathcal{O}_X)$ is induced by the map sending the identity in $\mathscr{H}om(I^\bullet, I^\bullet)$ to the canonical map $I^\bullet \to \mathcal{O}_X$. Therefore, we obtain the following commutative diagram

$$\begin{array}{ccccc} \mathscr{E}xt^i(I^\bullet,F) & \longrightarrow & \mathscr{E}xt^{i+1}(I^\bullet,I^\bullet) & \longrightarrow & \mathscr{E}xt^{i+1}(I^\bullet,\mathcal{O}_X) \\ & & \cong\downarrow & & \downarrow\cong \\ & & h^{i+1}(\mathcal{O}_X\oplus((A^\bullet)^\vee\otimes A^\bullet)_0)) & \longrightarrow & h^{i+1}(R\mathscr{H}om(I^\bullet,\mathcal{O}_X)) \\ & & & \overset{\mathrm{tr}}{\searrow} & \downarrow \\ & & & & \mathcal{O}_X \end{array}$$

and the desired statement follows. □

Therefore, a deformation of the stable pair (F, s) induces a deformation of the associated complex $I^\bullet$ with trivial determinant. Our goal is to show that these deformations are equal, so that we get in fact an isomorphism

$$\mathrm{Ext}^0(I^\bullet, F) \to \mathrm{Ext}^1(I^\bullet, I^\bullet)_0.$$

To do so, we first have to clarify what we mean by families of stable pairs and complexes.

Definition 3.5 ([30, p. 12])**.** Let B be a quasi-projective scheme. A *family of stable pairs over B* is a pair $s\colon \mathcal{O}_{X\times B} \to F$ on $X \times B$ such that F is flat over B and for all closed points $b \in B$, the restriction (F_b, s_b) to the fiber $X \times \{b\}$ is a stable pair.

A *family of complexes over B* is a perfect complex $I^\bullet$ on $X \times B$. By definition, a perfect complex is a complex locally being quasi-isomorphic to a finite complex of locally free sheaves. As $X \times B$ is quasi-projective, every perfect complex is in fact *strictly perfect* meaning that there is a quasi-isomorphism to a finite complex of locally free sheaves globally [32, Tag 0F8E].

Definition 3.6. Let B be a quasi-projective scheme and let J be an ideal sheaf of $\mathcal{O}_B$ satisfying $J^{N+1} = 0$. Let B_0 be the closed subscheme of B defined by J. Then the closed immersion $B_0 \hookrightarrow B$ is called an *N-th order thickening* of B_0.

Let $B_0 \hookrightarrow B$ be some N-th order thickening. A family of stable pairs (F, s) over B is a *deformation* of a family (F_0, s_0) over B_0 if the restriction of (F, s) to B_0 is isomorphic to (F_0, s_0). Likewise, a family of complexes $I^\bullet$ over B is a deformation of a family $I_0^\bullet$ over B_0 if the derived restriction of $I^\bullet$ to $X \times B_0$ is quasi-isomorphic to $I_0^\bullet$. For the rest of this section let

$$I_0^\bullet = \{\mathcal{O}_{X\times B_0} \xrightarrow{s_0} F_0\}$$

denote the family of complexes associated to a stable pair (F_0, s_0). The complex $I_0^\bullet$ is perfect.

Definition 3.7 ([7, p. 474])**.** Let R be a ring and let M be an R-module. The *projective dimension* $\mathrm{pd}_R(M)$ is the infimum of the lengths of projective resolutions of M.

In the following we will need the well-known Auslander–Buchsbaum formula.

Theorem 3.8 (Auslander–Buchsbaum formula, [7, Thm. 19.9])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. If M is a finitely generated R-module of finite projective dimension, then*

$$\mathrm{pd}_R(M) = \mathrm{depth}(R) - \mathrm{depth}(M).$$

In the forthcoming, a more general and derived version of Serre duality, called Grothendieck–Verdier duality, will be used.

Theorem 3.9 (Grothendieck–Verdier duality, [16, Thm. 3.34])**.** *Let $f\colon Y \to Z$ be a morphism of smooth schemes over a field k and let $F^\bullet \in D^b(Y)$. There exists an isomorphism*

$$Rf_* \circ \mathbb{D}_Y \cong \mathbb{D}_Z \circ Rf_*,$$

where $\mathbb{D}_Y$ is the dualizing functor

$$\mathbb{D}_Y\colon F^\bullet \mapsto R\mathscr{H}om(F^\bullet, K_Y[\dim Y]).$$

A couple of lemmas are necessary to get us prepared for the proof of the main theorem of this section starting with the following result.

Lemma 3.10 ([30, Lem. 2.5])**.** *Let $\iota\colon B_0 \hookrightarrow B$ be an n-th order thickening, and let F be a coherent sheaf on an open scheme $U \subset X \times B$. Let $F_0 = \iota^* F$ be the restriction of F to $U_0 = \iota^*(U)$. Then, F is flat over B if and only if*

(i) *$L\iota^* F \cong F_0$ and*
(ii) *F_0 is flat over B_0.*

Proof. For the implication that flatness of F implies (i) and (ii), note that the left derived pullback functor can be computed by taking the pullback on a flat resolution. If F is flat, it clearly follows that $L\iota^* F \cong \iota^* F = F_0$. Moreover, F_0 is flat over B_0 by base change.

For the converse we prove that

$$F \overset{L}{\otimes} M \cong F \otimes M \tag{12}$$

for any $\mathcal{O}_B$-module M to show that F is flat. Any quasi-coherent $\mathcal{O}_B$-module can be written as a direct limit of coherent $\mathcal{O}_B$-modules [14, Cor. 10.50]. As tensor products commute with direct limits, we can assume that M is coherent. Let $I = \ker(\iota^\sharp\colon \mathcal{O}_B \to \iota_* \mathcal{O}_{B_0})$, then there is a filtration

$$0 = M_{n+1} \subset M_n \subset \cdots \subset M_1 \subset M_0 = M,$$

where $M_a = I^a M$. Each quotient M_a / M_{a+1} is annihilated by I and can therefore be written as

$$M_a / M_{a+1} = \iota_* N_a$$

for some $\mathcal{O}_{B_0}$-module N_a. Suppose we have already shown that (12) holds for all modules of the form $\iota_* N_a$. Consider the case $n = 1$, then we have an exact sequence

$$0 \to \iota_* N_1 \to M \to \iota_* N_0 \to 0.$$

Tensoring with F gives a long exact sequence

$$\cdots \to \operatorname{Tor}_i(F, \iota_* N_1) \to \operatorname{Tor}_i(F, M) \to \operatorname{Tor}_i(F, \iota_* N_0) \to \cdots.$$

The first and last term vanish by assumption and the claim follows. Inductively, this holds for all $n \in \mathbb{N}$. We are left to show the statement for $\iota_* N_a$ where we have

$$F \overset{L}{\otimes} \iota_* N_a \cong \iota_*(L\iota^* F \overset{L}{\otimes} N_a) \overset{(i)}{\cong} \iota_*(F_0 \overset{L}{\otimes} N_a) \overset{(ii)}{\cong} \iota_*(F_0 \otimes N_a) \cong F \otimes \iota_* N_a.$$

Here, the first and last isomorphisms come from the projection formula [16, (3.11)]. □

The following lemma plays a crucial role for the proof of the main theorem.

Lemma 3.11 ([30, Lem. 2.1])**.** *Let $B_0 \hookrightarrow B$ be an N-th order thickening of quasi-projective schemes and let $I_0^\bullet$ be the perfect complex associated to a stable pair (F_0, s_0) on $X \times B_0$. Every deformation $I^\bullet$ over B of $I_0^\bullet$ is quasi-isomorphic to a two-term complex of sheaves $\{A \to E^1\}$ satisfying*

(i) *E^1 is locally free,*
(ii) *A is a pure sheaf of local projective dimension at most one,*
(iii) *$h^1(I^\bullet)$ has support of relative dimension zero over B,*
(iv) *$h^0(I^\bullet)$ is flat over B away from the support of $h^1(I^\bullet)$.*

Proof. The first step of the proof consists of showing that $I_0^\bullet$ is quasi-isomorphic to a three-term complex of locally free sheaves. The sheaf F_0 has relative dimension one on

$$\pi_0 \colon X \times B_0 \to B_0.$$

As its support on every fiber is a Cohen–Macaulay curve, F_0 has depth one on the fibers of π_0. Then by the Auslander–Buchsbaum formula, F_0 has local projective dimension equal to $\dim(X) - \operatorname{depth}(F_0) = 2$ on the fibers of π_0. Let $K^\bullet \to F_0 \to 0$ be a locally free resolution of F_0. By cutting this off after the term in degree -1 we obtain the exact sequence

$$0 \to \ker \to K^{-1} \to K^0 \to F_0 \to 0.$$

Breaking this sequence down into two short exact sequences

$$0 \to M \to K^0 \to F_0 \to 0$$

and

$$0 \to \ker \to K^{-1} \to M \to 0$$

and noting that as F_0 is flat, M also is and therefore ker is flat. Hence, restriction to the fiber $\pi_0^{-1}(b_0)$ preserves exactness. As on fibers, F_0 has projective dimension two we conclude that ker is locally free on fibers. But flat and locally free on fibers implies locally free (cf. [32, Tag 05P1]) and we have found a locally free resolution of F_0 on $X \times B_0$ of length two.

On a quasi-projective scheme Y any perfect complex $C^\bullet \in D^b(Y)$ with Tor-amplitude in $[a, b]$ is quasi-isomorphic to a complex consisting of locally free sheaves concentrated in degree $[a, b]$, see the construction in [32, Tag 0F8E]. We show that $I_0^\bullet$ has Tor-amplitude in $[-1, 1]$. Consider the exact triangle

$$\mathcal{O}_{X \times B_0} \to F_0 \to I_0^\bullet[1] \to \mathcal{O}_{X \times B_0}[1]$$

which is a rotated version of (5). Applying $\operatorname{Tor}_i(-, G)$ for some coherent sheaf G on $X \times B_0$ gives a long exact sequence

$$\begin{array}{lllllll}
\cdots \to & \color{blue}{\operatorname{Tor}_3(\mathcal{O}_{X\times B_0}, G)} & \to & \color{orange}{\operatorname{Tor}_3(F_0, G)} & \to & \color{red}{\operatorname{Tor}_2(I_0^\bullet, G)} & \to \\
\to & \color{blue}{\operatorname{Tor}_2(\mathcal{O}_{X\times B_0}, G)} & \to & \operatorname{Tor}_2(F_0, G) & \to & \operatorname{Tor}_1(I_0^\bullet, G) & \to \\
\to & \color{blue}{\operatorname{Tor}_1(\mathcal{O}_{X\times B_0}, G)} & \to & \operatorname{Tor}_1(F_0, G) & \to & \operatorname{Tor}_0(I_0^\bullet, G) & \to \\
\to & \operatorname{Tor}_0(\mathcal{O}_{X\times B_0}, G) & \to & \operatorname{Tor}_0(F_0, G) & \to & \operatorname{Tor}_{-1}(I_0^\bullet, G) & \to \\
\to & \color{blue}{\operatorname{Tor}_{-1}(\mathcal{O}_{X\times B_0}, G)} & \to & \color{orange}{\operatorname{Tor}_{-1}(F_0, G)} & \to & \color{red}{\operatorname{Tor}_{-2}(I_0^\bullet, G)} & \to \cdots
\end{array}$$

(note that $\operatorname{Tor}_i(I_0^\bullet[1], G) = \operatorname{Tor}_{i-1}(I_0^\bullet, G)$). Since F_0 has a locally free resolution of length two, the Tor-amplitude of F_0 is in $[-2, 0]$[1]; hence, the orange terms vanish. As $\mathcal{O}_{X \times B_0}$ is

[1] Recall that the Tor-amplitude is defined via the non-vanishing of $h^i(F_0 \overset{L}{\otimes} G)$ whereas $\operatorname{Tor}_i(F_0, G) = h^{-i}(F_0 \overset{L}{\otimes} G)$, so there is a difference in the signs.

locally free, all blue terms vanish. Therefore, the red terms vanish and we deduce that $I_0^\bullet$ has Tor-amplitude in $[-1,1]$. Thus, $I_0^\bullet$ is quasi-isomorphic to a three-term complex of locally free sheaves concentrated in $[-1,1]$.

Let
$$E^\bullet = \{E^m \to E^{m+1} \to \cdots \to E^{n-1} \to E^n\}$$
be a finite complex of locally free sheaves on $X \times B$ quasi-isomorphic to $I^\bullet$ with E^i sitting in degree i. The aim is to show that $E^\bullet$ can be shrunken to a length three complex. Suppose $n > 1$; as the restriction of $E^\bullet$ to $X \times B_0$ is quasi-isomorphic to $I_0^\bullet$ and the latter complex has vanishing cohomology in degree greater than one, the map
$$E^{n-1}|_{X\times B_0} \to E^n|_{X\times B_0}$$
is surjective. Then $E^{n-1} \to E^n$ is surjective on a neighborhood U of $X \times B_0$ inside $X \times B$ by Nakayama's Lemma. This neighborhood U must be equal to $X \times B$: we can reduce this to the affine case, so assume $X = \operatorname{Spec}(A), B = \operatorname{Spec}(R)$ and $B_0 = \operatorname{Spec}(R/I)$ with $I^{N+1} = 0$. Then $X \times B_0$ is given by $\operatorname{Spec}(A \otimes R/I(A \otimes R))$. Let $J = I(A \otimes R)$ denote this ideal, then $J^{N+1} = 0$. If an open U contains $V(J) = V(J^{N+1}) = V(0)$ it also contains $X \times B$. Therefore, the map $E^{n-1} \to E^n$ is surjective on all of $X \times B$. The kernel of a surjective map between locally free sheaves is again locally free, so we can replace $E^\bullet$ by the quasi-isomorphic complex
$$\cdots \to E^{n-2} \to \ker(E^{n-1} \to E^n) \to 0.$$
Repeating this argument we can assume that E^1 is the rightmost nonzero term of $E^\bullet$.

Now suppose $m < -1$. The cohomology of $I_0^\bullet$ over a point $x \times b_0$ can be computed by base changing the complex $E^\bullet|_{X\times B_0}$ to this point. As $I_0^\bullet$ is quasi-isomorphic to a complex of locally free sheaves with amplitude in $[-1,1]$, the cohomology of $I_0^\bullet$ on this point vanishes in degrees not contained in $[-1,1]$ and we obtain that
$$E^m|_{X\times B_0} \to E^{m+1}|_{X\times B_0}$$
is injective on all points. Again, by Nakayama's Lemma the injectivity can be extended to an open neighborhood of $X \times B_0$ (using that the E^i are locally free), so $E^m \to E^{m+1}$ is injective on every point. Let G denote the cokernel of the map $E^m \to E^{m+1}$, i.e.
$$0 \to E^m \to E^{m+1} \to G \to 0$$
is exact. Restricting to a point $x \times b$ gives a long exact sequence
$$\cdots \to \operatorname{Tor}_1^{X\times B}(E^{m+1}, \kappa(x\times b)) \to \operatorname{Tor}_1^{X\times B}(G, \kappa(x\times b)) \to E^m|_{x\times b} \to E^{m+1}|_{x\times b} \to \cdots.$$
The leftmost term vanishes as E^{m+1} is locally free and because of the injectivity of $E^m|_{x\times b} \to E^{m+1}|_{x\times b}$ we obtain that
$$\operatorname{Tor}_1^{X\times B}(G, \kappa(x\times b)) = 0$$
for every point. By the local criterion for flatness this implies that G is flat over $X \times B$ and hence locally free. Then we can replace $E^\bullet$ by the complex
$$0 \to G \to E^{m+2} \to \cdots$$
and repeating this procedure we see that $I^\bullet$ is quasi-isomorphic to the three-term complex
$$E^{-1} \to E^0 \to E^1$$
of locally free sheaves on $X \times B$. Furthermore, as $h^{-1}(I_0^\bullet) = 0$, the map
$$E^{-1}|_{X\times B_0} \to E^0|_{X\times B_0}$$

is injective as a map of sheaves and as before this implies injectivity on all of $X \times B$. Hence, $I^\bullet$ is quasi-isomorphic to

$$A = E^0/E^{-1} \to E^1,$$

where A has projective dimension less or equal than one since $E^{-1} \to E^0$ constitutes a projective resolution. Consider the relative curve

$$\operatorname{Supp}(F_0) =: \widetilde{C_0} \subset X \times B;$$

away from this curve the complex $I_0^\bullet$ becomes

$$\{0 \to \mathcal{O}_{X \times B_0} \to 0\}$$

so we can apply the argument from above to deduce that $E^{-1} \to E^0$ is actually injective on fibers and thus A is locally free away from $\widetilde{C_0}$.

The next thing to prove is the purity of A. Let $A' \subset A$ be a subsheaf with support of relative dimension at most two. Suppose A' was supported away from $\widetilde{C_0}$; as A is locally free there, A' would then be supported on all of $(X \times B) \setminus \widetilde{C_0}$, a contradiction. Thus, A' can only be supported on $\widetilde{C_0}$. To show that A' is in fact zero, we will show that

$$\operatorname{Hom}(A', A) = H^0(\pi_* \mathscr{H}om(A', A))$$

is zero, where π is the projection $\pi \colon X \times B \to B$. Applying Grothendieck–Verdier duality[2] 3.9 we see that

$$\begin{aligned} \mathbb{D}_B(R\pi_* R\mathscr{H}om(A', A)\,[3]) &\simeq R\pi_* R\mathscr{H}om(R\mathscr{H}om(A', A), K_{X \times B}) \\ &= R\pi_*(R\mathscr{H}om(A, A' \otimes K_{X \times B})), \end{aligned}$$

where the equality can be derived as follows:

$$\begin{aligned} R\mathscr{H}om(R\mathscr{H}om(A', A), K_{X \times B}) &= (R\mathscr{H}om(A', A))^\vee \overset{L}{\otimes} K_{X \times B} \\ &= (A'^\vee \overset{L}{\otimes} A)^\vee \overset{L}{\otimes} K_{X \times B} = R\mathscr{H}om(A, A' \otimes K_{X \times B}). \end{aligned}$$

Here, several times we made use of the fact that

$$R\mathscr{H}om(M, N) = M^\vee \overset{L}{\otimes} N \tag{13}$$

holds in the derived category, because one can choose a resolution of locally free modules for which the result is well-known. By the local-to-global spectral sequence 2.3 we get

$$E_2^{i,j} = R^i \pi_* \mathscr{E}xt^j(A, A' \otimes K_{X \times B}) \Rightarrow h^{i+j}(R\pi_*(R\mathscr{H}om(A, A' \otimes K_{X \times B}))).$$

As A has projective dimension less or equal than one, $\mathscr{E}xt^j(A, A' \otimes K_{X \times B})$ vanishes for $j \geq 2$. Since A' is supported in relative dimension at most one, also $\mathscr{E}xt^j(A, A' \otimes K_{X \times B})$ is supported in relative dimension at most one and $R^i \pi_* \mathscr{E}xt^j(A, A' \otimes K_{X \times B})$ vanishes for $i \geq 2$. In total, this term vanishes for $i + j \geq 3$, so $R\pi_*(R\mathscr{H}om(A, A' \otimes K_{X \times B}))$ is quasi-isomorphic to a complex with amplitude in $[0, 2]$. Applying the dualizing functor $\mathbb{D}_B$ gives a complex with amplitude in $[-2, 0]$ and shifting by $[-3]$ results in a complex supported in degrees at least one, quasi-isomorphic to $R\pi_* R\mathscr{H}om(A', A)$. In particular, its zeroth cohomology $\pi_* \mathscr{H}om(A', A)$ vanishes, showing that $\operatorname{Hom}(A', A) = 0$ and A is pure, proving part (ii).

[2]To apply the Grothendieck–Verdier duality as stated above we would have to assume that B is smooth. In fact, we can drop this assumption by replacing the dualizing sheaf with the dualizing complex, which exists even if B is not smooth.

Let $Q_0 = \operatorname{coker}(\mathcal{O}_{X\times B_0} \to F_0)$ and let $Z_0 = \operatorname{Supp}(Q_0)$; we define two open sets $U_0 = (X \times B_0) \setminus Z_0$ and $U = (X \times B) \setminus Z_0$. Note that U_0 is a closed subscheme of U. Since the map

$$\mathcal{O}_{X\times B_0\setminus Z_0} \to F_0|_{X\times B_0\setminus Z_0}$$

has zero cohomology, the map

$$E^0|_{U_0} \to E^1|_{U_0}$$

is surjective. Again by Nakayama's Lemma, we can extend the surjectivity to all of U, i.e.

$$E^0|_U \to E^1|_U$$

is surjective. Therefore, $h^1(I^\bullet)$ is only supported on Z_0, i.e. the support has relative dimension zero over B.

To show the last part of the lemma let $\ker|_U$ be the kernel of $E^0|_U \to E^1|_U$ which is locally free as the map is surjective. Recall that the map $E^{-1} \to E^0$ is injective. Therefore, $h^0(I^\bullet)|_U$ is quasi-isomorphic to

$$E^{-1}|_U \to \ker|_U .$$

Its restriction

$$E^{-1}|_{U_0} \to \ker|_{U_0} \tag{14}$$

is quasi-isomorphic to $h^0(I^\bullet)|_{U_0}$. Similarly, $h^0(I_0^\bullet)|_{U_0}$ is quasi-isomorphic to (14). Thus, we obtain an isomorphism

$$h^0(I_0^\bullet)|_{U_0} \cong h^0(I^\bullet)|_{U_0} .$$

Notice that the map

$$\mathcal{O}_{U_0} \to F_0|_{U_0}$$

is surjective with kernel $h^0(I_0^\bullet)|_{U_0}$. Both of the sheaves $\mathcal{O}_{U_0}$ and $F_0|_{U_0}$ are flat over B_0, so $h^0(I_0^\bullet)|_{U_0}$ is flat over B_0, too. Then Lemma 3.10 implies that $h^0(I^\bullet)|_U$ is flat over B. □

A proof of the following lemma can be found in [22].

Lemma 3.12 ([22, Lem. 6.13]). *Let $f\colon Y \to S$ be a smooth map of schemes and let E be a sheaf on Y flat over S.*

(i) *Assume that for every closed point $s \in S$ the restriction $E|_{Y_s}$ to its fiber is torsion-free of rank one. Then the double dual of E is locally free.*
(ii) *Assume that $E \cong E^{\vee\vee}$ and E is locally free away from a set of codimension three. Then E is locally free on all of Y.*

Remark 3.13. Let E be a pure sheaf on Y. If $\operatorname{Supp}(E) = Y$ then E is torsion-free. In fact, any coherent sheaf admits a unique *torsion filtration* (see [17, Def. 1.1.4])

$$0 \subset T_0(E) \subset \cdots \subset T_d(E) = E,$$

where $d = \dim(E)$ and $T_i(E)$ is the maximal subsheaf of E of dimension less than or equal to i. Therefore, if E is pure we see that $T_{d-1}(E) = 0$. The sheaf E is torsion-free if for each $y \in Y$ and $s \in \mathcal{O}_{Y,y} \setminus \{0\}$ multiplication by s is an injective map $E_y \to E_y$. This is equivalent to $T_{\dim(Y)-1}(E) = 0$. Hence, if $\dim(E) = \dim(Y)$ and E is pure, we conclude that E is torsion-free.

Lemma 3.14. *Let E be a pure sheaf on a Noetherian scheme Y with* $\mathrm{Supp}(E) = Y$. *Then the map from E to its double dual,*

$$\theta_E \colon E \to E^{\vee\vee},$$

is injective.

Proof. By Remark 3.13 we can find a closed subscheme $Z \subset Y$ of codimension at least one such that E is locally free on $U := Y \setminus Z$. In this case we get an isomorphism

$$\theta_E|_U : E|_U \to E^{\vee\vee}|_U .$$

Particularly, $\ker(\theta_E)$ is only supported on $Y \setminus U$ and has therefore codimension at least one. Since E is pure, we conclude that $\ker(\theta_E) = 0$. □

Definition 3.15 ([17, §1.1])**.** Let Y be a smooth projective variety of dimension n over a field k and let E be a coherent sheaf over Y of dimension d. Let $c = n - d$ denote the codimension of E. We say that E *satisfies property $S_{k,c}$* if for all $y \in \mathrm{Supp}(E)$ the following holds:

$$\mathrm{depth}(E_y) \geq \min\{k, \dim(\mathcal{O}_{Y,y}) - c\}.$$

Remark 3.16. The condition $S_{1,c}$ is equivalent to the purity of E. Indeed, let $y \in \mathrm{Supp}(E)$ be a point such that $\dim(\mathcal{O}_{Y,y}) > c$ and suppose $S_{1,c}$ holds. Then $\mathrm{depth}(E_y) \geq 1$ and this is equivalent to y not being an associated point of E. Hence, E is pure if and only if condition $S_{1,c}$ is satisfied.

Remark 3.17. If $\mathrm{Supp}(E) = Y$, i.e. the codimension c is zero, the condition $S_{1,c}$ implies that the set of singular points,

$$\{y \in Y \mid \mathrm{pd}(E_y) \neq 0\},$$

has codimension at least two, so E is locally free away from a closed codimension two subscheme.

The following is now a direct consequence of the statements above.

Lemma 3.18. *Let Y be a smooth projective variety and let $\pi \colon Y \times S \to S$ denote the canonical projection for some scheme S. Let E be a pure sheaf on $Y \times S$, flat over S with* $\mathrm{Supp}(E) = Y \times S$ *and let*

$$\theta_E \colon E \hookrightarrow E^{\vee\vee}$$

denote the embedding into its double dual. Then θ_E is an isomorphism away from a closed subscheme of codimension at least two.

Proof. By Remarks 3.16 and 3.17, the restriction of E to a fiber of π,

$$E|_{Y \times \{s\}},$$

for a point $s \in S$ is locally free away from a closed codimension two subscheme. As E is flat over S, E is locally free away from that subscheme. On the locus where E is locally free the map θ_E is clearly an isomorphism. □

The following lemma follows from results in [21, §2].

Lemma 3.19. *Let G and H be two coherent sheaves on a Noetherian scheme Y and let*

$$\phi\colon G \to H$$

be an injective map which is an isomorphism away from a closed subscheme of codimension at least two. Then there is an isomorphism

$$\det(G) \cong \det(H).$$

Proof. We have to check that there is a global map on the determinants of G and H induced by ϕ. Let $U(\phi)$ be the set of all points $y \in Y$ contained in an open neighborhood $V \subset Y$ such that ϕ restricted to V is an isomorphism (in $D^b(Y)$). Note that $U(\phi)$ contains every depth zero point of Y. Let $E_1^\bullet$ and $E_2^\bullet$ be two locally free resolutions of G and H, respectively. Moreover, let y be a point of Y with an open neighborhood V such that the $E_i^j|_V$ are free. By the choice of a basis on the $E_i^j|_V$ there is a chain of isomorphisms

$$\mathcal{O}_Y|_{V\cap U(\phi)} \xrightarrow{\sim} \det(E_1^\bullet)|_{V\cap U(\phi)} \xrightarrow{\sim} \det(G)|_{V\cap U(\phi)}$$
$$\xrightarrow{\sim} \det(H)|_{V\cap U(\phi)} \xrightarrow{\sim} \det(E_2^\bullet)|_{V\cap U(\phi)} \xrightarrow{\sim} \mathcal{O}_Y|_{V\cap U(\phi)}.$$

The composition of these isomorphisms is given by multiplication with a section

$$s \in H^0(V \cap U(\phi), \mathcal{O}_Y^*).$$

The zero locus of this section gives rise to a divisor $\mathrm{Div}(\phi)|_V$ defined on the whole V. Then the isomorphism on $U(\phi)$

$$\det(\phi)\colon \det(G)|_{U(\phi)} \xrightarrow{\sim} \det(H)|_{U(\phi)}$$

extends to a global isomorphism

$$\det(\phi)\colon \det(G)(\mathrm{Div}(\phi)) \xrightarrow{\sim} \det(H).$$

As ϕ is an isomorphism away from a codimension two subscheme, $\mathrm{Div}(\phi)$ is empty and the determinants are isomorphic. □

Definition 3.20. [17, Def. 1.2.2] Let E be a coherent sheaf on a d-dimensional scheme X. The Hilbert polynomial $P_E(k)$ can be uniquely written in the form

$$P_E(k) = \sum_{i=0}^{\dim(E)} \alpha_i(E)\frac{k^i}{i!}$$

with rational coefficients $\alpha_i(E)$. If E is of dimension d, then

$$\mathrm{rk}(E) := \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)}$$

is called the *rank* of E. If E is of dimension less than d, $\mathrm{rk}(E) := 0$.

Now we are able to prove the main theorem of this section, establishing a correspondence between deformations of complexes with trivial determinant and deformations of stable pairs.

Theorem 3.21 ([30, Thm. 2.7]). *Let $\iota\colon B_0 \hookrightarrow B$ be an N-th order thickening of quasi-projective schemes. Let $I^\bullet$ be a deformation over B of a complex $I_0^\bullet$ associated to a stable pair (F_0, s_0) on $X \times B_0$ with $\det(I^\bullet) \cong \mathcal{O}_{X\times B}$. Then $I^\bullet$ is quasi-isomorphic to a complex*

$$\{\mathcal{O}_{X\times B} \xrightarrow{s} F\},$$

where F is a flat deformation of F_0 with section s.

Proof. Let $I^\bullet$ be of the form as in Lemma 3.11, i.e.

$$I^\bullet = \{A \to E^1\}.$$

By Lemma 3.11 (iii), $Q := h^1(I^\bullet)$ has support of relative dimension zero over B; in particular, Q has rank zero on $X \times B$. As $h^0(I^\bullet|_{X\times B_0})$ has rank one, also $h^0(I^\bullet)$ has rank one. Being a subsheaf of the pure sheaf A (see Lemma 3.11 (ii)), $h^0(I^\bullet)$ is pure, too. Then Lemma 3.14 applies, so we get an injection

$$0 \to h^0(I^\bullet) \to h^0(I^\bullet)^{\vee\vee}.$$

The sequence

$$0 \to h^0(I^\bullet) \to A \to E^1 \to 0$$

is exact away from a codimension three subscheme. Therefore, the determinant

$$\det(h^0(I^\bullet)) \cong \det(I^\bullet) \cong \mathcal{O}_{X\times B}$$

is trivial, where the last isomorphism holds by assumption. By Lemmas 3.18 and 3.19, $h^0(I^\bullet)^{\vee\vee}$ has trivial determinant, too. The fourth statement of Lemma 3.11 states that $h^0(I^\bullet)$ is flat over B away from a codimension three subscheme. Therefore, by Lemma 3.12 (i), the double dual $h^0(I^\bullet)^{\vee\vee}$ is locally free away from a codimension three subscheme. As $h^0(I^\bullet)^{\vee\vee}$ is a reflexive sheaf, we conclude by Lemma 3.12 (ii) that $h^0(I^\bullet)^{\vee\vee}$ is in fact locally free. This implies that

$$h^0(I^\bullet)^{\vee\vee} \cong \mathcal{O}_{X\times B}$$

and $h^0(I^\bullet)$ is an ideal sheaf $\mathscr{I}_C \subset \mathcal{O}_{X\times B}$. Analogously to (6) there is an exact triangle

$$Q[-2] \to \mathscr{I}_C \to I^\bullet \to Q[-1]. \tag{15}$$

We can therefore regard $I^\bullet$ as the mapping cone of a map

$$\alpha \in \mathrm{Hom}(Q[-2], \mathscr{I}_C) = \mathrm{Ext}^2(Q, \mathscr{I}_C).$$

The ideal sheaf $\mathscr{I}_C$ comes with a short exact sequence

$$0 \to \mathscr{I}_C \to \mathcal{O}_{X\times B} \to \mathcal{O}_C \to 0. \tag{16}$$

Applying $\mathrm{Hom}(Q, -)$ to this sequence results in

$$\mathrm{Ext}^1(Q, \mathcal{O}_{X\times B}) \to \mathrm{Ext}^1(Q, \mathcal{O}_C) \to \mathrm{Ext}^2(Q, \mathscr{I}_C) \to \mathrm{Ext}^2(Q, \mathcal{O}_{X\times B}).$$

As Q has support of dimension zero, Lemma 2.4 yields the vanishing of $\mathscr{E}xt^i(Q, \mathcal{O}_{X\times B})$ for $i < 3$. Then by the local-to-global spectral sequence the first and last term of the above sequence are zero and we are left with an isomorphism

$$\mathrm{Ext}^1(Q, \mathcal{O}_C) \cong \mathrm{Ext}^2(Q, \mathscr{I}_C).$$

Consider the Yoneda cup product

$$\smile : \mathrm{Ext}^1(\mathcal{O}_C, \mathscr{I}_C) \times \mathrm{Ext}^1(Q, \mathcal{O}_C) \to \mathrm{Ext}^2(Q, \mathscr{I}_C)$$

obtained by concatenation of the sequences corresponding to the extensions. By the isomorphism above we conclude that the extension class $\alpha \in \mathrm{Ext}^2(Q, \mathscr{I}_C)$ can be written as a cup product $\alpha = \xi \smile \epsilon$, where ξ is the extension class corresponding to the short exact sequence (16) and ϵ is some class inside $\mathrm{Ext}^1(Q, \mathcal{O}_C)$. Therefore, α corresponds to an exact sequence

$$0 \to \mathscr{I}_C \to \mathcal{O}_{X\times B} \to F \to Q \to 0 \tag{17}$$

obtained from concatenation of the two sequences

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & F & \longrightarrow & Q \longrightarrow 0 \\ & & & & \| & & & & \\ 0 \longrightarrow \mathscr{I}_C & \longrightarrow & \mathcal{O}_{X\times B} & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 & & \end{array}$$

where F is the extension corresponding to ϵ. Comparing the sequence (17) with the exact triangle (15) we see that $I^\bullet$ is quasi-isomorphic to the complex

$$\{\mathcal{O}_{X\times B} \to F\}\,.$$

We are left to show that F is a flat deformation of F_0. By Lemma 3.10, it suffices to show that

$$L\iota^* F \cong F_0$$

to prove the flatness of F because F_0 is already flat over B_0. The sheaf F can be obtained from the mapping cone of the canonical map

$$I^\bullet \to \mathcal{O}_{X\times B} \tag{18}$$

occurring in the exact triangle

$$I^\bullet \to \mathcal{O}_{X\times B} \to F \to I^\bullet[1],$$

see the proof of Proposition 2.1. Therefore, its derived restriction to $X \times B_0$ can be obtained from the cone of the induced map

$$I_0^\bullet \xrightarrow{\phi} \mathcal{O}_{X\times B_0}.$$

Likewise, F_0 is obtained from the cone of the canonical map

$$I_0^\bullet \xrightarrow{\psi} \mathcal{O}_{X\times B_0},$$

so our task consists of showing that the maps ϕ and ψ agree. By Corollary 2.7, apart from $\mathscr{H}om(I^\bullet, \mathcal{O}_{X\times B}) \cong \mathcal{O}_{X\times B}$, all entries on the zeroth diagonal of the E_2-page of the local-to-global spectral sequence applied to $\mathscr{E}xt^i(I^\bullet, \mathcal{O}_{X\times B})$ vanish, so

$$\mathrm{Hom}(I^\bullet, \mathcal{O}_{X\times B}) \cong \Gamma(\mathcal{O}_{X\times B}).$$

Therefore, $\mathrm{Hom}(I^\bullet, \mathcal{O}_{X\times B})$ is generated by the canonical map in (18). Analogously,

$$\mathrm{Hom}(I_0^\bullet, \mathcal{O}_{X\times B_0}) \cong \Gamma(\mathcal{O}_{X\times B_0})$$

is generated by the canonical map ψ. Therefore, the map ϕ can be written as

$$\phi = \gamma\psi$$

for some map $\gamma \in \Gamma(\mathcal{O}_{X\times B_0})$. Away from the support of F, the canonical map (18) is the canonical isomorphism $I^\bullet \cong \mathcal{O}_{X\times B}$, and likewise for the canonical isomorphism $\psi\colon I_0^\bullet \to \mathcal{O}_{X\times B_0}$. Hence, γ is a unit away from support of F which has codimension two. This implies that γ is a unit globally, so the maps ϕ and ψ agree and $I^\bullet$ corresponds to a deformation F of the stable pair (F_0, s_0). □

As we have seen in Proposition 2.1, the map that associates to each stable pair the corresponding complex in the derived category is injective on objects. Now Theorem 3.21 tells us that the moduli space of stable pairs $P(X)$ is embedded into the moduli space of perfect complexes with trivial determinant.

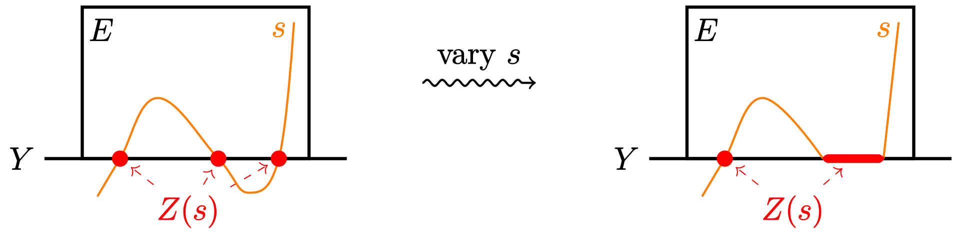

FIGURE 4.1. A vector bundle E on a scheme Y with a section s. For a generic section, the zero locus $Z(s)$ has the expected dimension (left-hand side). As s is varied, the dimension of $Z(s)$ can jump up and it can get non-equidimensional (right-hand side).

Remark 3.22. One might wonder if the moduli space of complexes is really bigger than $P(X)$ and the answer is yes. Let $I^\bullet$ be a complex associated to a stable pair. Then the complexes $I^\bullet[2n]$ for $n \in \mathbb{Z}$ are also perfect complexes with trivial determinant and rank one, pairwise non-quasi-isomorphic as the degree of the cohomology shifts by $2n$. Moreover, for $n \neq 0$ none of the complexes $I^\bullet[2n]$ is quasi-isomorphic to a complex associated to a stable pair as it has no cohomology in degree one and zero. Nevertheless, all these complexes yield the same numerical invariants.

4. A short introduction to virtual classes

The purpose of this section is to provide a short introduction to virtual classes as constructed by Behrend and Fantechi [4]. It is not intended to give many proofs, instead we focus on examples and explain the necessary parts to construct a virtual class for the moduli space of complexes with trivial determinant.

4.1. **First examples and definitions.** The first example illustrates the need for a virtual class and gives some first hints how the construction works.

Example 4.1. Let Y be a purely n-dimensional scheme and let E be a vector bundle of rank r on Y. Given a section $s \in H^0(Y, E)$, let $Z(s) \subset Y$ be the zero locus of s. For a *generic* section, i.e. a section intersecting the zero section transversely, $Z(s)$ is of dimension $n - r$. We call this the *expected* dimension. For example, if E is a rank one bundle on a curve we expect the zero locus to be just a collection of points. In this case we could easily count the number of these points (by integration against the fundamental class $[Z(s)]$).

But if we vary s it is possible that the dimension of $Z(s)$ jumps and increases. It can even occur that $Z(s)$ is not equidimensional (see Figure 4.1). Therefore, in general, the fundamental class $[Z(s)]$ does not yield meaningful countings. However, we would like to have a class for all $Z(s)$ that behaves very similar to the fundamental class if the space has the "correct" dimension. In particular, we want that integration against this class yields the correct counting result we also obtain in the case that the space has the expected dimension. Such a class is a virtual fundamental class. It sits in the Chow homology in the degree of the *virtual* dimension which equals the expected dimension.

One might ask why we do not just slightly "move" our objects so that intersections become transverse and counting becomes easy. Unfortunately, this is not possible in general. As an example, take a (-1)-curve C on a smooth surface. It is not possible to move C to be transverse to itself. Nevertheless, we can calculate the self-intersection number by integration of the Euler class against the curve class.

The following is a key definition from intersection theory.

Definition 4.2 ([11, App. B.6.1]). Let $Z \hookrightarrow Y$ be a closed embedding and let $\mathscr{I}$ be the ideal sheaf of Z inside Y. Then the *normal cone* $C_{Z/Y}$ to Z in Y is defined as

$$C_{Z/Y} = \mathbf{Spec}_Z \left(\bigoplus_{k \geq 0} \mathscr{I}^k / \mathscr{I}^{k+1} \right).$$

Lemma 4.3 ([11, App. B.6.6]). *Let Y be a purely n-dimensional scheme and Z a closed subscheme. Then $C_{Z/Y}$ is of pure dimension n.*

Proof. Consider the embedding

$$Z \subset Y \xhookrightarrow{\mathrm{id} \times 0} Y \times \mathbb{A}^1$$

and let $\mathscr{I}$ denote the ideal sheaf corresponding to the embedding $Z \hookrightarrow Y \times \mathbb{A}^1$. Then the normal cone $C_{Z/Y \times \mathbb{A}^1}$ to Z in $Y \times \mathbb{A}^1$ is given by

$$C_{Z/Y \times \mathbb{A}^1} = C_{Z/Y} \oplus 1.$$

Let

$$\pi \colon \operatorname{Bl}_Z(Y \times \mathbb{A}^1) = \mathbf{Proj} \left(\bigoplus_{k \geq 0} \mathscr{I}^k \right) \to Y \times \mathbb{A}^1$$

be the blow-up of $Y \times \mathbb{A}^1$ at Z. As Z is nowhere dense in $Y \times \mathbb{A}^1$, the blow-up $\operatorname{Bl}_Z(Y \times \mathbb{A}^1)$ is birational to $Y \times \mathbb{A}^1$ and therefore of pure dimension $n+1$. The exceptional divisor $E = \pi^{-1}(Z)$ is the projectivization of the normal cone, i.e.

$$E = \mathbf{Proj} \left(\bigoplus_{k \geq 0} \mathscr{I}^k / \mathscr{I}^{k+1} \right) = \mathbb{P}(C_{Z/Y} \oplus 1)$$

and is pure of dimension n being a Cartier divisor on $\operatorname{Bl}_Z(Y \times \mathbb{A}^1)$. Therefore, $C_{Z/Y}$ is pure of dimension n, too. $\square$

Coming back to the situation of Example 4.1, let $\mathscr{I}$ be the ideal sheaf of $Z = Z(s)$ inside Y and let $C_{Z/Y}$ be the normal cone to Z in Y. Let $\mathcal{E}^\vee$ be the locally free sheaf corresponding to E. There exists a surjection

$$\begin{array}{rcl} \mathcal{E}^\vee & \twoheadrightarrow & \mathscr{I} \\ f & \mapsto & f.s \in H^0(Y, E) \end{array}$$

which induces a surjective map

$$\bigoplus_{k \geq 0} \operatorname{Sym}^k \left(\mathcal{E}^\vee / \mathscr{I}.\mathcal{E}^\vee \right) \twoheadrightarrow \bigoplus_{k \geq 0} \left(\mathscr{I}^k / \mathscr{I}^{k+1} \right).$$

Taking relative spectrum of this map gives us a closed immersion

$$\mathbf{Spec}_Z \left(\bigoplus_{k \geq 0} \left(\mathscr{I}^k / \mathscr{I}^{k+1} \right) \right) \hookrightarrow \mathbf{Spec}_Z \left(\bigoplus_{k \geq 0} \operatorname{Sym}^k \left(\mathcal{E}^\vee / \mathscr{I}.\mathcal{E}^\vee \right) \right).$$

The left-hand side is the normal cone $C_{Z/Y}$, the right-hand side the restriction of the bundle E to Z. Therefore, we have obtained an embedding

$$C_{Z/Y} \hookrightarrow E|_Z .$$

The normal cone is of pure dimension n. Thus, by the above embedding we get a class

$$[C_{Z/Y}] \in A_n(E|_Z).$$

The pullback of this class along the section $s\colon Y \to E$ gives a class

$$s^*[C_{Z/Y}] \in A_{n-r}(Z).$$

This gives an example of a virtual class of Z in Y. Note that $n-r$ is the expected dimension of Z. If s is a generic section, i.e. it intersects the zero section transversely, and $i\colon Z(s) \hookrightarrow Y$ denotes the inclusion of the zero locus, then

$$i_* s^*[C_{Z/Y}] \in A_{n-r}(Y)$$

is equal to the Euler class

$$c_r(E) \frown [Y] = [Z]$$

of E (this follows from Proposition 4.12 below). This is a feature we wanted the virtual class to have: if the space has the expected dimension then the virtual class is equal to the fundamental class.

The second example illustrates how a virtual class on a moduli space is the correct tool for counting objects.

Example 4.4. Let Y_f be a quintic threefold, i.e. $Y_f = V(f) \subset \mathbb{P}^4$ for some homogeneous polynomial f of degree five. We consider the moduli space M_f of lines in Y_f, so M_f is the Hilbert scheme

$$M_f = \mathrm{Hilb}^{k+1}_{Y_f}.$$

It is a well-known fact that for a generic polynomial f_{gen} there are precisely 2785 lines in $Y_{f_{\mathrm{gen}}}$ [8, Cor. 6.35]. In this case, $M_{f_{\mathrm{gen}}}$ consists of 2785 isolated points. Particularly, the expected dimension of M_f is zero. How can we compute the number of lines using the fundamental class $[M_{f_{\mathrm{gen}}}]$ in this case? The moduli space of lines in Y can be regarded as a subspace of the moduli space of lines in $\mathbb{P}^4$, i.e.

$$M_f \subset \mathrm{Gr}(2,5).$$

A line L lies on Y_f if and only if

$$f|_L = 0 \in H^0(L, \mathcal{O}_L(5)). \tag{19}$$

Definition 4.5 ([11, App. B.5.7]). Let E be a vector bundle on a scheme Y of rank r. The Grassmann bundle

$$p\colon \mathrm{Gr}_d(E) \to Y$$

comes with a *tautological subbundle* $\mathcal{S} \subset p^*(E)$ which is a vector bundle of rank d whose fiber over a point $(y, V) \in \mathrm{Gr}_d(E)$ is the vector space $V \subset E_y$.

On $\mathrm{Gr}(2,5)$ there is the tautological subbundle $\mathcal{S}$ of rank two. The fiber of $\mathcal{S}^\vee$ at $[L] \in \mathrm{Gr}(2,5)$ is the space of linear forms on L, i.e. $H^0(\mathcal{O}_L(1))$. Then via the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \to H^0(\mathcal{O}_L(5)),$$

the polynomial f gives rise to a global section s_f of $\operatorname{Sym}^5 \mathcal{S}^\vee$ (which is a vector bundle of rank six) and the zero locus of s_f is precisely M_f by (19). If M_f has the expected dimension zero, i.e. f is generic, then the fundamental class is given by

$$[M_{f_{\text{gen}}}] = c_6(\operatorname{Sym}^5 \mathcal{S}^\vee) \in A_0(\operatorname{Gr}(2,5)).$$

In this case the number of lines in X_{gen} can be calculated via

$$\int_{[M_{f_{\text{gen}}}]} 1 = \deg([M_{f_{\text{gen}}}]) = 2785.$$

For details see [8, §6.2].

However, if we take for example f to be the Fermat quintic, i.e.

$$f_{\text{Fermat}} = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5$$

then there exist 50 one-dimensional families of lines in $Y_{f_{\text{Fermat}}}$ [1]. Our goal is to construct a class

$$[M_f]^{vir} \in A_0(M_f)$$

such that for any f (generic or not) we obtain

$$\int_{[M_f]^{vir}} 1 = \deg([M_f]^{vir}) = 2785. \tag{20}$$

Again, consider the embedding $M_f \subset \operatorname{Gr}(2,5)$ and the bundle $E := \operatorname{Sym}^5 \mathcal{S}^\vee$ with the global section

$$s_f\colon\ \operatorname{Gr}(2,5) \to E.$$

The following sequence of definitions culminates in the definition of the Gysin pullback.

Definition/Proposition 4.6 ([11, Thm. 3.3 & Def. 3.3]). Let

$$p\colon E \to Y$$

be a vector bundle on Y of rank r and let s be its zero section. The pullback map

$$p^*\colon A_{k-r}(Y) \to A_k(E)$$

is an isomorphism for all k. Then the *Gysin homomorphisms* are given by

$$s^*\colon A_k(E) \to A_{k-r}(Y),$$

where

$$s^*(\beta) := (p^*)^{-1}(\beta).$$

A useful property of these Gysin homomorphisms is that for any k-cycle β on E the pullback $s^*(\beta)$ is defined no matter how β intersects the zero section (i.e. transversely or not).

Definition 4.7 ([11, App. B.7.1]). A closed immersion $i\colon Z \hookrightarrow Y$ is called a *regular embedding of codimension d* if every point in Z has an affine open neighborhood $U \subset Y$ such that if A is the coordinate ring of U and I the ideal of A defining Z then I is generated by a regular sequence of length d.

Remark 4.8 ([11, App. B.7.1]). If $Z \hookrightarrow Y$ is a closed regular embedding then the normal bundle $N_{Z/Y}$ to Z in Y is isomorphic to the normal cone $C_{Z/Y}$.

Definition 4.9 ([11, §6.1]). Let $i\colon Z \hookrightarrow Y$ be a closed regular embedding of codimension d and let $N_{Z/Y}$ be the normal bundle. Let V be a purely k-dimensional scheme and let $f\colon V \to Y$ be a morphism. Denote $W := f^{-1}(Z)$ and consider the following fiber square:

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & Y \end{array}$$

Let $N := g^*N_{Z/Y}$ and let $p\colon N \to W$ be the projection; the cone $C := C_{W/V}$ embeds into N and gives rise to the following diagram:

$$\begin{array}{ccc} C & \hookrightarrow & N \\ & \searrow & \downarrow{\scriptstyle p} \\ & & W \end{array}$$

Let s be the zero section of N and consider the Gysin homomorphism

$$s^*\colon A_k(N) \to A_{k-d}(W).$$

Then we define the *intersection product* of Z and V as

$$Z \cdot V := s^*[C] \in A_{k-d}(W).$$

Definition 4.10 ([11, §6.2]). Let $i\colon Z \hookrightarrow Y$ be a closed regular embedding of codimension d and let $f\colon Y' \to Y$ be a morphism. Consider the following fiber square:

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & Y \end{array}$$

We define the *refined Gysin homomorphisms* or *Gysin pullbacks*

$$i^!\colon A_k(Y') \to A_{k-d}(Z')$$

by setting

$$i^![V] := Z \cdot V.$$

Proposition 4.11 ([11, Cor. 6.3]). *Let*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & Y \end{array}$$

be a fiber square with i a closed regular embedding of codimension d and normal bundle N. Assume i' is an isomorphism. Then

$$i^!(\alpha) = c_d(g^*N) \frown \alpha$$

for all $\alpha \in A_*(Y')$. As a special case one obtains

$$i^*i_*(\alpha) = c_d(N) \frown \alpha$$

for all $\alpha \in A_*(X)$ by setting $Y' = Y$ and $f = \mathrm{id}_Y$. This last statement is called the self-intersection formula.

Now we return to Example 4.4: Consider the fiber square

$$\begin{array}{ccc} M_f & \longrightarrow & \mathrm{Gr}(2,5) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle s_f} \\ \mathrm{Gr}(2,5) & \xrightarrow{0_E} & E \end{array}$$

where 0_E denotes the zero section of E and i the inclusion. The zero section defines a regular embedding of $\mathrm{Gr}(2,5)$ into E. Then we define the virtual class of M_f as

$$[M_f]^{vir} := 0_E^![\mathrm{Gr}(2,5)] \in A_0(M_f).$$

Let us unravel this expression: the Gysin pullback $0_E^![\mathrm{Gr}(2,5)]$ is defined as the intersection product $\mathrm{Gr}(2,5) \cdot \mathrm{Gr}(2,5)$ which is in turn defined as

$$s^*[C_{M_f/\,\mathrm{Gr}(2,5)}],$$

where s is the zero section of the bundle $i^*N_{\mathrm{Gr}(2,5)/E}$. The normal bundle $N_{\mathrm{Gr}(2,5)/E}$ is in this case just the restriction of E to M_f, so s is the zero section of $E|_{M_f}$ and we can also express $[M_f]^{vir}$ as

$$[M_f]^{vir} = 0^*_{E|_{M_f}}[C_{M_f/\,\mathrm{Gr}(2,5)}].$$

Why is this the class we wanted? First, let us check that the formula (20) is valid. The following formula will be very useful to us for the computation of virtual classes.

Proposition 4.12 ([11, Prop. 14.1]). *Let E be a vector bundle on a pure dimensional scheme Y, let s be a section of E and let $Z(s)$ be the zero locus of s. There is a fiber square*

$$\begin{array}{ccc} Z(s) & \longrightarrow & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{0_E} & E \end{array}$$

and we have

$$i_*0_E^![Y] = c_{\mathrm{top}}(E) \frown [Y] \in A(Y).$$

Proof. This follows essentially by the self-intersection formula from Proposition 4.11. For a detailed proof, see [11, Prop. 14.1]. □

In our case this gives the identity

$$i_*0_E^![\mathrm{Gr}(2,5)] = c_{\mathrm{top}}(E) \frown [\mathrm{Gr}(2,5)].$$

Then we can calculate

$$\int_{[M_f]^{vir}} 1 = \int_{[\mathrm{Gr}(2,5)]} i_*[M_f]^{vir} = \int_{[\mathrm{Gr}(2,5)]} c_{\mathrm{top}}(E) \frown [\mathrm{Gr}(2,5)] = 2785$$

which is what we were looking for.

4.2. **Towards a more general construction.** Let us review the ingredients used in the example above to cook up the virtual class and try to make a general recipe out of it.

First of all, we had an embedding of the space M_f into the smooth space $\mathrm{Gr}(2,5)$. This is a bit impractical because we do not want our virtual class to depend on this embedding. Luckily, the virtual class depends only on the embedding

$$C_{M_f/\,\mathrm{Gr}(2,5)} \hookrightarrow E|_{M_f}.$$

The goal is to make this notion intrinsic to M_f. Therefore, to construct a virtual class on a space X we will define

(i) the *intrinsic normal cone* $\mathfrak{C}_X$;
(ii) an embedding $\mathfrak{C}_X \hookrightarrow E$.

Here, E is an *obstruction bundle* (i.e. a vector bundle that admits an embedding of a cone); it is *not* intrinsic to X and our virtual class will depend on the obstruction theory we choose.

The construction of virtual classes makes use of the language of stacks due to the fact that one wants to consider quotients of a scheme Y under a group scheme action

$$G \times Y \to Y$$

which is in general not a scheme anymore. Instead, one can define a *stack quotient* $[Y/G]$. We will not introduce stacks in this thesis and try to avoid it where it is possible. The interested reader may be referred to [9] for an introduction.

How can we build obstruction bundles?

Construction 4.13 ([2, §4.1]). Let $i\colon Z \hookrightarrow Y$ be a closed embedding of schemes with ideal sheaf $\mathscr{I}_{Z/Y}$. Let $f_1, \dots, f_n$ be generators of $\mathscr{I}_{Z/Y}$; each of these f_j defines a Cartier divisor D_j. Then we get a surjection

$$\textstyle\bigoplus_{j=1}^n \mathscr{O}_Y(-D_j) \xrightarrow{(f_1,\dots,f_n)} \mathscr{I}_{Z/Y}$$

by multiplication with the f_j. Applying the pullback functor i^* and taking spectra results into the closed immersion

$$N_{Z/Y} = \mathbf{Spec}(\mathrm{Sym}(\mathscr{I}_{Z/Y}/\mathscr{I}_{Z/Y}^2)) \hookrightarrow \mathbf{Spec}\left(\mathrm{Sym}\left(\bigoplus_{j=1}^n i^*\mathscr{O}_Y(-D_j)\right)\right) =: E_{Z/Y}. \tag{21}$$

Note that there is a natural inclusion of the cone $C_{Z/Y}$ into the normal sheaf $N_{Z/Y}$ so we get an embedding

$$C_{Z/Y} \hookrightarrow E_{Z/Y}.$$

Remark 4.14. This construction depends on the choice of the generators f_j.

We now want to globalize this construction. The following definition is rather provisional and we will generalize it later.

Definition 4.15 ([2, Def. 5.3]). Let Z be a scheme. An *obstruction theory* $E_{Z/Y}$ on Z is given by the following data:

(i) an embedding $Z \hookrightarrow Y$, where Y is smooth,
(ii) a covering of Y by open sets Y_i, and on each $U_i = Z \cap Y_i$ a set of generators for the restriction to U_i of the ideal $\mathscr{I}_{U_i/Y_i}$; these induce obstruction bundles E_i by Construction 4.13,
(iii) gluing data for the sets of generators, i.e. isomorphisms

$$\phi_{ij}\colon E_i|_{U_{ij}} \to E_j|_{U_{ij}}$$

such that the ϕ_{ij} satisfy the cocycle condition, i.e. $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$, and the following diagram commutes:

$$\begin{array}{ccc} C_{U_i/Y_i}|_{U_{ij}} & \longrightarrow & E_i|_{U_{ij}} \\ {\scriptstyle \bar{\phi}_{ij}}\downarrow & & \downarrow{\scriptstyle \phi_{ij}} \\ C_{U_j/Y_j}|_{U_{ij}} & \longrightarrow & E_j|_{U_{ij}} \end{array}$$

Here, $\bar{\phi}_{ij}$ is the canonical isomorphism, and the horizontal maps are embeddings of the cones into the corresponding obstruction bundles.

This definition is not optimal because much information is needed: on every chart we have to choose generators of the ideal sheaf and give transition relations between these. Therefore, we want another definition that uses less "local" information. For this, observe the following: let

$$E_{Z/Y} = \mathbf{Spec}\left(\operatorname{Sym}\left(\bigoplus_{j=1}^{n} \mathcal{O}_Z(-D_j)\right)\right)$$

be an obstruction bundle as in Construction 4.13 and let δ denote the differentiation map

$$\delta\colon \mathscr{I}/\mathscr{I}^2 \to \Omega^1_Y|_Z$$

from the conormal exact sequence; this gives a morphism

$$\delta\colon T_Y|_Z \to N_{Z/Y}.$$

The composition of δ with the morphism (21) from above leads to a map

$$d = (\delta f_1, \ldots, \delta f_n)\colon T_Y|_Z \to E_{Z/Y}$$

defining an action

$$\begin{array}{ccc} T_Y|_Z \times E_{Z/Y} & \to & E_{Z/Y} \\ (\nu, \gamma) & \mapsto & d(\nu) + \gamma \end{array}$$

of the tangent bundle on the obstruction bundle. As one can show, both the cone and the normal sheaf $C_{Z/Y} \hookrightarrow N_{Z/Y} \hookrightarrow E_{Z/Y}$ are invariant under this action. Therefore, one obtains induced actions

$$T_Y|_Z \times N_{Z/Y} \to N_{Z/Y}$$

and

$$T_Y|_Z \times C_{Z/Y} \to C_{Z/Y}.$$

It turns out that such an action of the tangent bundle is really what is necessary for the construction of a virtual class.

4.3. **The general construction.**

Definition 4.16 ([4, Def. 3.10])**.** Let $Z \hookrightarrow Y$ be a closed embedding into a smooth scheme. The *intrinsic normal cone* $\mathfrak{C}_Z$ of Z is defined as the stack quotient

$$\mathfrak{C}_Z = [C_{Z/Y}/T_Y|_Z].$$

This gives an algebraic stack $\mathfrak{C}_Z$ of pure dimension zero.

Remark 4.17 ([28]). The definition of $\mathfrak{C}_Z$ is independent of the embedding of Z into Y. Suppose there are two embeddings of Z into smooth schemes Y_1 and Y_2 via maps i_1 and i_2, respectively. Then there is also an embedding of Z into $Y_1 \times Y_2$ and we obtain the following diagram:

$$\begin{array}{ccc} Z & \xhookrightarrow{(i_1,i_2)} & Y_1 \times Y_2 \\ & \searrow^{i_1} & \downarrow{\scriptstyle pr_1} \\ & & Y_1 \end{array}$$

Hence, we can restrict our attention to the situation

$$\begin{array}{ccc} Z & \stackrel{i'}{\hookrightarrow} & Y' \\ & \searrow_{i} & \downarrow f \\ & & Y \end{array}$$

where f is smooth. By [4, Prop. 3.1] there is an exact sequence

$$0 \to i'^* T_{Y'/Y} \to C_{Z/Y'} \to C_{Z/Y} \to 0.$$

Then

$$[C_{Z/Y'}/T_{Y'}|_Z] \cong \left[[C_{Z/Y'}/i'^* T_{Y'/Y}]/f^* T_Y|_Z\right] \cong [C_{Z/Y}/T_Y|_Z]$$

so in particular

$$[C_{Z/Y_1 \times Y_2}/T_{Y_1 \times Y_2}|_Z] \cong [C_{Z/Y_1}/T_{Y_1}|_Z]$$

and repeating the same argument for Y_2 shows the independence of the embedding.

Remark 4.18. The definition also works if the smooth embedding is just given locally.

Definition 4.19. Let $Z \hookrightarrow Y$ be a closed embedding into a smooth scheme. The *intrinsic normal sheaf* $\mathfrak{N}_Z$ of Z is defined as the stack quotient

$$\mathfrak{N}_Z = [N_{Z/Y}/T_Y|_Z].$$

Again, one can show similarly to Remark 4.17 that this definition is independent of the choice of the embedding.

The embedding

$$C_{Z/Y} \hookrightarrow N_{Z/Y}$$

induces an embedding

$$\mathfrak{C}_Z \hookrightarrow \mathfrak{N}_Z.$$

Now we want to establish the notion of a perfect obstruction theory. The idea is to find a two-term complex

$$E_{Z,\bullet} = \{E_0 \to E_1\}$$

such that there is an embedding

$$\mathfrak{C}_Z \hookrightarrow [E_1/E_0].$$

To define this, we first have to introduce the *cotangent complex* which generalizes the concept of Kähler differentials. Let

$$Z \xrightarrow{f} Y \to S$$

be morphisms of schemes. Recall that there exists the cotangent exact sequence

$$f^*\Omega^1_{Y/S} \to \Omega^1_{Z/S} \to \Omega^1_{Z/Y} \to 0$$

which is exact on the left if and only if f is smooth. If f is a closed immersion, $\Omega^1_{Z/Y}$ vanishes. Let $\mathscr{I}$ be the ideal sheaf of Z in Y. Then the cotangent exact sequence above can be extended to the left as

$$f^*\mathscr{I} \xrightarrow{\delta} f^*\Omega^1_{Y/S} \to \Omega^1_{Z/S} \to 0,$$

where δ is the differentiation map as already seen above. All this suggests that there is something more going on at the left side of the cotangent exact sequence and we just see the end of a long exact cohomology sequence. Indeed, it is the last part of the long exact sequence resulting from taking cohomology at the cotangent complex $\mathbb{L}^\bullet_{Z/Y}$ which is a complex in non-positive degrees. From our considerations, we can immediately extract some properties of the cotangent complex:

(i) For every triple

$$Z \xrightarrow{f} Y \to S$$

there exists an exact triangle

$$f^*\mathbb{L}^\bullet_{Y/S} \to \mathbb{L}^\bullet_{Z/S} \to \mathbb{L}^\bullet_{Z/Y} \to f^*\mathbb{L}^\bullet_{Y/S}[1]$$

in the derived category $D(Z)$.

(ii) If $Z \to Y$ is smooth, $h^i(\mathbb{L}^\bullet_{Z/Y}) = 0$ for $i \leq -1$.

We do not give the definition of the cotangent complex here; the interested reader is referred to [19].

Let $\tau^{\geq -1}\mathbb{L}^\bullet_{Z/Y}$ denote the $[-1,0]$-truncation of the cotangent complex. For the construction of a virtual class in our case we deal with an embedding $Z \hookrightarrow Y$ where Y is smooth (i.e. a *global* embedding; in general, the construction also works if we have such embeddings only locally). Therefore, the following proposition fits our situation.

Proposition 4.20 ([32, Tag 08R6]). *Given a triple*

$$Z \overset{i}{\hookrightarrow} Y \to S,$$

where i is a closed immersion with ideal sheaf $\mathscr{I}$ and $Y \to S$ is smooth, there is a natural quasi-isomorphism

$$\tau^{\geq -1}\mathbb{L}^\bullet_{Z/Y} \to \left\{\mathscr{I}/\mathscr{I}^2 \xrightarrow{\delta} \Omega^1_Y|_Z\right\},$$

where δ is the differentiation map.

In fact, the truncated cotangent complex is independent of the ambient space Y, i.e. given two smooth embeddings into Y_1 and Y_2, the corresponding truncated cotangent complexes $\mathbb{L}^\bullet_{Z/Y_1}$ and $\mathbb{L}^\bullet_{Z/Y_2}$ are quasi-isomorphic. Therefore, we may also omit the ambient space from our notation and just write $\mathbb{L}^\bullet_Z$.

Definition 4.21. Let

$$E_\bullet = \left\{E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} \cdots\right\}$$

be a complex of vector bundles over Z. Then we define $h^1/h^0(E_\bullet)$ to be the quotient

$$h^1/h^0(E_\bullet) = [\ker(d_1)/E_0].$$

One can show this only depends on the isomorphism-class of $E_\bullet$ in $D^{\geq 0}(Z)$, see [4, Prop. 2.1].

Proposition 4.22 ([4, Prop. 2.4–2.6]). *Let*

$$\phi\colon E^\bullet \to L^\bullet$$

be a morphism in $D^{\leq 0}(Z)$ between complexes of vector bundles. Then there exists an induced morphism

$$\phi^\vee\colon h^1/h^0((L^\bullet)^\vee) \to h^1/h^0((E^\bullet)^\vee).$$

The morphism $\phi^\vee$ is a closed immersion if and only if

(i) *$h^0(\phi)$ is an isomorphism and*

(ii) *$h^{-1}(\phi)$ is surjective.*

Proposition 4.23 ([4, §3]). *Let $\tau^{\geq -1}\mathbb{L}_Z^\bullet$ be the truncated cotangent complex. Then the quotient*

$$h^1/h^0((\tau^{\geq -1}\mathbb{L}_Z^\bullet)^\vee) = \mathfrak{N}_Z$$

equals the intrinsic normal sheaf.

This motivates the definition of a perfect obstruction theory.

Definition 4.24 ([4, Def. 4.4]). A *perfect obstruction theory* for Z is a complex $E_Z^\bullet \in D^b(Z)$ which is locally quasi-isomorphic to a complex of locally free sheaves with amplitude in $[-1, 0]$, together with a morphism

$$\phi\colon E_Z^\bullet \to \tau^{\geq -1}\mathbb{L}_Z^\bullet$$

such that

(i) $h^0(\phi)\colon h^0(E_Z^\bullet) \to h^0(\tau^{\geq -1}\mathbb{L}_Z^\bullet)$ is an isomorphism and
(ii) $h^{-1}(\phi)\colon h^{-1}(E_Z^\bullet) \to h^{-1}(\tau^{\geq -1}\mathbb{L}_Z^\bullet)$ is surjective.

Propositions 4.22 and 4.23 give rise to the following statement which is the reason behind the previous definition.

Proposition 4.25 ([4, §5]). *Let $E_Z^\bullet$ be a perfect obstruction theory on Z and let $E_\bullet$ denote its dual; then there is an embedding*

$$\mathfrak{C}_Z \hookrightarrow h^1/h^0(E_\bullet) \simeq [E_1/E_0].$$

In particular, the special obstruction theory from Definition 4.15 gives rise to a perfect obstruction theory. To see this, note that by Construction 4.13 we get a map of complexes locally on an open set U

$$\begin{array}{ccc} (\tau^{\geq -1}\mathbb{L}_U^\bullet)^\vee \simeq \{T_Y|_U & \longrightarrow & N_{U/Y}\} \\ \downarrow & & \downarrow \\ \{T_Y|_U & \longrightarrow & E_{U/Y}\}; \end{array}$$

its dual map fulfills both conditions of Definition 4.24. As we are also given gluing data, these local maps glue to a perfect obstruction theory for Z.

How can we use a perfect obstruction theory to construct a virtual class?

Definition 4.26 ([3, §2]). Let Z be a scheme with a perfect obstruction theory $E_Z^\bullet$ and let $E_\bullet$ denote its dual complex. The class $[\mathfrak{C}_Z]$ sits inside $A_0([E_1/E_0])$. Then we define the *virtual class* of Z with respect to the obstruction theory $E_Z^\bullet$ as

$$[Z, E_Z^\bullet]^{vir} = 0^!_{[E_1/E_0]}[\mathfrak{C}_Z] \in A_{vd(E_Z^\bullet)}(Z),$$

where $vd(E_Z^\bullet) = \mathrm{rk}(E_0) - \mathrm{rk}(E_1)$ is the *virtual dimension.*

Notation 4.27. When it it is clear from the context we may also omit the perfect obstruction theory from the notation and just write $[Z]^{vir}$.

Remark 4.28. The original construction by Behrend–Fantechi was slightly different as the definition of Chow groups for Artin stacks was unavailable at that time.

Example 4.29 ([2, Rem. 5.12.2]). Let Z be a smooth scheme and Y a point. Then $\mathbb{L}^\bullet_{Z/Y} \simeq \Omega_Z$ and for any vector bundle E over Z, the complex

$$\left\{E \xrightarrow{0} \Omega_Z\right\}$$

is a perfect obstruction theory. We can regard Z as the zero locus of the zero section of E. Then by Proposition 4.12, the virtual class of Z is just

$$[Z]^{vir} = c_{\text{top}}(E) \frown [Z]$$

which is what we would expect.

This is an instance of a more general statement.

Definition 4.30 ([3, Def. 2.1]). Let Z be a scheme with a perfect obstruction theory $E^\bullet_Z$. We denote by

$$\text{Obs}_Z := h^1((E^\bullet_Z)^\vee)$$

the *obstruction sheaf* of Z (with respect to $E^\bullet_Z$).

Proposition 4.31 ([28]). *Let Z be a smooth scheme and let $E^\bullet_Z$ be a perfect obstruction theory with $\dim(Z) \geq vd(E^\bullet_Z)$ and obstruction sheaf Obs_Z. Then Obs_Z is a vector bundle of rank $vd(E^\bullet_Z) - \dim(Z)$ and*

$$[Z]^{vir} = c_{\text{top}}(\text{Obs}_Z) \frown [Z].$$

Proof. As Z is smooth, we can choose as an embedding the identity $\text{id}\colon Z \to Z$. Assume that $E^\bullet_Z$ is a two-term complex of locally free sheaves,

$$E^\bullet_Z = \{E^{-1} \to E^0\}.$$

The considerations about the cotangent complex above yield

$$\tau^{\geq -1}\mathbb{L}^\bullet_Z = \{0 \to \Omega_Z\}.$$

Then there is a map ϕ between the complexes

$$\begin{array}{ccc} E_Z^{-1} & \longrightarrow & E_Z^0 \\ {\scriptstyle \phi^{-1}}\downarrow & & \downarrow{\scriptstyle \phi^0} \\ 0 & \longrightarrow & \Omega_Z \end{array}$$

with the property that $h^{-1}(\phi)$ is surjective and $h^0(\phi)$ is an isomorphism. This yields an exact sequence

$$0 \to \ker(E^{-1} \to E^0) \to E^{-1} \to E^0 \to \Omega_Z \to 0.$$

Since Ω_Z is locally free, also the kernel on the left-hand side is locally free. Dualizing the sequence gives

$$0 \to T_Z \to E_0 \to E_1 \to \text{Obs}_Z \to 0$$

and we see that Obs_Z is locally free.
There exists a fiber square

$$\begin{array}{ccc} \left[\frac{C_{Z/Z} \times E_0}{T_Z}\right] & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ \mathfrak{C}_Z = \left[C_{Z/Z}/T_Z\right] & \longrightarrow & [E_1/E_0] \end{array} \tag{22}$$

and since $C_{Z/Z} = Z$ we see that the term in the upper left corner is $[E_0/T_Z] =: D$. Applying Proposition 4.11 gives

$$[Z]^{vir} = 0^!_{[E_1/E_0]}(\mathfrak{C}_Z) = 0^!_{E_1}(D) = c_{\text{top}}(E_1/D) \frown [Z] = c_{\text{top}}(\text{Obs}_Z) \frown [Z]. \qquad \square$$

Example 4.32 ([2, Ex. 3.3 & 6.3]). Let $Z = V(xz, yz) \subset \mathbb{P}^3 = \mathbf{Proj}(\mathbb{C}[x, y, z, w])$ be the union of the plane $H = \{z = 0\}$ with the line $L = \{x = y = 0\}$. First, we want to describe the normal cone $C_{Z/\mathbb{P}^3}$. On the affine chart $U_w = \{w \neq 0\}$, the coordinate ring of Z is given by

$$R = \mathbb{C}[x, y, z]/(xz, yz).$$

Consider the surjective map

$$\begin{array}{rcl} R[A, B] & \to & \bigoplus_{k \geq 0} \mathscr{I}^k_{Z/\mathbb{P}^3}/\mathscr{I}^{k+1}_{Z/\mathbb{P}^3} \\ A & \mapsto & xz \\ B & \mapsto & yz; \end{array}$$

then locally, on the chart U_w, the normal cone $C_{Z/\mathbb{P}^3}|_{U_w}$ is given by the kernel of this map,

$$C_{Z/\mathbb{P}^3}|_{U_w} = \text{Spec}(R[A, B]/(yA - xB)).$$

Globally, this gives an embedding

$$C_{Z/\mathbb{P}^3} \hookrightarrow \mathcal{O}_Z(2) \oplus \mathcal{O}_Z(2). \tag{23}$$

The normal cone has two components: over the line $L = \{x = y = 0\}$, the ideal generated by $yA - xB$ becomes zero, so

$$C_{Z/\mathbb{P}^3}|_L \cong \mathcal{O}_L(2) \oplus \mathcal{O}_L(2).$$

Over the plane $H = \{z = 0\}$ two situations can appear: over the origin the cone is isomorphic to the rank two vector bundle $\mathcal{O}_O(2) \oplus \mathcal{O}_O(2)$. Away from the origin, at least one of x and y is nonzero, so the cone is defined by $\text{Spec}(R[A])$ or $\text{Spec}(R[B])$ and is isomorphic to a line bundle $\mathcal{O}_H(2)$.

As we have already found an embedding of the cone into a vector bundle in (23),

$$E^\bullet_Z = \left\{\mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(-2) \xrightarrow{\delta} \Omega^1_{\mathbb{P}^3}|_Z\right\}$$

together with the obvious map

$$\phi\colon E^\bullet_Z \to \tau^{\geq -1}\mathbb{L}^\bullet_Z = \left\{\mathscr{I}_{Z/\mathbb{P}^3}/\mathscr{I}^2_{Z/\mathbb{P}^3} \to \Omega^1_{\mathbb{P}^3}|_Z\right\}$$

is a perfect obstruction theory. We can now calculate the virtual class of Z with respect to $E^\bullet_Z$. Using the analogous fiber diagram to (22) and applying Proposition 4.11, we obtain

$$i_*[Z, E^\bullet_Z]^{vir} = i_* 0^!_{[E_1/E_0]}(\mathfrak{C}_Z) = i_* 0^!_{E_1}\left(\left[\frac{C_{Z/\mathbb{P}^3} \times E_0}{T_{\mathbb{P}^3}|_Z}\right]\right)$$

$$= c_{\text{top}}(\mathcal{O}_Z(2) \oplus \mathcal{O}_Z(2)) \frown [Z] \in A_*(\mathbb{P}^3).$$

This last term can be computed as $[L] + 3[L']$ where $[L']$ is the class of a line in $H = \{z = 0\}$, see [2, Ex. 6.3] for details.

5. Obstruction theory and a virtual class for $P(X)$

The goal of this section is to construct a perfect obstruction theory for the moduli space of stable pairs to define a virtual fundamental class on $P(X)$. The results of Proposition 2.1 and Theorem 3.21 yield an embedding of $P(X)$ into the moduli space of perfect complexes in $D^b(X)$ with trivial determinant. The obstruction theory of the latter space is governed by $\mathrm{Ext}^*(I^\bullet, I^\bullet)_0$ and we will use this to define a virtual class on $P(X)$.

As the moduli space $P(X)$ is fine, it comes with a universal family of stable pairs

$$\mathcal{O}_{X\times P(X)} \to \mathbb{F}$$

on $X \times P(X)$. Its associated complex is denoted by

$$\mathbb{I}^\bullet = \{\mathcal{O}_{X\times P(X)} \to \mathbb{F}\} \in D^b(X \times P(X)).$$

Let π_X and $\pi_{P(X)}$ be the projections

$$\pi_X \colon X \times P(X) \to X \quad \text{resp.} \quad \pi_{P(X)} \colon X \times P(X) \to P(X)$$

and let

$$\omega_\pi = \pi_X^* K_X$$

be the relative dualizing sheaf of $\pi_{P(X)}$. Let $A^\bullet$ be a finite complex of locally free sheaves quasi-isomorphic to $\mathbb{I}^\bullet$. As already shown in the proof of Claim 3.4, the trace map induces a splitting

$$(A^\bullet)^\vee \otimes A^\bullet \cong \mathcal{O}_{X\times P(X)} \oplus ((A^\bullet)^\vee \otimes A^\bullet)_0,$$

where $((A^\bullet)^\vee \otimes A^\bullet)_0$ is the kernel of the trace map. The quasi-isomorphism class of trace-free endomorphisms is then defined by

$$R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \simeq ((A^\bullet)^\vee \otimes A^\bullet)_0.$$

The following is the key result in the construction of a virtual class.

Theorem 5.1. *There exists a map*

$$R\pi_{P(X)*}(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_\pi)[2] \to \tau^{\geq -1}\mathbb{L}^\bullet_{P(X)}$$

providing a perfect obstruction theory for $P(X)$.

Remark 5.2. The moduli space $P(X)$ is a projective variety, in particular there exists an embedding into a smooth ambient space Y. Then by Proposition 4.20 the truncated cotangent complex is given by

$$\tau^{\geq -1}\mathbb{L}^\bullet_{P(X)} = \left\{\mathscr{I}/\mathscr{I}^2 \xrightarrow{\delta} \Omega^1_Y|_{P(X)}\right\},$$

where δ is the differentiation map and $\mathscr{I}$ the ideal sheaf from the embedding $P(X) \hookrightarrow Y$.

In their original paper, Pandharipande and Thomas showed the existence of the map in Theorem 5.1 and that the complex on the left-hand side is quasi-isomorphic to a two-term complex of locally free sheaves with amplitude in $[-1, 0]$; this is what we also show in this thesis. The complete proof of Theorem 5.1 was given in [18, Cor. 4.3].

Lemma 5.3 ([30, Lem. 2.10])**.** *The complex $R\pi_{P(X)*}R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$ on $P(X)$ is quasi-isomorphic to a two-term complex of locally free sheaves $\{E_1 \to E_2\}$ with amplitude in* $[1, 2]$.

Proof. Let $B^\bullet$ be a finite locally free resolution of the complex $R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \simeq ((A^\bullet)^\vee \otimes A^\bullet)_0$ with the B^i sufficiently negative. For example, if $((A^\bullet)^\vee \otimes A^\bullet)_0$ consists of only one sheaf E, then there exist $m, N \in \mathbb{N}$ such that there is a surjection

$$\mathcal{O}_{X\times P(X)}^N \twoheadrightarrow E(m)$$

and the first term B^{-1} of the complex $B^\bullet$ is $\mathcal{O}_{X\times P(X)}(-m)^N$. Now $R^i\pi_{P(X)*}\mathcal{O}_{X\times P(X)}(-m)^N$ is isomorphic to $H^i(X, \mathcal{O}_X(-m)^N) \otimes \mathcal{O}_{P(X)}$ and therefore vanishes for all $i \neq \dim(X) = 3$. For $i = 3$ we obtain

$$R^3\pi_{P(X)*}\mathcal{O}_{X\times P(X)}(-m)^N \cong H^3(X, \mathcal{O}_X(-m)^N) \otimes \mathcal{O}_{P(X)} \cong H^0(X, \mathcal{O}_X(m)^N \otimes K_X)^\vee \otimes \mathcal{O}_{P(X)}$$

which is locally free. The same argument applies for all B^j if we choose the resolution to be sufficiently negative and also without the assumption that $((A^\bullet)^\vee \otimes A^\bullet)_0$ consists of only one sheaf. Therefore, we have that $R^3\pi_{P(X)*}B^j$ is locally free and $R^{\leq 2}\pi_{P(X)*}B^j = 0$ for all j. We define the complex $E^\bullet$ as

$$E^k := R^3\pi_{P(X)*}B^{k+3};$$

this is a finite complex consisting of locally free sheaves and is quasi-isomorphic to

$$E^\bullet \simeq R\pi_{P(X)*}R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0.$$

By the flat base change theorem, for any point $[I^\bullet] \in P(X)$ the restriction of $E^\bullet$ to this point is a complex of vector spaces computing $\mathrm{Ext}^*(I^\bullet, I^\bullet)_0$. By Lemma 2.5 we know $\mathcal{E}xt^{\leq -1}(I^\bullet, I^\bullet) = 0$ and Corollary 2.6 states $\mathcal{E}xt^{\leq -1}(I^\bullet, I^\bullet \otimes K_X) = 0$. Using the local-to-global spectral sequence and Serre duality, this implies that

$$\mathrm{Ext}^i(I^\bullet, I^\bullet) = 0 \quad \text{for } i \notin [0, 3].$$

By the second part of Corollary 2.6, $\mathcal{H}om(I^\bullet, I^\bullet \otimes K_X) \cong K_X$ and applying the local-to-global spectral sequence and the vanishing of the Ext-terms in negative degrees we get an isomorphism

$$H^0(K_X) \to \mathrm{Hom}(I^\bullet, I^\bullet \otimes K_X)$$

induced by the map i on $I^\bullet$, see Remark 3.3. Dualizing this yields an isomorphism

$$\mathrm{Ext}^3(I^\bullet, I^\bullet) \to H^3(\mathcal{O}_X)$$

which is precisely the trace map. Therefore, the trace-free part $\mathrm{Ext}^3(I^\bullet, I^\bullet)_0$ vanishes. Again, by Remark 3.3 the composition

$$\mathbb{C} \xrightarrow{i} \mathrm{Hom}(I^\bullet, I^\bullet) \xrightarrow{\mathrm{tr}} H^0(\mathcal{O}_X)$$

is multiplication by $\mathrm{rk}(I^\bullet) = 1$, so the trace map is an isomorphism and $\mathrm{Hom}(I^\bullet, I^\bullet)_0 = 0$. Therefore,

$$\mathrm{Ext}^i(I^\bullet, I^\bullet)_0 = 0 \quad \text{for } i \notin [1, 2]$$

and by base change to a point $[I^\bullet] \in P(X)$, the complex $E^\bullet$ has nonzero cohomology only in degree one and two. Now the technique already used in the proof of Lemma 3.11 can be applied to show that $E^\bullet$ is quasi-isomorphic to a complex of locally free sheaves concentrated in degree one and two: if E^n, $n \geq 3$ is the rightmost nonzero term of $E^\bullet$, the map

$$E^{n-1} \to E^n$$

is surjective on any point $[I^\bullet]$ as the cohomology $\mathrm{Ext}^n(I^\bullet, I^\bullet)_0$ vanishes. By Nakayama's Lemma this map is surjective globally and hence its kernel is locally free. Therefore, the complex $E^\bullet$ is quasi-isomorphic to

$$\cdots \to E^{n-2} \to \ker(E^{n-1} \to E^n) \to 0$$

and repeating this argument we can assume $n = 2$. Let E^m, $m \leq 0$ be the leftmost term of $E^\bullet$, then the map

$$E^m \to E^{m+1}$$

is injective over each point by the cohomology result above. Then analogously to the proof of Lemma 3.11 the cokernel is flat and hence locally free. Therefore, the complex $E^\bullet$ is quasi-isomorphic to

$$0 \to \mathrm{coker}(E^m \to E^{m+1}) \to E^{m+2} \to \cdots$$

and inductively we can assume $m = 1$. Concluding, $E^\bullet$ is quasi-isomorphic to a two-term complex $\{E_1 \to E_2\}$. □

We are now going to construct a map from $R\pi_{P(X)*}(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_\pi)[2]$ to the truncated cotangent complex.

Construction 5.4. Let Z denote the product $Z = X \times P(X)$, let Y be a smooth ambient variety and let $\mathscr{I}$ be the ideal sheaf of Z in Y. By Proposition 4.20, the truncated cotangent complex of Z is given by

$$\tau^{\geq -1}\mathbb{L}_Z^\bullet = \left\{\mathscr{I}/\mathscr{I}^2 \to \Omega_Y^1|_Z\right\}.$$

Let $\mathscr{J}_{\Delta_Y}$ be the ideal sheaf of the image of the diagonal embedding $i_{\Delta_Y}\colon Y \to Y \times Y$. There exists a map of complexes

$$\begin{array}{ccccccc} \mathcal{O}_{\Delta_Z} & \simeq & \{i_{\Delta_Z*}(\mathscr{I}/\mathscr{I}^2) & \longrightarrow & \mathscr{J}_{\Delta_Y}|_{Z\times Z} & \longrightarrow & \mathcal{O}_{Z\times Z}\} \\ \downarrow & & \| & & \downarrow & & \\ i_{\Delta_Z*}\tau^{\geq -1}\mathbb{L}_Z^\bullet[1] & \simeq & \{i_{\Delta_Z*}(\mathscr{I}/\mathscr{I}^2) & \to & \mathscr{J}_{\Delta_Y}/\mathscr{J}_{\Delta_Y}^2|_{Z\times Z}\} & & \end{array}$$

giving a class

$$\alpha_Z \in \mathrm{Ext}^1_{Z\times Z}(\mathcal{O}_{\Delta_Z}, i_{\Delta_Z*}\tau^{\geq -1}\mathbb{L}_Z^\bullet)$$

(see [18, §2] for a proof of well-definedness). Let $\pi_i\colon Z \times Z \to Z$ denote the projection to the i-th factor. Then by tensoring the class α_Z by $\pi_1^*(\mathbb{I}^\bullet)$ and pushing forward along π_{2*} (in other terms, we view α_Z as a map of Fourier–Mukai kernels, see [16, §5]) gives a class

$$A(\mathbb{I}^\bullet) \in \mathrm{Ext}^1_Z(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \tau^{\geq -1}\mathbb{L}_Z^\bullet)$$

called the *truncated Atiyah class* of $\mathbb{I}^\bullet$ [18, Def. 2.6]. The cotangent complex $\mathbb{L}_Z^\bullet$ of the product $Z = X \times P(X)$ is equal to the sum

$$\mathbb{L}_Z^\bullet = \mathbb{L}_{X\times P(X)}^\bullet = \pi_X^*\mathbb{L}_X^\bullet \oplus \pi_{P(X)}^*\mathbb{L}_{P(X)}^\bullet$$

(see [32, Tag 09DL]). Therefore, $A(\mathbb{I}^\bullet)$ is a class in

$$\begin{aligned} &\mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \pi_X^*\tau^{\geq -1}\mathbb{L}_X^\bullet) \oplus \mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \pi_{P(X)}^*\tau^{\geq -1}\mathbb{L}_{P(X)}^\bullet) \\ &\quad = \mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \pi_X^*\tau^{\geq -1}\mathbb{L}_X^\bullet) \oplus \mathrm{Ext}^1(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet), \pi_{P(X)}^*\tau^{\geq -1}\mathbb{L}_{P(X)}^\bullet), \end{aligned}$$

where we made use of (13). Hence, $A(\mathbb{I}^\bullet)$ projects to a class in

$$\mathrm{Ext}^1(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0, \pi_{P(X)}^*\tau^{\geq -1}\mathbb{L}_{P(X)}^\bullet)$$

which is isomorphic to

$$\mathrm{Ext}^{-2}(R\pi_{P(X)*}(R\mathscr{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\otimes\omega_\pi),\tau^{\geq -1}\mathbb{L}^\bullet_{P(X)})$$

by Grothendieck–Verdier duality. This class gives a map

$$\phi\colon R\pi_{P(X)*}(R\mathscr{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\otimes\omega_\pi)[2]\to\tau^{\geq -1}\mathbb{L}^\bullet_{P(X)}$$

which provides an obstruction theory. Note that by Lemma 5.3 and Grothendieck–Verdier duality,

$$R\pi_{P(X)*}(R\mathscr{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\otimes\omega_\pi)[2]$$

is quasi-isomorphic to a two-term complex of locally free sheaves with amplitude in $[-1,0]$. The proof that $h^{-1}(\phi)$ is surjective and $h^0(\phi)$ is an isomorphism is more complicated and was done in [18, Thm. 4.1], one year after the original paper by Pandharipande and Thomas.

The existence of a perfect obstruction theory immediately gives us a virtual class.

Corollary 5.5 ([30, Thm. 2.14]). *Every component $P_n(X,\beta)$ of the moduli space $P(X)$ carries a virtual fundamental class*

$$[P_n(X,\beta)]^{vir}\in A_{c_\beta}(P_n(X,\beta))$$

with respect to the perfect obstruction theory

$$\phi\colon R\pi_{P(X)*}(R\mathscr{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\otimes\omega_\pi)[2]\to\tau^{\geq -1}\mathbb{L}^\bullet_{P(X)}$$

constructed above. The virtual dimension is given by

$$c_\beta := -\chi(R\mathrm{Hom}(I^\bullet,I^\bullet)_0)=\int_\beta c_1(X) \tag{24}$$

for some complex $I^\bullet\in P_n(X,\beta)$.

Proof. The only thing to prove here is the statement about the virtual dimension. Let $E^\bullet$ denote the complex

$$E^\bullet = R\pi_{P(X)*}(R\mathscr{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\otimes\omega_\pi)[2]$$

and let $E_\bullet$ be its dual complex. By the construction of the virtual class, the virtual dimension is given by $\mathrm{rk}(E_0)-\mathrm{rk}(E_1)$. On the component $P_n(X,\beta)$ this is equal to $-\chi(R\mathrm{Hom}(I^\bullet,I^\bullet)_0)$.

The second equality follows from the Hirzebruch–Riemann–Roch Theorem. Let (F,s) be the stable pair corresponding to $I^\bullet$. Note that by Lemma 5.6 below we have $\mathrm{ch}_1(F)=c_1(F)=0$, so

$$\mathrm{ch}(F)=-c_2(F)+\frac{c_3(F)}{2}.$$

Then we compute

$$\begin{aligned}
\chi(R\mathrm{Hom}(I^\bullet,I^\bullet)) &= \chi((I^\bullet)^\vee\overset{L}{\otimes} I^\bullet)\\
&=\int_X \mathrm{ch}(I^\bullet)\mathrm{ch}((I^\bullet)^\vee)\mathrm{td}(X)\\
&=\int_X(1-\mathrm{ch}(F))(1-\mathrm{ch}(F^\vee))\mathrm{td}(X)\\
&=\int_X\left(1+c_2(F)-\frac{c_3(F)}{2}\right)\left(1+c_2(F)+\frac{c_3(F)}{2}\right)\mathrm{td}(X)\\
&=\int_X(1+2c_2(F))\left(1+\frac{c_1(X)}{2}+\mathrm{td}_2(X)+\mathrm{td}_3(X)\right).
\end{aligned}$$

Since by assumption

$$\begin{aligned}\chi(F(k)) &= k\int_\beta c_1(L) + n \\ &= \int_X \mathrm{ch}(F)(1 + kc_1(L))\mathrm{td}(X) \\ &= k\int_X -c_2(F)c_1(L) + \int_X \frac{1}{2}(-c_2(F)c_1(X) + c_3(F)),\end{aligned}$$

we see that

$$\int_X c_2(F)c_1(X) = -\int_\beta c_1(X)$$

and the computation above yields

$$\chi(R\mathrm{Hom}(I^\bullet, I^\bullet)) = -\int_\beta c_1(X) + \int_X \mathrm{td}_3(X) = -\int_\beta c_1(X) + \chi(\mathcal{O}_X).$$

Recall the splitting

$$R\mathscr{H}om(I^\bullet, I^\bullet) \cong \mathcal{O}_X \oplus R\mathscr{H}om(I^\bullet, I^\bullet)_0$$

from (11); therefore, we conclude

$$-\chi(R\mathrm{Hom}(I^\bullet, I^\bullet)_0) = \int_\beta c_1(X).$$

□

Lemma 5.6. *Let E be a d-dimensional coherent sheaf on an n-dimensional smooth projective scheme Y. Then, for $k < n-d$, the k-th graded piece $\mathrm{ch}_k(E)$ of the Chern character vanishes.*

Proof. Let Z be the closed subscheme $Z = \mathrm{Supp}(E) \subset Y$ and let $U = Y \setminus Z$ be its complement. By excision, there is an exact sequence

$$A^k(Z) \to A^k(Y) \xrightarrow{i^*} A^k(U) \to 0$$

for any k, where i denotes the inclusion. In particular, for $k < n - d$ the map i^* is an isomorphism. Since

$$i^*(\mathrm{ch}(E)) = \mathrm{ch}(E|_U) = 0,$$

we see that $\mathrm{ch}_k(E) = 0$ for $k < n - d$. □

The stable pair invariants defined above are deformation invariant with respect to smooth deformations of the target space X. This is proven by introducing a relative virtual class for a smooth family of projective threefolds and showing its compatibility with the virtual class on the fiber. The following proposition makes this precise. A proof is given in [18, Cor. 4.3] and relies on a pullback property for virtual classes [4, Prop. 7.2].

Proposition 5.7 ([30, Thm. 2.15]). *Let*

$$\pi\colon \mathcal{X} \to B$$

be a smooth family of projective threefolds over a base B and let

$$\mathcal{P}_n(\mathcal{X}, \beta) \to B$$

be the relative moduli space of stable pairs (i.e. for each fiber $\mathcal{X}_b$ of π we recover the moduli space $P_n(\mathcal{X}_b, \beta)$). We denote by

$$X = \mathcal{X} \times_B \{0\}$$

the central fiber and by

$$i_0 \colon P_n(X,\beta) \to \mathcal{P}_n(\mathcal{X},\beta)$$

the inclusion on the central fiber. Then there exists a π-relative virtual class

$$[\mathcal{P}_n(\mathcal{X},\beta)]^{vir} \in A_{c_\beta+\dim(B)}(\mathcal{P}_n(\mathcal{X},\beta))$$

such that

$$i_0^![\mathcal{P}_n(\mathcal{X},\beta)]^{vir} = [P_n(X,\beta)]^{vir}.$$

6. Computations and comparisons

If the space X is a Calabi–Yau threefold, i.e. $K_X = \mathcal{O}_X$, by the formula (24) we see that the virtual dimension c_β of $P_n(X,\beta)$ is zero for any integer n and any curve class β. Therefore, it makes sense to define counting invariants by taking the degree of the virtual class. These are the stable pair invariants.

Definition 6.1 ([30, Def. 2.16])**.** If $c_\beta = 0$ (in particular, if X is Calabi–Yau), the *stable pair invariants* $P_{n,\beta} \in \mathbb{Z}$ are defined to be the degree of the virtual class

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1.$$

6.1. **Computations.** Now we want to do some calculations. For this purpose it is very useful to observe that the obstruction theory used to define stable pair invariants is symmetric.

Definition 6.2 ([3, Def. 3.5])**.** A perfect obstruction theory $E_Z^\bullet \to \tau^{\geq -1}\mathbb{L}_Z^\bullet$ for a scheme Z is *symmetric* if $E_Z^\bullet$ is endowed with a non-degenerate symmetric bilinear form

$$\theta \colon E_Z^\bullet \to (E_Z^\bullet)^\vee[1].$$

Remark 6.3 ([3, Rem. 3.7]). If Z is smooth with a symmetric perfect obstruction theory $E_Z^\bullet \to \tau^{\geq -1}\mathbb{L}_Z^\bullet$, the obstruction sheaf computes as

$$\mathrm{Obs}_Z = h^1((E_Z^\bullet)^\vee) = h^0((E_Z^\bullet)^\vee[1]) \cong h^0(E_Z^\bullet) \cong \Omega_Z.$$

Lemma 6.4. *If X is a Calabi–Yau threefold, the perfect obstruction theory*

$$R\pi_{P(X)*}(R\mathscr{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0 \otimes \omega_\pi)[2] \to \tau^{\geq -1}\mathbb{L}_{P(X)}^\bullet$$

is symmetric.

Proof. As X is Calabi–Yau, the relative dualizing sheaf ω_π is trivial. Let $E^\bullet$ denote the complex

$$E^\bullet = R\pi_{P(X)*}(R\mathscr{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0)[2].$$

Then

$$(E^\bullet)^\vee[1] = (R\pi_{P(X)*}(R\mathscr{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0)[3])[-1] = E^\bullet$$

by Grothendieck–Verdier duality. □

Now we return to Example 1.22 where X is the total space of the bundle

$$\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to \mathbb{P}^1.$$

We have already seen that the moduli space $P_n(X,[\mathbb{P}^1])$ is isomorphic to

$$P_n(X,[\mathbb{P}^1]) \cong \mathbb{P}^{n-1}.$$

As this moduli space is smooth, we can use Proposition 4.31 and compute the virtual class via the obstruction sheaf $\mathrm{Obs}_{\mathbb{P}^{n-1}} \cong \Omega_{\mathbb{P}^{n-1}}$ (by Remark 6.3). Therefore,

$$[P_n(X,[\mathbb{P}^1])]^{vir} = c_{\mathrm{top}}(\Omega_{\mathbb{P}^{n-1}}) \frown [\mathbb{P}^{n-1}] = (-1)^{n-1}c_{n-1}(T_{\mathbb{P}^{n-1}}) \frown [\mathbb{P}^{n-1}]$$

and the stable pair invariants are

$$P_{n,[\mathbb{P}^1]} = \int_{[P_n(X,[\mathbb{P}^1])]^{vir}} 1 = (-1)^{n-1}\int_{[\mathbb{P}^{n-1}]} c_{n-1}(T_{\mathbb{P}^{n-1}}) = (-1)^{n-1}\chi_{\mathrm{top}}(\mathbb{P}^{n-1}) = (-1)^{n-1}n,$$

where χ_{top} denotes the topological Euler characteristic.

We can arrange these invariants in a generating series

$$Z_{X,[\mathbb{P}^1]}(q) := \sum_n P_{n,[\mathbb{P}^1]}q^n = q - 2q^2 + 3q^3 \mp \cdots = \frac{q}{(1+q)^2}.$$

These generating series play a crucial role in the study of curve counting invariants as it is possible to convert different invariants, e.g. Donaldson–Thomas, Gromov–Witten and stable pair invariants, into each other by manipulation of the corresponding generating series, see for instance [30, §3] and the discussion in §6.2 below.

For a smooth curve $C \subset X$ the space $P_n^C(X,[C])$ parametrizing stable pairs with fixed support C is a component of $P_n(X,[C])$ and we call the invariants given by $\int_{[P_n^C(X,[C])]^{vir}} 1$ the *contribution* of C to the stable pair invariants. In the following we compute the corresponding generating series.

Proposition 6.5 ([30, §4.2]). *Let X be a Calabi–Yau threefold and let $C \subset X$ be a smooth embedded curve of genus g. Then the contribution of C to the stable pair invariants is given by*

$$Z^C_{X,[C]}(q) = q^{1-g}(1+q)^{2g-2}.$$

Proof. Let (F,s) be a stable pair on X with $\mathrm{Supp}(F) = C$. The Hilbert polynomial of F is given by

$$P_F(k) = k\int_{[C]} c_1(\mathcal{O}_X(1)) + n = k + n$$

so F is pure of rank one. Hence, F is a line bundle $\mathcal{O}_C(D)$ for an effective divisor D on C and s is the canonical section s_D. By the Riemann–Roch Theorem, the constant term is given by

$$n = \chi(\mathcal{O}_C(D)) = 1 - g + d,$$

where $d = \deg(D)$. Therefore, the moduli space $P^C_{1-g+d}(X,[C])$ is just given by the space of effective divisors of degree d on C, i.e.

$$P^C_{1-g+d}(X,[C]) = \mathrm{Sym}^d(C).$$

This is a smooth space, so by Remark 6.3 the obstruction sheaf is given by $\mathrm{Obs}_{\mathrm{Sym}^d(C)} = T^\vee_{\mathrm{Sym}^d(C)}$ and by Proposition 4.31 we obtain

$$Z^C_{X,[C]} = \sum_{d\geq 0} q^{1-g+d}(-1)^d\chi_{\mathrm{top}}(\mathrm{Sym}^d(C)).$$

The desired statement follows once we have shown

$$\sum_{d\geq 0}(-1)^d\chi_{\mathrm{top}}(\mathrm{Sym}^d(C))q^d = (1+q)^{2g-2}.$$

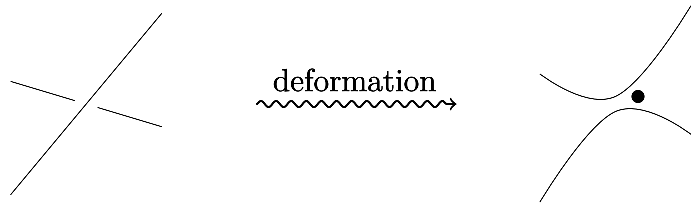

FIGURE 6.1. A point breaking off from a curve in a deformation process, see [31, p. 18]).

We will now show that

$$\sum_{d\geq 0} \chi_{\mathrm{top}}(\mathrm{Sym}^d(Y))q^d = (1-q)^{-\chi_{\mathrm{top}}(Y)}$$

holds for any topological space Y. As $\chi_{\mathrm{top}}(C) = 2-2g$, the proof is then finished.

First note that since

$$\mathrm{Sym}^d(Y) = Y^d/\mathfrak{S}_d,$$

the Euler characteristic $\chi_{\mathrm{top}}(\mathrm{Sym}^d(Y))$ only depends on $\chi_{\mathrm{top}}(Y)$. The identity

$$\mathrm{Sym}^d(Y \sqcup \{p\}) \cong \mathrm{Sym}^d(Y) \sqcup \mathrm{Sym}^{d-1}(Y \sqcup \{p\})$$

holds and by the additivity of the Euler characteristic we get

$$\sum_{d\geq 0} \chi_{\mathrm{top}}(\mathrm{Sym}^d(Y))q^d = (1-q)\sum_{d\geq 0} \chi_{\mathrm{top}}(\mathrm{Sym}^d(Y \sqcup \{p\}))q^d.$$

As $\chi_{\mathrm{top}}(Y \sqcup \{p\}) = \chi_{\mathrm{top}}(Y)+1$, by repeating the above procedure we can assume $\chi_{\mathrm{top}}(Y) \geq 0$. Since the desired statement only depends on $\chi_{\mathrm{top}}(Y)$, we may therefore assume Y consists only of $\chi_{\mathrm{top}}(Y)$ many points. In this case, $\mathrm{Sym}^d(Y)$ consists of unordered tuples of d points chosen from $\chi_{\mathrm{top}}(Y)$ many points (with repetition allowed), so

$$\chi_{\mathrm{top}}(\mathrm{Sym}^d(Y)) = |\,\mathrm{Sym}^d(Y)| = \binom{\chi_{\mathrm{top}}(Y)-1+d}{d}$$

and the generating series becomes

$$\sum_{d\geq 0} \chi_{\mathrm{top}}(\mathrm{Sym}^d(Y))q^d = \sum_{d\geq 0}\binom{\chi_{\mathrm{top}}(Y)-1+d}{d}q^d = (1-q)^{-\chi_{\mathrm{top}}(Y)}.$$

□

6.2. **Relations to other curve counting invariants.** If we want to count smooth curves embedded into a space X there are other approaches than stable pair invariants originating in different compactifications of the moduli space of curves. Note that if we want deformation invariant counting invariants, it is not reasonable to consider only the moduli space of smooth curves as smooth curves can degenerate to singular curves under deformations. In the following we will very briefly discuss Donaldson–Thomas and BPS invariants without giving proofs; for a general overview of curve counting invariants, see [31], which is also the main reference for this section.

Let $I_n(X,\beta)$ be the Hilbert scheme parametrizing subschemes $Z \subset X$ with $\chi(\mathcal{O}_Z) = n$ and $[Z] = \beta \in H_2(X,\mathbb{Z})$. For X a Calabi–Yau threefold one can construct a virtual class

$[I_n(X,\beta)]^{vir}$ of virtual dimension zero. The *Donaldson–Thomas invariants* (in the following named DT invariants for short) are then defined by taking the degree of this class, i.e.

$$I_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1.$$

The space $I_n(X,\beta)$ does not only parametrize curves but also points breaking off in the deformation process, see Figure 6.1.

Roughly speaking, the difference between stable pair and DT invariants is that the latter allow points wandering freely on X whereas the former restrict points to the curve.

To compare DT with stable pair invariants we introduce the generating series

$$Z^{DT}_{X,\beta}(q) = \sum_n I_{n,\beta} q^n$$

for the DT invariants. As we are not interested in the free points we consider the reduced series

$$\widetilde{Z}^{DT}_{X,\beta} = \frac{Z^{DT}_{X,\beta}(q)}{Z^{DT}_{X,0}(q)}.$$

The series $Z^{DT}_{X,0}(q)$ computing the zero degree invariants is actually of a very nice form as the following proposition states.

Proposition 6.6 ([5, Thm. 4.12])**.** *The zero degree DT invariants are given by*

$$Z^{DT}_{X,0}(q) = M(-q)^{\chi_{top}(X)},$$

where M is the MacMahon function

$$M(q) = \prod_{n\geq 1} (1-q^n)^{-n}.$$

The following relationship between DT and stable pair invariants was first conjectured by Pandharipande and Thomas [30, Conj. 3.3] and proven by Bridgeland [6, Thm. 1.1].

Theorem 6.7 (DT/pairs correspondence)**.** *Let X be a Calabi–Yau threefold. Then for any curve class $\beta \in H_2(X,\mathbb{Z})$, the reduced generating series for Donaldson–Thomas invariants and the generating series for stable pair invariants coincide,*

$$\widetilde{Z}^{DT}_{X,\beta}(q) = Z^{PT}_{X,\beta}(q).$$

It is worth mentioning that there is also a conjectured correspondence with the generating series of Gromov–Witten invariants via the change of variable $-q = e^{iu}$, proven in a large class of cases [29].

There is another sort of counting invariants related to curve counting motivated by string theory, namely BPS state counts. Gopakumar and Vafa defined them by counting certain D-branes[3] [12, 13]. BPS states, representing in certain cases extremal black holes[4], are promising candidates for the microstates associated to the entropy of a black hole. The entropy is then related to the number of these states and indeed it is possible to calculate the entropy of a black hole via BPS invariants [33].

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[3]In string theory, D-branes are the objects to which the ends of an open string satisfying the Dirichlet boundary conditions are fixed.

[4]If the mass of a black hole is above a certain bound it radiates and looses mass until it reaches the bound and becomes stationary—these are extremal black holes.

Unfortunately, up to now there is no rigorous mathematical framework to make the physical notions above precise [31, §2$\frac{1}{2}$] (as far as the author of this thesis is informed, this has not changed since the publication of [31]). Nevertheless, the string theoretic Gopakumar–Vafa formula [13, §2] relates the BPS state counts to Gromov–Witten invariants (historically, this was also the reason why one expected Gromov–Witten theory to be governed by integers). The correspondence with stable pair invariants discussed above motivates the following new definition for BPS invariants via a formula very similar to the one obtained by Gopakumar and Vafa.

Definition/Proposition 6.8 ([30, §3.4])**.** Let X be a Calabi–Yau threefold and let $F_X^{PT}(q,v) \in \mathbb{Q}(\!(q,v)\!)$ be the Laurent series defined by

$$F_X^{PT}(q,v) = \log\left(1 + \sum_{\beta\neq 0} Z_{X,\beta}^{PT}(q)v^\beta\right).$$

Then $F_X^{PT}(q,v)$ can be uniquely written as

$$F_X^{PT}(q,v) = \sum_{g>-\infty}\sum_{\gamma\neq 0}\sum_{d\geq 1} n_{g,\gamma}\frac{(-1)^{1-g}}{d}(-q)^{d(1-g)}(1-(-q)^d)^{2g-2}v^{d\gamma}.$$

The BPS invariants are defined as the $n_{g,\gamma}$ in the above expression.

Remark 6.9. One can show that the $n_{g,\gamma}$ are integer-valued for all $g, \gamma \neq 0$ [30, Thm. 3.20].

Lastly, we like to mention the following expression.

Proposition 6.10 ([31, Eqn. (4.9)])**.** *Let X be a Calabi–Yau threefold. If the curve class $\beta \in H_2(X,\mathbb{Z})$ is irreducible, i.e. any nontrivial algebraic curve classes $\alpha, \gamma \in H_2(X,\mathbb{Z})$ do not sum up to $\alpha + \gamma = \beta$, then the following holds:*

$$Z_{X,\beta}^{PT}(q) = \sum_{g\geq 0} n_{g,\beta}q^{1-g}(1+q)^{2g-2}.$$

In view of Proposition 6.5, we can therefore say that stable pair invariants "recognize" an irreducible class β as a disjoint union of $n_{g,\beta}$ many smooth curves of genus g.

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