Nodal quintic surfaces and the Fano variety of lines

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Nodal quintic surfaces and the Fano variety of lines

$X \subset \mathbb{P}^5$ smooth cubic fourfold $\rightsquigarrow$ $F(X)$= Fano variety of lines $L \subset X$

$L \in F(X)$ (generic) line $\rightsquigarrow$ $F_L := \{L' \mid L \cap L' \neq \emptyset\} \subset F(X)$

Today: Survey of old & new results concerning F_L ... and some open questions

History: Fano 1904, ..., Beauville 1979, Catanese 1981, ...,

..., **Voisin** [Théorème de Torelli pour les cubiques de $\mathbb{P}^5$, 1986],

..., Izadi 1999, Shen 2012, ...

(i) Geometry of F_L as a surface: H^*, CH^*, $\mathfrak{h}^*$, ...

(ii) Link $X \leftrightarrow F_L$.

Comparison with two other situations: $A \to A/\pm$ & $C_L \subset F(Y \subset \mathbb{P}^4)$

(A) A= principally polarized abelian surface

$$\rightsquigarrow \quad A \to S := A/\pm \subset \mathbb{P}^3,$$

$\in |\mathcal{O}(4)|$ singular Kummer surface with 16 ODP

$$H^*(A,\mathbb{Z})^- = H^1(A,\mathbb{Z}) \oplus H^3(A,\mathbb{Z}) \quad \& \quad \mathrm{CH}^*(A)^- \cong A \times \hat{A}$$

(B) $Y \subset \mathbb{P}^4$ smooth cubic threefold $\rightsquigarrow$ $C_L := \{L' \mid L \cap L' = \mathrm{pt}\}$

$$\rightsquigarrow \quad C_L \to D_L := C_L/\iota \subset \mathbb{P}^2,$$

$\in |\mathcal{O}(5)|$ smooth curve, $g(D_L) = 6$

$$H^*(C_L,\mathbb{Z})^- \cong H^3(Y,\mathbb{Z})(1) \quad \& \quad \mathrm{CH}(C_L)^- \cong \mathrm{Prym}(C_L/D_L)$$

Comparison: Abelian surfaces vs Fano variety of lines I

(A, Θ)= pp abelian surface

- $\varphi_{2\Theta} : A \to S = A/\pm \subset \mathbb{P}^3$ & 16 ODP
- Start with $S \subset \mathbb{P}^3$, $\in |\mathcal{O}(4)|$ & 16 ODP

$\tilde{S} = \mathrm{Bl}_{\{x_i\}}(S) \to S$, $E_i \mapsto x_i \quad \Rightarrow \quad \tilde{S} = \mathrm{K3}$

$b_2(\tilde{S}) = 22$, $b_2(S) = 6$

Automatic: $\mathcal{O}(\bigcup E_i) \cong \mathcal{L}^2$

- $\rightsquigarrow$ $\begin{array}{ccc} \tilde{A} & \longrightarrow & A \\ \downarrow & & \downarrow \pi \\ \tilde{S} & \longrightarrow & S \end{array}$ $\circlearrowright \pm$ 16 fixed points

$\omega_A \cong \pi^* \omega_S \cong \mathcal{O}_A$

$\pi^* \mathcal{O}(1) \cong \mathcal{O}(2\Theta)$

- Start with $D \subset \mathbb{P}^3$, $\in |\mathcal{O}(5)|$ & 16 ODP

$\tilde{D} = \mathrm{Bl}_{\{x_i\}}(D) \to D$, $E_i \mapsto x_i \Rightarrow \tilde{D}$ general type

$b_2(\tilde{D}) = 53$, $b_2(D) = 37$

Condition: $\mathcal{O}(\bigcup E_i) \cong \mathcal{L}^2$

Catanese (forthcoming): Not automatic!

- $\rightsquigarrow$ $\begin{array}{ccc} \tilde{F} & \longrightarrow & F \\ \downarrow & & \downarrow \pi \\ \tilde{D} & \longrightarrow & D \end{array}$ $\circlearrowright \iota$ 16 fixed points

$\omega_F \cong \pi^* \omega_D \cong \pi^* \mathcal{O}(1)$

$\pi^* \mathcal{O}(1)$ primitive: use $(\pi^* \mathcal{O}(1))^2 = 2 \cdot 5$

Comparison: Abelian surfaces vs Fano variety of lines II

- $\langle E_i \rangle \subset K_S \subset H^2(\tilde{S}, \mathbb{Z})$, $\mathrm{disc}(K_S) = 2^6$
- $H^2(A,\mathbb{Z}) \cong \bigwedge^2 H^1(A,\mathbb{Z}) \cong \mathbb{Z}^{\oplus 6}$
- $H^2(A,\mathbb{Z})^+ = H^2(A,\mathbb{Z}) \quad \cong U^{\oplus 3}$

  $\cong (H^2(S,\mathbb{Z})\ \&\ 2(\ .\ ))$

- $H^2(A,\mathbb{Z})^+_{\mathrm{pr}} \cong U^{\oplus 2} \oplus \mathbb{Z}(-2)$
- $H^*(A,\mathbb{Z})^- = H^1(A,\mathbb{Z}) \oplus H^3(A,\mathbb{Z})$

  $= \mathrm{Ker}(1+\iota^*) \supsetneq \mathrm{Im}(1-\iota^*)$

$H^*(A,\mathbb{Z})^+ \oplus H^*(A,\mathbb{Z})^- = H^*(A,\mathbb{Z})$

- $\langle E_i \rangle \subset K_D \subset H^2(\tilde{D}, \mathbb{Z})$, $\mathrm{disc}(K_D) = \ ??$
- $H^2(F,\mathbb{Z}) \cong \mathbb{Z}^{\oplus 60}$ torsion free
- $H^2(F,\mathbb{Z})^+ \cong \mathbb{Z}^{\oplus 37} \quad \cong\ ??$, sign=(9,28)

  $\supset (H^2(D,\mathbb{Z})\ \&\ 2(\ .\ ))$

  $=$ on algebraic part

- $H^2(F,\mathbb{Z})^+_{\mathrm{pr}} \cong\ ??$, sign=(8,28)
- $H^*(F,\mathbb{Z})^- = H^2(F,\mathbb{Z})^- \cong\ ??$

  $= \mathrm{Ker}(1+\iota^*) = \mathrm{Im}(1-\iota^*)$

$H^*(F,\mathbb{Z})^+ \oplus H^*(F,\mathbb{Z})^- \subset H^*(F,\mathbb{Z})$

Coker = $(\mathbb{Z}/2\mathbb{Z})^{\oplus 23}$

Proposition $F \xrightarrow{2:1} D \subset \mathbb{P}^3$ even 16-nodal quintic

$\Rightarrow H^2(F,\mathbb{Z})^-$ = K3 Hodge structure: $p_g^- = 1$, sign=(2,21)

Comparison: Abelian surfaces vs Fano variety of lines III

- $\pi_1(S) = \{1\}$
- $\pi_1(A) = \mathbb{Z}^{\oplus 4}$
- $H_1(A, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 4}$

- $\mathrm{CH}(A)^- \cong A \times \hat{A}$
- $\mathfrak{h}(A)^- \cong \mathfrak{h}^1(A) \oplus \mathfrak{h}^3(A)$

- $\pi_1(D) = \{1\}$
- $\pi_1^{\mathrm{alg}}(F) = \{1\}$, Ottem: $\pi_1(F) = \{1\}$ via Fano
- $H_1(F, \mathbb{Z}) = 0$

- $\mathrm{CH}(F)^- \cong \mathrm{CH}_0(F)^- \oplus \mathrm{CH}_1(F)^-$
- $\mathfrak{h}(F)^- \cong \mathfrak{h}^2(F)^-$

Naive and obvious questions:

(i) $\exists$ Beauville–Voisin class $c_{\mathrm{BV}} \in \mathrm{CH}(F)^+$?

(ii) $\mathrm{Pic}(F)^- \times \mathrm{Pic}(F)^- \to \mathbb{Z} \cdot c_{\mathrm{BV}}$?

(iii) $\dim \mathfrak{h}^2(F)^- < \infty$??

Comparison: Cubic threefolds vs cubic fourfolds I

$Y \subset \mathbb{P}^4$ smooth cubic

- $F = F(Y)$ surface, $\omega_{F(Y)} \cong \mathcal{O}(1)$ ample
- $H^*(F,\mathbb{Q}) \cong \bigwedge^* H^1(F,\mathbb{Q})/(P_3)$
- $\bigwedge^2 H^1(F,\mathbb{Z}) \hookrightarrow H^2(F,\mathbb{Z}) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$

[Beauville]

- $L \in F(Y)$ generic

$$\rightsquigarrow \quad C_L := \{L' \mid L \cap L' = \mathrm{pt}\}$$

- C_L smooth curve, $g(C_L) = 11$
- $\mathcal{O}(2)|_{C_L} \cong \omega_{C_L} \otimes q^*\mathcal{O}(2)$
- $(C_L.C_L) = 5, \quad 3[C_L] = g_{\mathrm{Pl}}$

$$C_L \xrightarrow[q]{5:1} L$$

- Y general $\Rightarrow$

$$H^{1,1}(F(Y),\mathbb{Q}) = \mathbb{Q}\cdot[C_L] = \mathbb{Q}\cdot g_{\mathrm{Pl}}$$

$X \subset \mathbb{P}^5$ smooth cubic

- $F = F(X)$ smooth 4-fold, $\omega_F \cong \mathcal{O}$, HK $\approx \mathrm{K3}^{[2]}$ [BD]
- $H^*(F,\mathbb{Q}) \cong S^*H^2(F,\mathbb{Q})/(H_3)$
- $S^2H^2(F,\mathbb{Z}) \hookrightarrow H^4(F,\mathbb{Z}) \twoheadrightarrow (\mathbb{Z}/2)^{\oplus 23} \times \mathbb{Z}/5$

[B,N-W,S]

- $L \in F(X)$ generic

$$\rightsquigarrow \quad F_L := \{L' \mid L \cap L' \neq \emptyset\}$$

- F_L smooth surface, ω_{F_L} ample, $p_g = 5$
- $\mathcal{O}(1)|_{F_L} \cong \boldsymbol{\omega_{F_L}} \otimes q^*\mathcal{O}(1)$
- $(F_L.F_L) = 5$

$$F_L \overset{q}{\dashrightarrow} L \quad (g=4)$$

- X general $\Rightarrow$

$$H^{2,2}(F(Y),\mathbb{Q}) = \mathbb{Q}\cdot[F_L] \oplus \mathbb{Q}\cdot g_{\mathrm{Pl}}^2$$

Comparison: Cubic threefolds vs cubic fourfolds II

$$\begin{array}{ccc} L' & C_L & \circlearrowright \iota \\ \downarrow & \downarrow & \text{2:1, étale} \\ \overline{LL'} \cap \mathbb{P}^2 & D_L & \subset \mathbb{P}^2, \in |\mathcal{O}(5)| \end{array}$$

D_L smooth, $g(D_L) = 6$

- $H^*(C_L, \mathbb{Z})^- = H^1(C_L, \mathbb{Z})^-$

$$H^1(C_L,\mathbb{Z})^+ \hookrightarrow H^1(C_L,\mathbb{Z}) \twoheadrightarrow H^1(C_L,\mathbb{Z})^-$$
$\cup$ ind=2
$$H^1(D_L,\mathbb{Z})$$

- $H^1(C_L,\mathbb{Z})^+ \oplus H^1(C_L,\mathbb{Z})^- \subset H^1(C_L,\mathbb{Z})$

  Coker = $(\mathbb{Z}/2\mathbb{Z})^{\oplus 10}$

- $\mathrm{CH}^-(C_L) = \mathrm{Prym}(C_L/D_L)$

  $= \mathrm{Im}(1-\iota^*) \subsetneq \mathrm{Ker}(1+\iota^*)$

- $\mathfrak{h}^-(C_L) = \mathfrak{h}^1(C_L)^-$

$$\begin{array}{ccc} L' & F_L & \circlearrowright \iota \quad \text{with 16 fixed points} \\ \downarrow & \downarrow & \text{2:1} \\ \overline{LL'} \cap \mathbb{P}^3 & D_L & \subset \mathbb{P}^3, \in |\mathcal{O}(5)| \end{array}$$

D_L has 16 ODP, $p_g(D_L) = 5$

- Recall: Dimension count

$$H^2(F_L,\mathbb{Z})^+ \hookrightarrow H^2(F_L,\mathbb{Z}) \twoheadrightarrow H^2(F_L,\mathbb{Z})^-$$
$\cup$ ind= ??
$$H^2(D_L,\mathbb{Z})$$

- $H^2(F_L,\mathbb{Z})^+ \oplus H^2(F_L,\mathbb{Z})^- \subset H^2(F_L,\mathbb{Z})$

  Coker = $(\mathbb{Z}/2\mathbb{Z})^{\oplus 23}$

- $\mathrm{CH}^-(F_L) = \mathrm{CH}_0(F_L)^- \oplus \mathrm{CH}_1(F_L)^-$

  $= \mathrm{Im}(1-\iota^*) = \mathrm{Ker}(1+\iota^*)$

- $\mathfrak{h}^-(F_L) = \mathfrak{h}^2(F_L)^-$

Comparison: Cubic threefolds vs cubic fourfolds III

- $H^1(F(Y),\mathbb{Z}) \overset{[M]}{\cong} H^1(C_L,\mathbb{Z})^-$ pol.

$\wr\|$

$H^3(Y,\mathbb{Z})(1)$

Global Torelli

For generic lines $L \subset Y$ & $L' \subset Y'$:

$Y \cong Y' \Leftrightarrow \exists\, H^1(C_L,\mathbb{Z})^- \cong H^1(C_{L'},\mathbb{Z})^-$
Hodge isometry

[Beauville,Clemens–Griffiths, Mumford,Tyurin]

$g_{\mathrm{Pl}} \rightsquigarrow H^2(F_L,\mathbb{Z})^{\pm}_{\mathrm{pr}} \subset H^2(F_L,\mathbb{Z})^{\pm}$

- $(H^2(F(X),\mathbb{Z})_{\mathrm{pr}}, q_{\mathrm{BBF}}) \cong (H^2(F_L,\mathbb{Z})^-_{\mathrm{pr}}, \frac{1}{2}(\ .\ ))$

$\wr\|$ [BD] [Izadi–Shen–H.]

$H^4(X,\mathbb{Z})_{\mathrm{pr}}(1)$ [Kuznetsov: Conic bundles]

- $H^2(F_L,\mathbb{Z})^-_{\mathrm{pr}} \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2(-1)$ sign=(2,20)

$\cap$

$H^2(F_L,\mathbb{Z})^- \cong$?? sign=(2,21)

Global Torelli

For generic lines $L \subset X$ & $L' \subset X'$:

$X \cong X' \Leftrightarrow \exists\, H^2(F_L,\mathbb{Z})^-_{\mathrm{pr}} \cong H^2(F_{L'},\mathbb{Z})^-_{\mathrm{pr}}$
Hodge isometry

Note: $H^2(F_L,\mathbb{Z})^-_{\mathrm{pr}} \equiv \mathrm{const}$ for $L \subset X = \mathrm{fixed}$

Comparison: Cubic threefolds vs cubic fourfolds IV

- $\mathrm{CH}_0(C_L)^- \cong \mathrm{CH}_1(Y)_{\mathrm{hom}}$ [Murre]

- $\mathfrak{h}^2(C_L)^- \cong \mathfrak{h}^3(Y)$

  [Manin, Sermenev, Nagel–Saito]

- $\mathrm{CH}_0(F_L)^- \cong \mathrm{CH}_1(X)_{\mathrm{hom}}$

If Kuznetsov component $\mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(S = \mathrm{K3})$

$\Rightarrow \quad \mathrm{CH}_0(F_L)^- \cong \mathrm{CH}_1(X)_{\mathrm{hom}} \cong \mathrm{CH}_0(S)_{\mathrm{hom}}$

- $\mathfrak{h}^2(F_L)^-_{\mathrm{pr}} \cong \mathfrak{h}^4(X)_{\mathrm{pr}}(1)$

If Kuznetsov component $\mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(S = \mathrm{K3})$

$\Rightarrow \quad \mathfrak{h}^2(F_L)^-_{\mathrm{pr}} \cong \mathfrak{h}^4(X)_{\mathrm{pr}}(1) \cong \mathfrak{h}^2(S)_{\mathrm{pr}}$

Conclusion The Fano variety $F(X)$ is always associated with the 'half' F_L^- of F_L which behaves like a K3 surface.

Open questions

Lattice & Hodge theory $H^2(F_L,\mathbb{Z})^+ \cong H^2(D_L,\mathbb{Z})$? Kummer lattice $K_D = $?

Deformations $H^1(F_L,\mathcal{T}) = $?

Hassett divisors $X \in \mathcal{C}_d$ s.t. $(**)$

$$\rightsquigarrow \quad H^2(T = \mathrm{K3},\mathbb{Z})_{\mathrm{pr}} \hookrightarrow H^2(F(X),\mathbb{Z})_{\mathrm{pr}} \cong H^2(F_L,\mathbb{Z})^-_{\mathrm{pr}}$$

Are there geometric realizations of $T \leftrightarrow F_L$ (surfaces) ? Typically, $T \rightsquigarrow$ divisor in $F(X)$.

Where is $\mathrm{D^b}(\mathrm{Coh})$ **?**

Recall [BKR]: $\mathrm{D^b}(C_L)^+ \cong \mathrm{D^b}(D_L)$ and $\mathrm{D^b}(A)^+ \cong \mathrm{D^b}(\tilde{S} = \mathrm{K3})$ and $\mathrm{D^b}(F_L)^+ \cong \mathrm{D^b}(\tilde{D}_L)$.

Are there natural categories $\mathrm{D^b}(C_L)^-$ or $\mathrm{D^b}(A)^-$ or $\mathrm{D^b}(F_L)^-$?

Cubic fourfolds: Kuznetsov component $\mathcal{A}_X \subset \mathrm{D^b}(X)$: $\mathcal{A}_X \leftrightarrow \mathrm{D^b}(F_L)^-$?