# HYPER-KÄHLER MANIFOLDS OF GENERALIZED KUMMER TYPE AND THE KUGA-SATAKE CORRESPONDENCE

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Abstract. We first describe the construction of the Kuga-Satake variety associated to a (polarized) weight 2 Hodge structure of hyper-Kähler type. We describe the classical cases where the Kuga-Satake correspondence between a hyper-Kähler manifold and its Kuga-Satake variety has been proved to be algebraic. We then turn to recent work of O'Grady and Markman which allow to prove that the Kuga-Satake correspondence is algebraic for projective hyper-Kähler manifolds of generalized Kummer deformation type.

## 1. Introduction

The Kuga-Satake construction associates to any $K3$ surface, and more generally to any weight 2 Hodge structure of hyper-Kähler type, a complex torus which is an abelian variety when the Hodge structure is polarized. This construction allows to realize the Hodge structure on degree 2 cohomology of a projective hyper-Kähler manifold as a direct summand of the $H^2$ of an abelian variety. Although the construction is formal and not known to be motivic, it has been used by Deligne in [2] to prove deep results of a motivic nature, for example the Weil conjectures for $K3$ surfaces can be deduced from the Weil conjectures for abelian varieties.

Section 2 of the notes is devoted to the description of the original construction, and to the presentation of a few classical examples where the Kuga-Satake correspondence is known to be algebraic, i.e. realized by a correspondence between the hyper-Kähler manifold and its Kuga-Satake variety. In Section 3, we will focus on the case of hyper-Kähler manifolds of a generalized Kummer type, and present a few recent results. If $X$ is a (very general) projective hyper-Kähler manifold of generalized Kummer type, the Kuga-Satake variety $KS(X)$ built on $H^2(X,\mathbb{Z})_{\rm tr}$ is a sum of copies of an abelian fourfold $KS(X)_c$ of Weil type. There is another abelian fourfold associated to $X$, namely the intermediate Jacobian $J^3(X)$ which is defined as the complex torus

$$J^3(X) = H^{1,2}(X)/H^3(X,\mathbb{Z})$$

where $b_3(X) = 8$. Here we use the fact that $H^{3,0}(X) = 0$ and the projectivity of $X$ guarantees that $J^3(X)$ is an abelian variety. O'Grady [10] proves the following result.

**Theorem 1.1.** *The two abelian varieties $J^3(X)$ and $KS(X)_c$ are isogenous.*

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We will also prove in Section 3.2 a more general statement concerning hyper-Kähler manifolds with $b_3(X) \neq 0$. Section 3.3 is devoted to the question of the algebraicity of the Kuga-Satake correspondence. We will prove, using results of Markman and Theorem 1.1 that the Kuga-Satake correspondence is algebraic for hyper-Kähler manifolds of generalized Kummer type:

**Theorem 1.2.** *There exists a codimension $2n$ cycle $\mathcal{Z} \in \mathrm{CH}^{2n}(\mathrm{KS}(X)_c \times X)_{\mathbb{Q}}$ such that*

$$[\mathcal{Z}]_* : H_2(\mathrm{KS}(X)_c, \mathbb{Q}) \longrightarrow H_2(X, \mathbb{Q}) \tag{1.1}$$

*is surjective.*

## 2. The Kuga-Satake construction

2.1. **Main Construction.** In this section, we recall the construction and the properties of the Kuga-Satake variety associated to a Hodge structure of *Hyper-Kähler type.* This construction is due to due to Kuga and Satake in [5]. For a complete introduction see [4, Ch. 4] and [12]

**Definition 2.1.** A pair $(V, q)$ is a Hodge structure of *Hyper-Kähler type* if the following conditions hold: $V$ is a rational Hodge structure of level two with $\dim V^{2,0} = 1$, and $q : V \otimes V \longrightarrow \mathbb{Q}(-2)$ is a morphism of Hodge structures which defines a non-degenerate quadratic form on $V$, whose extension to $V_{\mathbb{R}}$ is negative definite on $(V^{2,0} \oplus V^{0,2}) \cap V_{\mathbb{R}}$.

**Remark 2.2.** Notice that if $X$ is an Hyper-Käler manifold and $q$ is the Beauville-Bogomolow quadratic form, the pair $(H^2(X, \mathbb{Q}), -q)$ is indeed a Hodge structure of *Hyper-Kähler type.*

Let $(V, q)$ be a Hodge structure of Hyper-Käler type, and consider the tensor algebra of the underlying rational vector space $V$:

$$T(V) := \bigoplus_{i \geqslant 0} V^{\otimes i}$$

where $V^{\otimes 0} := \mathbb{Q}$.
Considering $q$ as a quadratic form of $V$, the *Clifford algebra* of $(V, q)$ is the quotient algebra

$$\mathrm{Cl}(V) := \mathrm{Cl}(V, q) := T(V)/I(q),$$

where $I(q)$ is the two-sided ideal of $T(V)$ generated by elements of the form $v \otimes v - q(v)$ for $v \in V$.
Since $I(q)$ is generated by elements of even degree, the natural $\mathbb{Z}/2\mathbb{Z}$-grading on $T(X)$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Cl}(V)$. Write

$$\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^-(V),$$

where $\mathrm{Cl}^+(V)$ is the even part and $\mathrm{Cl}^-(V)$ is the odd part. Notice that $\mathrm{Cl}^+(V)$ is still a $\mathbb{Q}$-algebra, it is called *even Clifford algebra.*

We now use the assumption that $(V,q)$ is a Hodge structure of Hyper-Kähler type to define a complex structure $\mathrm{Cl}^+(V)_\mathbb{R}$:
Consider the following decomposition of the real vector space $V_\mathbb{R}$ :

$$V_\mathbb{R} = V_1 \oplus V_2, \quad \text{with} \quad V_1 := V^{1,1} \cap V_\mathbb{R},\ V_2 := \{V^{2,0} \oplus V^{0,2}\} \cap V_\mathbb{R}.$$

The $\mathbb{C}$-linear span of $V_2$ is $V^{2,0} \otimes V^{0,2}$, which is two-dimensional and by hypothesis, $q$ is negative definite on $V_2$.
Pick a generator $\sigma = e_1 + ie_2$ of $V^{2,0}$ with $e_1, e_2 \in V_2$ and $q(e_1) = -1$. Since $q(\sigma) = 0$, we deduce that $q(e_1, e_2) = 0$ and $q(e_2) = -1$. Therefore, $e_1, e_2$ is an orthonormal basis of $V_2$. From this, it is straightforward to check that $e_1 \cdot e_2 = -e_2 \cdot e_1 \in \mathrm{Cl}(V_\mathbb{R})$. Therefore left multiplication with $J := e_1 \cdot e_2$ induces a complex structure on the real vector space $\mathrm{Cl}(V)_\mathbb{R}$ which preserves the real subspaces $\mathrm{Cl}^+(V)_\mathbb{R}$ and $\mathrm{Cl}^-(V)_\mathbb{R}$. Since giving a complex structure on a real vector space is equivalent to giving an Hodge structure of level one on the rational vector space, we have the following definition:

**Definition 2.3.** The *Kuga-Satake Hodge structure* on $\mathrm{Cl}^+(V)$ is the Hodge structure of level one given by

$$\rho\colon \mathbb{C}^* \longrightarrow \mathrm{GL}(\mathrm{Cl}^+(V)_\mathbb{R}),\ x + yi \longrightarrow x + yJ,$$

where $x + yJ$ acts on $\mathrm{Cl}^+(V)_\mathbb{R}$ via left multiplication.

Therefore, starting from a rational Hodge structure of Hyper-Kähler type of level two $(V,q)$, we constructed a rational Hodge structure of level one on $\mathrm{Cl}^+(V)$. This determines naturally an isogeny class of complex tori: Let $\Gamma \subseteq \mathrm{Cl}^+(V)$ be a lattice in the rational vector space $\mathrm{Cl}^+(V)$. Then, the *Kuga-Satake variety* associated to $(V,q)$ is the (isogeny class of) the complex torus

$$\mathrm{KS}(X) := \mathrm{Cl}^+(V)_\mathbb{R}/\Gamma,$$

where $\mathrm{Cl}^+(V)_\mathbb{R}$ is endowed with the complex structure induced by left multiplication by $J$.
Notice that $\mathrm{KS}(V)$ is determined only up isogeny class. On the other hand, if we started from an integral Hodge structure of Hyper-Kähler type, there would be a natural way to determine the lattice $\Gamma$, therefore $\mathrm{KS}(V)$ would not be determined only up to isogeny.
By construction, one has the following:

$$H^1_{\mathrm{KS}}(V) := H^1(\mathrm{KS}(V), \mathbb{Q}) \simeq \mathrm{Cl}^+(V)^* \simeq \mathrm{Cl}^+(V),$$

where the isomorphism between $\mathrm{Cl}^+(V)$ and its dual is induced by the non-degenerate form $q$.

**Remark 2.4.** Consider the case where $V$ can be written as a direct sum of Hodge structures $V = V^1 \oplus V^2$. Since $\dim V^{2,0} = 1$, we must have that either $V_1$, or $V_2$ has to be pure of type $(1,1)$. We may then assume that $V_2^{2,0} = 0$. Then, one can check that the Kuga-Satake Hodge structure $\mathrm{Cl}^+(V)$ is isomorphic to the product of $2^{n_2-1}$ copies of $\mathrm{Cl}^+(V_1) \oplus \mathrm{Cl}^-(V_1)$. In particular:

$$\mathrm{KS}(V_1 \oplus V_2) \sim \mathrm{KS}(V_1)^{2^{n_2}}.$$

**Remark 2.5.** Since the Kuga-Satake Hodge structure on $\mathrm{Cl}^+(V)$ is induced by left multiplication by $J \in \mathrm{Cl}^+(V)$, right multiplication of $\mathrm{Cl}^+(V)$ is compatible with the Hodge structure. Therefore, we have an embedding

$$\mathrm{Cl}^+(V) \hookrightarrow \mathrm{End}_{Hdg}(\mathrm{Cl}^+(V)) \simeq \mathrm{End}(\mathrm{KS}(V)) \otimes \mathbb{Q}.$$

Since the dimension of $\mathrm{Cl}^+(V)$ is $2^{n-1}$, where $n := \dim V$, we deduce that in general, the endomorphism algebra of $KS(V)$ is big. This is connected with the fact that the Kuga-Satake variety of a Hodge structure of Hyper-Kähler type is in general not simple, but it is isogenus to some power of a torus.

A remarkable property of the Kuga-Satake construction, is the fact that realizes the starting Hodge structure of level two as a sub-Hodge structure of the tensor product of two Hodge structures of level one:

**Theorem 2.6.** *Let $(V, q)$ be a Hodge structure of Hyper-Kähler type, then there is an embedding of Hodge structures:*

$$V \hookrightarrow \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V),$$

*where $\mathrm{Cl}^+(V)$ is endowed with the level one Hodge structure of Definition 2.3.*

*Proof.* We recall here just the definition of the desired map, for all the details we refer to [4, Prop. 3.2.6].
Fix an element $v_0 \in V$ such that $q(v_0) \neq 0$ and consider the following map:

$$\varphi \colon V \longrightarrow \mathrm{End}(\mathrm{Cl}^+(V)).$$
$$v \longrightarrow f_v \colon w \longrightarrow v \cdot w \cdot v_0$$

The fact that $f_v$ is a morphism of Hodge structure follows from Remark 2.5. The injectivity of $\varphi$ follows from the equality: $f_v(v' \cdot v_0) = q(v_0)(v \cdot v')$ for any $v' \in V$. See the reference for the proof of the fact that $\varphi$ is a morphism of Hodge structures.
Finally, the desired embedding is given by the composition of $\phi$ and the isomorphisms

$$\mathrm{End}_{Hdg}(\mathrm{Cl}^+(V)) \simeq \mathrm{Cl}^+(V)^* \otimes \mathrm{Cl}^+(V) \simeq \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V),$$

where the isomorphism $\mathrm{Cl}^+(V)^* \simeq \mathrm{Cl}^+(V)$ is induced by $q$. □

**Remark 2.7.** Notice that the morphism of Theorem 2.6 is not canonical, in the sense that it depends on the choice of $v_0 \in V$. Nevertheless, choosing another $v_0' \in V$ changes the embedding by the automorphism of $\mathrm{Cl}^+(V)$ which sends $w \longrightarrow 1/q(v_0) w \cdot v_0 \cdot v_0'$.

Theorem 2.6 shows that any Hodge structure of Hyper-Kähler type can be realized as a sub-Hodge structure of $W \otimes W$ for some level one Hodge structure $W$. On the other hand, in [2, Sec. 7], Deligne proves that for a very general Hodge structure of level two, the same conclusion does not hold. We recall here a version of this fact as presented in [12, Prop. 4.2]:

**Theorem 2.8.** *Let $(V, q)$ be a polarized Hodge structure of level two such that* $\mathrm{MT}(V) = \mathrm{SO}(q)$*. Then, if* $\dim V^{2,0} > 1$*, $V$ cannot be realized as a sub-Hodge structure of $W \otimes W$ for any level one Hodge structure $W$.*

**Remark 2.9.** One can show that the technical condition $\mathrm{MT}(V) = \mathrm{SO}(q)$ of Theorem 2.8 is satisfied for a very general Hodge structure, see [2, Sec. 7] and [14, Cor. 4.12]. The proof goes as follows: Given a $\pi\colon \mathcal{X} \twoheadrightarrow B$ a smooth projective morphism, one shows that for very general $t \in B$, the Mumford-Tate group $\mathrm{MT}(\mathcal{X}_t)$ contains a finite index subgroup of the monodromy group of the base. Already in the case of hypersurfaces in an even dimensional projective space, this shows that for a very general hypersurface, the Mumford Tate group is maximal.

To conclude this section, we recall the fact that if the Hodge structure of Hyper-Kähler type is polarized, then also the resulting Kuga-Satake Hodge structure on the even Clifford algebra is naturally polarized:

**Theorem 2.10.** *Let $(V, q)$ be a Hodge structure of Hyper-Kähler type such that $q$ is a polarization for $V$, then the Kuga-Satake Hodge structure on* $\mathrm{Cl}^+(V)$ *has a natural polarization. In particular, the Kuga-Satake torus* $\mathrm{KS}(V)$ *is an abelian variety.*

2.2. **Some examples.** Let $X$ be an Hyper-Kähler variety (*resp.* a two dimensional complex torus). The pair $(H^2(X, \mathbb{Q}), -q_{BB})$ where $q_{BB}$ is the Beauville-Bogomolov form (*resp.* the intersection pairing) is an Hodge structure of Hyper-Kähler type. Therefore, we can apply the Kuga-Satake construction to it and we get the *Kuga-Satake variety* of $X$:

$$\mathrm{KS}(X) := \mathrm{KS}(H^2(X, \mathbb{Q})).$$

Since $-q_{BB}$ is not a polarization on the whole $H^2(X, \mathbb{Q})$, the variety $\mathrm{KS}(X)$ is not necessarily an abelian variety, but it is just a complex torus.
On the other hand, if $X$ is projective and $l$ is an ample class on $X$, the primitive part

$$H^2(X, \mathbb{Q})_p := l^\perp \subseteq H^2(X, \mathbb{Q})$$

is a sub-Hodge structure which is polarized by the restriction of the form $-q_{BB}$. Therefore, by Theorem 2.10, the Kuga-Satake variety of $H^2(X, \mathbb{Q})_p$ is an abelian variety. Moreover, by Remark 2.4, we have

$$\mathrm{KS}(X) := \mathrm{KS}(H^2(X, \mathbb{Q})) \sim \mathrm{KS}(H^2(X, \mathbb{Q})_p)^2.$$

In particular, in the projective case, $\mathrm{KS}(X)$ is an abelian variety.
A similar argument can be applied to the transcendental lattice $T(X) \subseteq H^2(X, \mathbb{Q})$, to deduce that $\mathrm{KS}(X)$ is isogenous to some power of $\mathrm{KS}(T(X))$.
On the other hand, if $X$ is not projective, the torus $\mathrm{KS}(X)$ need not be polarized.

**Theorem 2.11.** [9] *Let $A$ a complex torus of dimesion two. Then there exists an isogeny*

$$\mathrm{KS}(A) \sim (A \times \hat{A})^4,$$

*where $\hat{A}$ is the dual complex torus.*
*In particular, if $A$ is an abelian surface*

$$\mathrm{KS}(A) \sim A^8 \quad \textit{and} \quad \mathrm{KS}(\mathrm{Kum}(A)) \sim A^{2^{19}},$$

*where* $\mathrm{Kum}(A)$ *is the Kummer surface associated to $A$.*

**Definition 2.12.** Let $K$ be the field $\mathbb{Q}(\sqrt{-d})$ for some positive rational number $d$ and let $A$ be an abelian variety of dimension $2n$. The abelian variety $A$ is called of $K$-Weil type if $K \subseteq \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and if the action of $\sqrt{-d}$ on the tangent space at the origin of $A$ has eigenvalues $\sqrt{-d}$ and $-\sqrt{-d}$ both with multiplicity $n$.

Associated to an abelian of $K$-Weil type one can associate naturally an element $\delta \in \mathbb{Q}/\mathrm{N}(K)$, where $\mathrm{N}(K)$ is the set of norms of $K$. The element $\delta$ is called the discriminant of $A$.
The next result is due to Lombardo [6], we recall here the version presented in [12, Thm. 9.2]

**Theorem 2.13.** *Let $A$ be an abelian fourfold of $\mathbb{Q}(\sqrt{-d})$-Weil type of discriminant $\delta = 1$ for some positive rational number d. Then $A^4$ is the Kuga-Satake of some polarized Hodge structure of Hyper-Kähler type of dimension six. Conversely, given a Hodge structure of Hyper-Kähler type of dimension six, its Kuga-Satake variety is isogenous to $A^4$ for some abelian fourfold of Weil-type.*

2.3. **Kuga-Satake Hodge conjecture.** In this section, we analyze the morphism of Hodge structures

$$V \hookrightarrow \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V)$$

of Theorem 2.6, in the case where $V = T(X)$, the transcendental lattice of a projective Hyper-Kähler variety $X$.
Since we are in the projective setting, there exists a natural projection from $H^2(X, \mathbb{Q})$ to $T(X)$. On the other hand, recall that $\mathrm{Cl}^+(T(X)) \simeq H^1_{\mathrm{KS}}(T(X))$. Therefore, we can apply Künneth decomposition to embed

$$H^1_{\mathrm{KS}}(T(X)) \otimes H^1_{\mathrm{KS}}(T(X)) \hookrightarrow H^2(\mathrm{KS}(T(X))^2, \mathbb{Q})$$

Composing these morphisms, we obtain a morphisms of Hodge structures

$$H^2(X, \mathbb{Q}) \longrightarrow H^2(\mathrm{KS}(T(X))^2, \mathbb{Q}),$$

which is called *Kuga-Satake correspondence.* This morphisms corresponds via Poincaré duality to an Hodge class

$$\kappa \in H^{2n,2n}(X \times \mathrm{KS}(T(X)) \times \mathrm{KS}(T(X)),$$

where $2n = \dim X$.
The Hodge conjecture applied to this special case gives us the following:

**Conjecture 2.14** (Kuga-Satake Hodge conjecture)**.** *Let $X$ be a projective Hyper-Kähler variety or a complex projective surface with $h^{2,0} = 1$, then the class $\kappa$ is algebraic.*

**Remark 2.15.** In the case where $X$ is an abelian surface or a Kummer surface, the Kuga-Satake Hodge conjecture can be deduced from Theorem 2.11, using the fact that the Hodge conjecture is known for self products of any given abelian surface [8].

In [11], Paranjape shows the Kuga-Satake Hodge conjecture for the following family of K3 surfaces: Let $L_1, \dots, L_6$ be six lines in $\mathbb{P}^2$ in general position, and let $\pi : Y \longrightarrow \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along the six lines. Then, the resolution of singularities of $\pi$ is a K3 surface. This way, one constructs a family of K3 surfaces over the four dimensional moduli space of configurations of six lines in the projective plane.
For six lines in general position, the Picard number of the resulting K3 surface is 16, where a basis of the Neron-Severi group is given by the 15 exceptional divisors over the intersection points of the lines, together with the pullback of the ample line of $\mathbb{P}^2$ via the map $X \longrightarrow \mathbb{P}^2$. In particular, the transcendental lattice of $X$ is six dimensional, and hence satisfies the hypotheses of Theorem 2.13. Its Kuga-Satake variety is therefore isogenous to the fourth power of some abelian fourfold. In [11], the author shows that this abelian fourfold is the Prym Variety of some $4:1$ cover $C \longrightarrow E$ where $C$ is a genus 5 curve and $E$ is an elliptic curve, and finds a cycle in the product of $X$ and the Prym variety which realizes the Kuga-Satake correspondence.

## 3. The case of hyper-Kähler manifolds of generalized Kummer type

3.1. **Cup-product: generalization of a result of O'Grady.** Let $X$ be a hyper-Kähler manifold of dimension $2n$ with $n \geqslant 2$. The Beauville-Bogomolov quadratic form $q_X$ is a non-degenerate quadratic form on $H^2(X, \mathbb{Q})$, whose inverse gives an element of $\mathrm{Sym}^2 H^2(X, \mathbb{Q})$. By Verbitsky [13], the later space imbeds by cup-product in $H^4(X, \mathbb{Q})$, hence we get a class

$$c_X \in H^4(X, \mathbb{Q}). \tag{3.1}$$

The O'Grady map $\phi : \bigwedge^2 H^3(X, \mathbb{Q}) \longrightarrow H^{4n-2}(X, \mathbb{Q})$ is defined by

$$\phi(\alpha \wedge \beta) = c_X^{n-2} \cup \alpha \cup \beta. \tag{3.2}$$

The following result was first proved by O'Grady [10] in the case of a hyper-Kähler manifold of generalized Kummer deformation type.

**Theorem 3.1.** *([10], [15]) Let $X$ be a hyper-Kähler manifold of dimension $2n$. Assume $H^3(X, \mathbb{Q}) \neq 0$. Then the O'Grady map $\phi : \bigwedge^2 H^3(X, \mathbb{Q}) \longrightarrow H^{4n-2}(X, \mathbb{Q})$ is surjective.*

*Proof.* We can choose the complex structure on $X$ to be general, so that $\rho(X) = 0$, and this implies that the Hodge structure on $H^2(X, \mathbb{Q})$ (or equivalently $H^{4n-2}(X, \mathbb{Q})$ as they are isomorphic by combining Poincaré duality and the Beauville-Bogomolov form) is simple. As the morphism $\phi$ is a morphism of Hodge structures, its image is a Hodge substructure of

$H^{4n-2}(X,\mathbb{Q})$, hence either $\phi$ is surjective, or it is 0. Theorem 3.1 thus follows from the next proposition. □

**Proposition 3.2.** *The map $\phi$ is not identically* 0.

*Sketch of proof.* Let $\omega \in H^2(X,\mathbb{R})$ be a Kähler class. Then we know that the $\omega$-Lefschetz intersection pairing $\langle\, ,\, \rangle_\omega$ on $H^3(X,\mathbb{R})$, defined by

$$\langle \alpha, \beta\rangle_\omega := \int_X \omega^{2n-3} \cup \alpha \cup \beta$$

is nondegenerate. This implies that the cup-product map

$$\psi : \bigwedge^2 H^3(X,\mathbb{Q}) \longrightarrow H^6(X,\mathbb{Q})$$

has the property that $\operatorname{Im}\psi$ pairs nontrivially with the image of $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})$ in $H^{4n-6}(X,\mathbb{Q})$. Note that the Hodge structure on $H^3(X,\mathbb{Q})$ has Hodge level 1, so that the Hodge structure on the image of $\operatorname{Im}\psi$ in $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})^*$ is a Hodge structure of level at most 2. One checks by a Mumford-Tate group argument that for a very general complex structure on $X$, the only level 2 Hodge substructure of $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})$ is $c_X^{n-2}H^2(X,\mathbb{Q})$, where we see here $c_X$ as an element of $\operatorname{Sym}^2H^2(X,\mathbb{Q})$. It follows that the image of $\operatorname{Im}\psi$ in $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})^*$ pairs nontrivially with $c_X^{n-2}H^2(X,\mathbb{Q})$, which concludes the proof. □

3.2. **Intermediate Jacobian and the Kuga-Satake variety.**

3.2.1. *Universal property of the Kuga-Satake construction.* The following result is proved in [1]. Using the Mumford-Tate group, this is a statement in representation theory.

**Theorem 3.3.** *Let $(H^2, (\, ,\, ))$ be a polarized Hodge structure of hyper-Kähler type. Let $H$ be a simple effective weight* 1 *Hodge structure, such that there exists a surjective morphism of Hodge structures of bidegree* $(-1,-1)$

$$H^2 \hookrightarrow \operatorname{End}(H).$$

*Then $H$ is a direct summand of the Kuga-Satake Hodge structure $H^1_{KS}(H^2)$.*

The statement is easier to prove when the Mumford-tate group of the considered Hodge structure of hyper-Kähler type is the orthogonal group (see [3]). In that case, one knows that the Kuga-Satake weight 1 Hodge structure is a power of a simple weight 1 Hodge structure of dimension $2^{E(\frac{b_2-2}{2})}$, where $b_2 = \dim H^2$, hence one gets as a consequence an inequality (see [1])

$$\dim H \geqslant 2^{E(\frac{b_2-2}{2})}.$$

*Proof of Theorem 1.1.* Let $X$ be a very general projective hyper-Kähler manifold of generalized Kummer type of dimension $\geqslant 4$. We apply Theorem 3.3 to the O'Grady map (3.2) that we know to a surjective morphism of Hodge structures by Theorem 3.1, or rather to its dual. We then conclude that $H^3(X,\mathbb{Q})$ contains a direct summand of $H^1_{KS}(H^2(X,\mathbb{Q})_{\mathrm{tr}})$. As $H^1_{KS}(H^2(X,\mathbb{Q})_{\mathrm{tr}})$ is a power of a simple weight 1 Hodge structure $H^1_{K-S}(H^2(X,\mathbb{Q})_{\mathrm{tr}})_c$ of dimension 8, and $b_3(X)=8$, we conclude that $H^3(X,\mathbb{Q})\simeq H^1_{KS}(H^2(X,\mathbb{Q})_{\mathrm{tr}})_c$ as Hodge structures. Hence the two corresponding abelian varieties are isogenous. □

### 3.3. Algebraicity of the Kuga-Satake correspondence for HK's of generalized Kummer type .

3.3.1. *A result of Markman.* For a projective manifold $X$ with $h^{3,0}(X)\neq 0$, it is expected from the Hodge conjecture that there exists a cycle $\mathcal{Z}\in \mathrm{CH}^2(J^3(X)\times X)_{\mathbb{Q}}$ such that $[\mathcal{Z}]_*: H_1(J^3(X),\mathbb{Q})\longrightarrow H^3(X,\mathbb{Q})$ is the natural isomorphism. Indeed, the map $[\mathcal{Z}]_*$ is an isomorphism of Hodge structures, hence provides a degree 4 Hodge class on $J^3(X)\times X$. Equivalently, after replacing $\mathcal{Z}$ by a multiple that makes it integral, the Abel-Jacobi map

$$\Phi_{\mathcal{Z}}: J^3(X)\longrightarrow J^3(X),\ \Phi_{\mathcal{Z}}:=\Phi_X\circ\mathcal{Z}_*,$$

where $\Phi_X:\mathrm{CH}^2(X)_{\mathrm{alg}}\longrightarrow J^3(X)$ is the Abel-Jacobi map for codimension 2 cycles algebraically equivalent to zero on $X$, is a nonzero multiple of the identity and in particular $\Phi_X$ is surjective.

**Theorem 3.4.** *(Markman [7]) Let $X$ be a projective hyper-Kähler manifold of generalized Kummer type. Then there exists a codimension 2 cycle $\mathcal{Z}\in \mathrm{CH}^2(J^3(X)\times X)_{\mathbb{Q}}$ satisfying the property above.*

The proof of this theorem uses a deformation argument starting from a generalized Kummer manifold, using the fact that $J^3(X)$ can be realized as a moduli space of sheaves on $X$ in that case.

We now use the result of Markman to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let $\mathcal{Z}$ be the Markman codimension 2 cycle of Theorem 3.4. We choose a cycle $C_X\in \mathrm{CH}^2(X)_{\mathbb{Q}}$ of class $[C_X]=c_X$ (it exists by results of Markman [7]). We now consider the cycle

$$\Gamma=\mathcal{Z}^2\cdot \mathrm{pr}_X^*C_X^{n-2}\in \mathrm{CH}^{2n}(J^3(X)\times X)\mathbb{Q}.$$

One checks using the Künneth decomposition that $[\Gamma]_*: H_2(J^3(X),\mathbb{Q})\longrightarrow H_2(X,\mathbb{Q})$ is the O'Grady map. By Theorem 1.1, this is also the surjective morphism of Hodge structures (1.1). □

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