K3 surfaces and modular curves

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Contents

Introduction

K3 surfaces occupy a central position in algebraic and complex geometry. First studied systematically in the mid-twentieth century, they form a distinguished class of smooth, compact complex surfaces characterized by trivial canonical bundle and vanishing irregularity. Their topology is deceptively simple—indeed, every K3 surface is diffeomorphic to the Fermat quartic—yet their algebraic, arithmetic, and categorical structures are astonishingly rich.

A powerful modern perspective on the geometry of K3 surfaces arises from their derived categories of coherent sheaves. For any smooth complex projective variety X, its bounded derived category of coherent sheaves $\mathrm{D^b}(X)$ comes equipped with a group $\mathrm{Aut}(\mathrm{D^b}(X))$ of exact $\mathbb{C}$-linear autoequivalences, whose structure is, in general, difficult to determine. Beyond the foundational results of Bondal and Orlov in [BO01, Thm. 3.1] and [Orl02, Thm. 4.14] in situations where the canonical bundle is ample or anti-ample, or when X is an abelian variety, very few complete descriptions are known. In particular, the group $\mathrm{Aut}(\mathrm{D^b}(X))$ of a general K3 surface X remains hitherto unresolved.

In analogy to the global Torelli theorem—which describes the group $\mathrm{Aut}(X)$ of automorphisms of a K3 surface X via its action on the second singular cohomology $H^2(X, \mathbb{Z})$, cf. [Huy16, Cor. 2.3]—one begins by studying autoequivalences through their action on the full cohomology lattice $\widetilde{H}(X, \mathbb{Z})$. A classical result of Orlov in [Orl97, Prop. 3.5], building on earlier work due to Mukai in [Muk87, Thm. 4.9], establishes that every autoequivalence of $\mathrm{D^b}(X)$ induces a Hodge isometry of $\widetilde{H}(X, \mathbb{Z})$. This construction yields a natural representation $\varpi\colon \mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}))$, whose image was settled in a difficult result due to Huybrechts, Macrì, and Stellari in [HMS09, Cor. 3]. To determine the group $\mathrm{Aut}(\mathrm{D^b}(X))$, it thus remains to study the kernel of ϖ, which we shall denote by $\mathrm{Aut}_0(\mathrm{D^b}(X))$. This group is highly nontrivial i.a. due to the existence of spherical twist functors, see Section 1.3. Bridgeland's theory of stability conditions on K3 surfaces, developed in [Bri08], provides a conjectural description of the group $\mathrm{Aut}_0(\mathrm{D^b}(X))$, see Conjecture 1.3.8.

A major breakthrough was achieved by Bayer and Bridgeland in [BB17, Thm. 1.3], where they proved Bridgeland's conjecture for K3 surfaces of Picard number 1. In this setting they further established the isomorphism

$$\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2] \cong \pi_1^{\mathrm{orb}}\left(\left[\,\mathcal{Q}_0^+(X)\Big/\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))\,\right]\right),$$

where $\mathcal{Q}_0^+(X)$ is a period space to be introduced later in Section 2.1, and $\mathrm{Aut}_s(\mathrm{D^b}(X))$ (resp. $\mathrm{Aut}_s(\widetilde{H}(X, \mathbb{Z}))$) denotes the group of symplectic autoequivalences of $\mathrm{D^b}(X)$ (resp. symplectic Hodge isometries of $\widetilde{H}(X, \mathbb{Z})$). Fan and Lai later observed in [FL23, Sec. 4.3] that in the special case of degree 2, the correspondence requires a modification.

This isomorphism marks the starting point of the present thesis, which builds on the analysis of Fan and Lai in [FL23]. For a K3 surface X with Picard number 1, our main objective is to classify finite subgroups of autoequivalences of the derived category $\mathrm{D^b}(X)$ up to even shifts. Along the way, we also derive a description of the group structure of the kernel $\mathrm{Aut}_0(\mathrm{D^b}(X))$.

The thesis is organized as follows. Chapter 1 collects the necessary preliminaries: lattice and Hodge theory for K3 surfaces, the derived category and autoequivalences, and basic facts on modular curves.

Chapter 2 contains the core technical work: We establish that the orbifold $[\mathcal{Q}_0^+(X)/\,\mathrm{Aut}_s^+(\widetilde{H}(X, \mathbb{Z}))]$ is diffeomorphic to the more accessible modular curve $[\mathbb{H}_0/\Gamma_0^+(n)]$. The proof requires a detailed analysis of the period domain, of the action of symplectic Hodge isometries, and of the geometry of the punctured upper half plane $\mathbb{H}_0$. We have chosen to present explicit maps and to avoid non-canonical isomorphic identifications; this choice, although sometimes at the expense of readability, renders a transparent and precise construction.

In Chapter 3, the thesis picks up where the reference paper [FL23] by Fan and Lai begins: Exploiting the classical theory of modular curves, we compute the orbifold fundamental group $\pi_1^{\mathrm{orb}}([\mathbb{H}_0/\Gamma_0^+(n)])$, which in turn yields the group structure of $\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$. Finally, a careful analysis of the finite subgroups of free products along with the consideration of the passage from symplectic autoequivalences to general autoequivalences culminates with the classification of maximal finite subgroups of $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$ in Theorem 3.2.7.

Throughout the exposition, whenever in doubt, we have preferred to err on the side of thoroughness, so that the reader can freely choose the level of detail at which to engage with the material rather than regretting the absence of certain details.

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Deutsche Zusammenfassung

K3-Flächen nehmen eine zentrale Stellung in der algebraischen und komplexen Geometrie ein. Obwohl sie topologisch durch eine einfache Struktur gekennzeichnet sind — jede K3-Fläche ist diffeomorph zur Fermat-Quartik — weisen sie eine außerordentlich reiche algebraische und arithmetische Geometrie auf. Ein moderner Zugang besteht in der Untersuchung der abgeleiteten Kategorie der kohärenten Garben $\mathrm{D^b}(X)$ und insbesondere ihrer Gruppe von Autoäquivalenzen $\mathrm{Aut}(\mathrm{D^b}(X))$. Während diese in gewissen Fällen — etwa für abelsche Varietäten oder Flächen mit (anti-)amplen kanonischen Bündel — durch Ergebnisse von Bondal und Orlov in [BO01, Thm. 3.1] und [Orl02, Thm. 4.14] vollständig beschrieben ist, bleibt die Struktur von $\mathrm{Aut}(\mathrm{D^b}(X))$ für eine K3-Fläche X weitgehend unbekannt.

Durch Arbeiten von Mukai und Orlov in [Muk87, Thm. 4.9] und [Orl97, Prop. 3.5] ist bekannt, dass jede Autoäquivalenz von $\mathrm{D^b}(X)$ eine Hodge-Isometrie des Mukai-Gitters $\widetilde{H}(X,\mathbb{Z})$ induziert. Huybrechts, Macrì und Stellari lieferten in [HMS09, Cor. 3] eine vollständige Beschreibung des Bildes der entsprechenden Darstellung $\varpi\colon \mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))$. Die Bestimmung der Gruppe der Autoäquivalenzen reduziert sich damit auf das Studium des Kerns $\ker\varpi =: \mathrm{Aut}_0(\mathrm{D^b}(X))$ dieser Abbildung, welcher unter anderem von sphärischen Twists erzeugt wird und von Bridgeland im Rahmen seiner Theorie von Stabilitätsbedingungen in [Bri08] konjektural beschrieben wurde.

Einen entscheidenden Fortschritt erzielten Bayer und Bridgeland in [BB17], wo sie Bridgelands Vermutung für K3-Flächen X vom Picard-Rang 1 bewiesen und dabei zeigten, dass

$$\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2] \cong \pi_1^{\mathrm{orb}}\left(\left[\,\mathcal{Q}_0^+(X) \Big/ \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))\right]\right).$$

Diese Arbeit baut auf diesem Isomorphismus und auf den Resultaten von Fan und Lai in [FL23] auf und verfolgt das Ziel, endliche Untergruppen von Autoäquivalenzen von $\mathrm{D^b}(X)$ bis auf gerade Verschiebungen zu klassifizieren. Kapitel 1 führt die notwendigen Grundlagen ein: Gitter- und Hodge-Theorie von K3-Flächen, die abgeleitete Kategorie und Autoäquivalenzen sowie elementare Aspekte der Theorie der Modulkurven. Kapitel 2 enthält den technischen Kern und zeigt, dass die Orbifaltigkeit $[\mathcal{Q}_0^+(X)/\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))]$ diffeomorph zur Modulkurve $[\mathbb{H}_0/\Gamma_0^+(n)]$ ist. Kapitel 3 baut schließlich auf der Theorie klassischer Modulkurven zur Berechnung der zugehörigen Orbifaltigkeit-Fundamentalgruppe $\pi_1^{\mathrm{orb}}([\mathbb{H}_0/\Gamma_0^+(n)])$ auf und liefert daraus eine Klassifikation maximaler endlicher Untergruppen von $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$.

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1 Preliminaries

1.1 K3 surfaces, lattice theory and Hodge structures

While several equivalent definitions exist, for the purposes of this thesis, we shall adopt the following:

Definition 1.1.1. A *K3 surface* over $\mathbb{C}$ is a projective, smooth and irreducible surface X over $\mathbb{C}$ that satisfies the following conditions:

(i) The irregularity of X is zero, i.e. $H^1(X, \mathcal{O}_X) = 0$.

(ii) The canonical bundle of X is trivial, i.e. $\omega_X \coloneqq \Omega_X^2 \cong \mathcal{O}_X$.

Example 1.1.2. The Fermat quartic $V(x_0^4+x_1^4+x_2^4+x_3^4) \subseteq \mathbb{P}^3$ is a K3 surface, see [Huy16, Ex. 1.1.3].

Interestingly, as a consequence of a result due to Kodaira, one can show that every K3 surface is diffeomorphic to Fermat's quartic, cf. [Huy16, Thm. 7.1.1]. Thus, topological invariants alone will not help us distinguish between K3 surfaces. Using standard tools in algebraic topology and complex geometry, one finds that the integral singular cohomology of a K3 surface X is given by

$$H^0(X, \mathbb{Z}) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22},$$

with all other cohomology groups vanishing.

Moreover, observe that X can be viewed as a compact oriented real four-dimensional manifold. As such, we can endow it with a unimodular intersection form on the second integral cohomology—this form arises naturally from the cup product:

$$\begin{aligned} H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) &\longrightarrow \mathbb{Z} \\ (a, b) &\longmapsto (a.b) \coloneqq \langle a \smile b, [X] \rangle. \end{aligned}$$

Remark 1.1.3. Observe that, by extending scalars, we obtain an intersection pairing $H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \to \mathbb{R}, (a, b) \mapsto (a.b)$. Since X is smooth, we can reinterpret this form in terms of the de Rham cohomology: If a and b are represented by 2-forms α and β, it holds that

$$(a.b) = ([\alpha].[\beta]) = \int_X \alpha \wedge \beta.$$

The intersection pairing on $H^2(X, \mathbb{Z})$ defines an even symmetric bilinear form and motivates, hence, the following definition:

Definition 1.1.4. A *lattice* Λ is a free $\mathbb{Z}$-module of finite rank endowed with a symmetric bilinear form $(-,-)\colon \Lambda \times \Lambda \to \mathbb{Z}$. We refer to the determinant of the Gram matrix of $(\,,\,)$ with respect to an arbitrary $\mathbb{Z}$-basis as its *discriminant* $\operatorname{disc} \Lambda$. Furthermore, the *orthogonal group* of Λ, i.e. the group of isometries $\Lambda \xrightarrow{\sim} \Lambda$, is denoted $\mathrm{O}(\Lambda)$.

As before, upon extending scalars, we obtain a real vector space $\Lambda_{\mathbb{R}} \coloneqq \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, which inherits a symmetric bilinear form $(\,,\,)_{\mathbb{R}}$ induced by the $\mathbb{R}$-linear extension of $(\,,\,)$.

Definition 1.1.5. Let Λ be a lattice. The *signature* of Λ is defined to be the pair (n_+, n_-), where n_+ (resp. n_-) denotes the positive (resp. negative) index of inertia of the bilinear form $(\,,\,)_{\mathbb{R}}$ on $\Lambda_{\mathbb{R}}$.

For future reference, we also introduce the discriminant group of a lattice:

Definition 1.1.6. Let Λ be a lattice and consider the additive subgroup $\Lambda^* \coloneqq \{x \in \Lambda_{\mathbb{Q}} : (x.\Lambda) \subseteq \mathbb{Z}\} \subseteq \Lambda_{\mathbb{Q}}$. The *discriminant group* A_Λ of Λ is the cokernel of the natural inclusion $i_\Lambda \colon \Lambda \hookrightarrow \Lambda^*$, i.e.

$$A_\Lambda \coloneqq \Lambda^*/\Lambda.$$

A lattice Λ is said to be *unimodular* if A_Λ is trivial.

Remark 1.1.7. The discriminant group of a lattice Λ is a finite group of order $|\operatorname{disc}\Lambda|$, see [Huy16, Sec. 14.0.1].

Example 1.1.8. The second integral cohomology $H^2(X,\mathbb{Z})$ of a K3 surface X, together with the aforementioned bilinear form, defines a lattice. Building on this, we can also endow the total integral cohomology

$$\widetilde{H}(X,\mathbb{Z}) \coloneqq H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \cong \mathbb{Z}^{24}$$

with a lattice structure by orthogonally extending the intersection form on $H^2(X,\mathbb{Z})$ as follows:

$$\begin{aligned} \widetilde{H}(X,\mathbb{Z}) \times \widetilde{H}(X,\mathbb{Z}) &\to \mathbb{Z} \\ ((r_1,D_1,s_1),(r_2,D_2,s_2)) &\mapsto (D_1.D_2) - r_1s_2 - r_2s_1. \end{aligned}$$

As a result of the classification theory of unimodular lattices, one finds that

$$H^2(X,\mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3},$$

where U denotes the hyperbolic plane and $E_8(-1)$ represents the standard E_8-lattice with the quadratic form modified by a sign change, see [Huy16, Prop. 1.3.5]. Therefore, $H^2(X,\mathbb{Z})$ carries the structure of an even unimodular lattice of signature $(3,19)$. From this, one readily verifies that

$$\widetilde{H}(X,\mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 4},$$

so $\widetilde{H}(X,\mathbb{Z})$ becomes an even unimodular lattice of signature $(4,20)$.

Let now V be a free $\mathbb{Z}$-module. By $V_{\mathbb{C}}$ we denote the vector space $V \otimes_{\mathbb{Z}} \mathbb{C}$ obtained by scalar extension and refer to it as the *complexification* of V. Since $\mathbb{Z}$ is a subring of $\mathbb{R}$, it follows that the complex vector space

$$V_{\mathbb{C}} \coloneqq V \otimes_{\mathbb{Z}} \mathbb{C} \cong V \otimes_{\mathbb{Z}} (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}) \cong (V \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \eqqcolon V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong V_{\mathbb{R}} \otimes_{\mathbb{R}} (\mathbb{R} \oplus \mathbf{i}\mathbb{R})$$

inherits a real structure, i.e. conjugation yields a well-defined $\mathbb{R}$-linear isomorphism $V_{\mathbb{C}} \xrightarrow{\sim} V_{\mathbb{C}}$. Moreover, if V is endowed with a bilinear form $(\,,)$, we can equip $V_{\mathbb{C}}$ with its $\mathbb{C}$-linear extension, which we shall also denote $(\,,)$ by abuse of notation. It then holds that $\overline{(a,b)} = (\bar{a},\bar{b})$ for a, $b \in V_{\mathbb{C}}$.

Definition 1.1.9. Let V be a free $\mathbb{Z}$-module. A *Hodge structure of weight* $n \in \mathbb{N}$ on V is a decomposition of the complex vector space $V_{\mathbb{C}}$ into direct summands:

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q},$$

in such a way that $\overline{V^{p,q}} = V^{q,p}$ and $V^{p,q} = 0$ whenever $p < 0$ or $q < 0$.

The complexification of the second cohomology $H^2(X,\mathbb{Z})$ of a K3 surface X can be endowed with a Hodge structure of weight 2,

$$H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

where the subspaces $H^{p,q}(X)$ are given by the Dolbeault cohomology, see [Huy05, Cor. 3.2.12]. We refer to this decomposition as the *natural Hodge structure of weight* 2 of $H^2(X,\mathbb{Z})$. Its importance in the study of K3 surfaces becomes apparent in the classical global Torelli theorem, which establishes that two K3 surfaces are isomorphic if and only if there exists a Hodge isometry between their respective Hodge lattices, cf. [Huy16, Thm. 3.2.4].

Definition 1.1.10. Let Λ and Λ' be two lattices and let $\Lambda_\mathbb{C} = \bigoplus \Lambda^{p,q}$ and $\Lambda'_\mathbb{C} = \bigoplus (\Lambda')^{p,q}$ be Hodge structures of weight n on Λ resp. Λ'. A linear isomorphism $\varphi\colon \Lambda \to \Lambda'$ is called a *Hodge isomorphism* if $(\varphi \otimes \mathrm{id}_\mathbb{C})(\Lambda^{p,q}) = (\Lambda')^{p,q}$ for all (p,q). A *Hodge isometry* is a Hodge isomorphism $\Lambda \xrightarrow{\sim} \Lambda'$ that preserves their bilinear forms. For a lattice Λ with a given Hodge structure of weight n as above, we denote the group of Hodge isometries $\Lambda \xrightarrow{\sim} \Lambda$ by $\mathrm{Aut}(\Lambda)$.

For a K3 surface X, the natural Hodge structure of weight 2 on $H^2(X,\mathbb{Z})$ induces a natural Hodge structure of weight 2 on the total cohomology lattice $\widetilde{H}(X,\mathbb{Z})$ by setting $\widetilde{H}(X,\mathbb{C}) = \widetilde{H}^{2,0}(X) \oplus \widetilde{H}^{1,1}(X) \oplus \widetilde{H}^{0,2}(X)$ with

$$\widetilde{H}^{2,0}(X) \coloneqq H^{2,0}(X), \quad \widetilde{H}^{1,1}(X) \coloneqq H^{1,1}(X) \oplus H^0(X,\mathbb{C}) \oplus H^4(X,\mathbb{C}) \quad \text{and} \quad \widetilde{H}^{0,2}(X) \coloneqq H^{0,2}(X).$$

For future reference, let us introduce three sublattices of $\widetilde{H}(X,\mathbb{Z})$ which will play an important role in the further development of this thesis. Let us start with the Néron-Severi group.

Definition 1.1.11. Let the *Néron-Severi group* $\mathrm{NS}(X)$ of a K3 surface X be

$$\mathrm{NS}(X) \coloneqq H^{1,1}(X) \cap H^2(X,\mathbb{Z}) \subseteq H^2(X,\mathbb{Z}).$$

Furthermore, define the *Picard number* $\rho(X)$ of a K3 surface to be the rank $\mathrm{rk}(\mathrm{NS}(X))$ of its Néron-Severi group.

Remark 1.1.12. For an algebraic surface S, one usually defines its Néron-Severi group as the quotient $\mathrm{NS}(S) \coloneqq \mathrm{Pic}(S)/\mathrm{Pic}^0(S)$. Suppose now that S defines a K3 surface. Then, one can show that there are no non-trivial line bundles that are algebraically equivalent to zero, so the natural surjection becomes an isomorphism $\mathrm{Pic}(S) \xrightarrow{\sim} \mathrm{NS}(S)$, cf. [Huy16, Prop. 1.2.4]. Finally, the Lefschetz theorem on $(1,1)$-classes establishes that $\mathrm{Pic}(S) \cong H^{1,1}(S) \cap H^2(S,\mathbb{Z})$, which is the definition we employ. We also note that the Picard number is well-defined, as the Néron-Severi group $\mathrm{NS}(X)$ of a K3 surface X can be shown to be finitely generated, see [Huy16, Prop. 1.2.1].

Since the Néron-Severi group $\mathrm{NS}(X)$ of a K3 surface X is a subgroup of the free $\mathbb{Z}$-module $H^2(X,\mathbb{Z})$, it is itself a free $\mathbb{Z}$-module as well. Hence, we see that $\mathrm{NS}(X)$ inherits a lattice structure upon restricting the bilinear form $(\,.\,)$ on $H^2(X,\mathbb{Z})$ to $\mathrm{NS}(X)$. As a consequence of the Hodge index theorem, we obtain that:

Proposition 1.1.13. *The signature of the intersection form on* $\mathrm{NS}(X)$ *is* $(1,\rho(X)-1)$.

Definition 1.1.14. The *numerical Grothendieck group* $\mathcal{N}(X)$ of a K3 surface X is given by

$$\mathcal{N}(X) \coloneqq H^0(X,\mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X,\mathbb{Z}) = \widetilde{H}^{1,1}(X) \cap \widetilde{H}(X,\mathbb{Z}) \subseteq \widetilde{H}(X,\mathbb{Z}).$$

Remark 1.1.15. By definition, $H^0(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$ forms a hyperbolic plane on the lattice $\widetilde{H}(X,\mathbb{Z})$. As with $\mathrm{NS}(X)$, the numerical Grothendieck group inherits a lattice structure from $\widetilde{H}(X,\mathbb{Z})$, which, thus, has signature $(2,\rho(X))$.

Finally, we introduce:

Definition 1.1.16. The transcendental lattice $T(X)$ of a K3 surface X is defined as the minimal primitive sub-Hodge structure $T(X) \subseteq H^2(X,\mathbb{Z})$ such that $H^{2,0}(X) \subseteq T(X)_\mathbb{C}$. In other words, we require that the natural Hodge structure of weight 2 on $H^2(X,\mathbb{Z})$ induces a Hodge structure on the sublattice $T(X) \subseteq H^2(X,\mathbb{Z})$ via the decomposition $T^{p,q}(X) \coloneqq T(X)_\mathbb{C} \cap H^{p,q}(X)$, with the following conditions:

(i) It holds that $H^{2,0}(X) \subseteq T(X)_\mathbb{C}$, i.e. $T^{2,0}(X) = H^{2,0}(X)$,

(ii) the quotient $H^2(X,\mathbb{Z})/T(X)$ is torsion-free (primitivity) and

(iii) the lattice $T(X)$ is minimal with respect to these properties.

Remark 1.1.17. The primitivity requirement ensures that minimality can be achieved. Besides, as it turns out, one can show that $T(X)$ is given by the orthogonal complement $\mathrm{NS}(X)^\perp \subseteq H^2(X,\mathbb{Z})$ of the Néron-Severi group of X in $H^2(X,\mathbb{Z})$, see [Huy16, Lem. 3.3.1]. If one views $T(X)$ as a sublattice of $\widetilde{H}(X,\mathbb{Z}) \supseteq H^2(X,\mathbb{Z})$, we similarly find that $T(X) = \mathcal{N}(X)^\perp \subseteq \widetilde{H}(X,\mathbb{Z})$.

As a consequence of the definition of $T(X)$, we infer the following elementary albeit useful result:

Proposition 1.1.18. [Huy16, Lem. 3.3.3] *Let $\varphi \in \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))$ be a Hodge isometry $\widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X,\mathbb{Z})$ and let $\varphi_\mathbb{C} := \varphi \otimes \mathrm{id}_\mathbb{C}$. It then holds that*

$$\varphi_\mathbb{C}\big|_{\widetilde{H}^{2,0}(X)} = \mathrm{id}_{\widetilde{H}^{2,0}(X)} \qquad \textit{if and only if} \qquad \varphi_\mathbb{C}\big|_{T(X)_\mathbb{C}} = \mathrm{id}_{T(X)_\mathbb{C}},$$

i.e. $\varphi_\mathbb{C}$ restricts to the identity on $\widetilde{H}^{2,0}(X)$ if and only if it also acts trivially on $T(X)_\mathbb{C}$.

Proof. Since, by definition, $\widetilde{H}^{2,0}(X) := H^{2,0}(X) \subseteq T(X)_\mathbb{C}$, we only have to show that if $\varphi_\mathbb{C}$ restricts to the identity on $\widetilde{H}^{2,0}(X)$, then it necessarily does so as well on $T(X)_\mathbb{C}$. For the sake of contradiction, suppose that this is not the case. Consider then

$$T' := \ker\left(\varphi_\mathbb{C} - \mathrm{id}_{\widetilde{H}(X,\mathbb{Z})}\right) \cap H^2(X,\mathbb{Z}) \subsetneq T(X) \subseteq H^2(X,\mathbb{Z}).$$

By assumption, we have that $H^{2,0}(X) \subseteq T'$, from which it follows that $H^{0,2}(X) = \overline{H^{2,0}(X)} \subseteq T'$, as $\varphi_\mathbb{C} = \varphi_\mathbb{R} \otimes_\mathbb{R} \mathrm{id}_\mathbb{C}$. Furthermore,

$$H^2(X,\mathbb{Z})/T' \cong \mathrm{im}\left(\varphi_\mathbb{C} - \mathrm{id}_{\widetilde{H}(X,\mathbb{Z})}\right) \cap H^2(X,\mathbb{Z}) \subseteq H^2(X,\mathbb{Z})$$

is free abelian, for it is a subgroup of the free abelian group $H^2(X,\mathbb{Z})$. In particular, we infer that $T' \subseteq H^2(X,\mathbb{Z})$ is a primitive sub-Hodge structure of $H^2(X,\mathbb{Z})$ with $H^{2,0}(X) \subseteq T'_\mathbb{C}$, which is properly contained in $T(X)$. This contradicts the definition of the transcendental lattice $T(X)$. ☐

We refer to these isometries as symplectic:

Definition 1.1.19. A Hodge isometry $\varphi \in \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))$ is said to be *symplectic* if $\varphi_\mathbb{C}$ restricts to the identity on $\widetilde{H}^{2,0}(X)$. We denote the group of symplectic Hodge isometries $\widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X,\mathbb{Z})$ by $\mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}))$.

1.2 The period domain

In this section, we follow Huybrechts' approach in [Huy16, Ch. 6] and present two realizations of the period domain. Henceforth, let Λ be a non-degenerate lattice with bilinear form $(\,.\,)$ and signature (n_+, n_-).

Definition 1.2.1. The *period domain* $\mathcal{D}_\Lambda$ associated with Λ is defined as

$$\mathcal{D}_\Lambda := \{[x] \in \mathbb{P}(\Lambda_\mathbb{C}) : (x.x) = 0,\ (x.\bar{x}) > 0\} \subseteq \mathbb{P}(\Lambda_\mathbb{C}).$$

Remark 1.2.2. Observe that the second condition is indeed well-defined, for $(\lambda x.\overline{\lambda x}) = |\lambda|^2(x,\bar{x})$. Also, note that $\mathcal{D}_\Lambda$ is an open subset (with respect to the analytic topology) of the zero locus $\{[x] \in \mathbb{P}(\Lambda_\mathbb{C}) : (x.x) = 0\}$, which, in turn, defines a smooth quadric as $(\,.\,)$ is non-degenerate. Thus, we can regard the period domain as a complex manifold, cf. [Huy05, Ch. 2.1].

Take now $[x] \in \mathcal{D}_\Lambda$. Since $\Lambda_\mathbb{C}$ comes with a real structure, we can consider the real and imaginary parts $\operatorname{Re} x, \operatorname{Im} x \in \Lambda_\mathbb{R}$ and the 2-plane they span:

$$P_x \coloneqq \operatorname{span}(\operatorname{Re} x, \operatorname{Im} x) \subseteq \Lambda_\mathbb{R}.$$

One can easily verify that the defining constraints of the period domain ensure that P_x is positive definite with respect to the $\mathbb{R}$-linear extension of $(\,.\,)$ and that $\operatorname{Re} x, \operatorname{Im} x$ are orthogonal to each other:

$$\begin{aligned} 4(\operatorname{Re} x.\operatorname{Re} x) &= (x+\bar{x}.x+\bar{x}) = (x.x) + \overline{(x.x)} + 2(x.\bar{x}) = 2(x.\bar{x}) > 0, \\ 4(\operatorname{Im} x.\operatorname{Im} x) &= -(x-\bar{x}.x-\bar{x}) = -(x.x) - \overline{(x.x)} + 2(x.\bar{x}) = 2(x.\bar{x}) > 0, \\ 4\mathbf{i}(\operatorname{Re} x.\operatorname{Im} x) &= (x+\bar{x}.x-\bar{x}) = (x.x) - \overline{(x.x)} = 0. \end{aligned}$$

Conversely, for an oriented positive definite 2-plane $P \subseteq \Lambda_\mathbb{R}$, there exists an oriented orthonormal basis (u, v), e.g. by Gram-Schmidt. It then holds that $[u + \mathbf{i}v] \in \mathcal{D}_\Lambda$, and the oriented orthonormal basis (u, v) is unique up to the action of $\mathrm{SO}(2) \cong \mathrm{U}(1)$.

Let $\mathrm{Gr}^{\mathrm{p}}(2, \Lambda_\mathbb{R})$ denote the open set of the Grassmannian $\mathrm{Gr}(2, \Lambda_\mathbb{R})$ consisting of all 2-planes $P \subset \Lambda_\mathbb{R}$ on which the bilinear form $(\,.\,)$ restricts to a positive definite form. Furthermore, let $\mathrm{Gr}^{\mathrm{po}}(2, \Lambda_\mathbb{R})$ denote the manifold of all such planes equipped with a choice of orientation. Summarizing the preceding discussion we obtain:

Proposition 1.2.3. [Huy16, Prop. 6.1.5] *The following map defines a diffeomorphism:*

$$\begin{aligned} \eta_\Lambda \colon \mathcal{D}_\Lambda &\longrightarrow \mathrm{Gr}^{\mathrm{po}}(2, \Lambda_\mathbb{R}) \\ [x] &\longmapsto P_x \coloneqq \operatorname{span}(\operatorname{Re} x, \operatorname{Im} x). \end{aligned}$$

Remark 1.2.4. For $n_+ = 2$, we see that $\mathrm{Gr}^{\mathrm{po}}(2, \Lambda_\mathbb{R})$ has two connected components depending on the orientation of the planes—see Remark 1.3.6 on how the orientations of two oriented positive definite 2-planes in $\Lambda_\mathbb{R}$ can be compared. Thus, for $n_+ = 2$, we find that $\mathcal{D}_\Lambda$ decomposes as the disjoint union $\mathcal{D}_\Lambda^+ \sqcup \mathcal{D}_\Lambda^-$ of its two connected components $\mathcal{D}_\Lambda^+$ and $\mathcal{D}_\Lambda^-$.

Finally, we present the *tube domain realization* of the period domain. For this purpose, we first have to assume that $n_+, n_- > 0$. Sylvester's law of inertia yields the existence of an orthogonal basis $B \coloneqq \{b_1, \ldots, b_{n_+}, c_1, \ldots, c_{n_-}\}$ of $\Lambda_\mathbb{R}$ such that $(b_i.b_i) = 1$ and $(c_j.c_j) = -1$. Let $e \coloneqq \frac{1}{2}(b_1 + c_1)$, $f \coloneqq (c_1 - b_1)$ and note that $(e.e) = (f.f) = 0$ and $(e.f) = -1$, i.e. $U_\mathbb{R} \coloneqq \operatorname{span}(e, f)$ is a hyperbolic plane. We then consider the orthogonal decomposition $\Lambda_\mathbb{R} = U_\mathbb{R} \oplus W$, where $W \coloneqq U_\mathbb{R}^\perp$.

Definition 1.2.5. The *tube domain* H_Λ of Λ with respect to the decomposition $\Lambda_\mathbb{C} = U_\mathbb{C} \oplus W_\mathbb{C}$ is defined as

$$H_\Lambda \coloneqq \{z \in W_\mathbb{C} : (\operatorname{Im} z.\operatorname{Im} z) > 0\}.$$

Remark 1.2.6. The tube domain comes with a natural complex manifold structure.

Proposition 1.2.7. [Huy16, Prop. 6.1.7] *Assume that $n_+ = 2$. Then, the following map defines a biholomorphism:*

$$\begin{aligned} H_\Lambda &\longrightarrow \mathcal{D}_\Lambda \\ z &\longmapsto \left[e + \tfrac{1}{2}(z.z)f + z\right]. \end{aligned}$$

Proof. Let us first confirm that the map takes values within D_Λ. To this end, let $z \in H_\Lambda$ and note that

$$\left(e + \tfrac{1}{2}(z.z)f + z \,.\, e + \tfrac{1}{2}(z.z)f + z\right) = (z.z)(e.f) + (z.z) = 0,$$

and that

$$\left(e + \tfrac{1}{2}(z.z)f + z \,.\, \overline{e + \tfrac{1}{2}(z.z)f + z}\right) = \left(e + \tfrac{1}{2}(z.z)f + z \,.\, e + \tfrac{1}{2}(\bar{z}.\bar{z})f + \bar{z}\right)$$

$$= \tfrac{1}{2}((z.z) + (\bar{z}.\bar{z}))(e.f) + (z.\bar{z})$$
$$= -\tfrac{1}{2}(z - \bar{z}.z - \bar{z}) = 2(\operatorname{Im} z.\operatorname{Im} z) > 0,$$

since $z \in H_\Lambda$. Moreover, it is straightforward to verify that the map is injective, leaving only the surjectivity to be established. Take $[x] \in \mathcal{D}_\Lambda \subseteq \mathbb{P}(\Lambda_\mathbb{C})$ with $x = \alpha e + \beta f + z$ for suitable $\alpha, \beta \in \mathbb{C}$ and $z \in W_\mathbb{C}$. Let us first address the case $\alpha = 0$. By Proposition 1.2.3, one finds that $\operatorname{Re}(\beta f + z)$ and $\operatorname{Im}(\beta f + z)$ span a positive definite plane in $\mathbb{R}f \oplus W_\mathbb{R} \subseteq \Lambda_\mathbb{R}$ with respect to the corresponding restriction of the $\mathbb{R}$-linear extension of $(\,.\,)$. Note, however, that f is isotropic in $\mathbb{R}f \oplus W_\mathbb{R}$, so it follows that $\mathbb{R}f \oplus W_\mathbb{R}$ has at most $n_+ - 1 = 1$ positive eigenvalues. Hence, $\alpha \neq 0$ and we may assume that $\alpha = 1$. Thus, we are left to show that $\beta = \frac{1}{2}(z.z)$ and that $z \in H_\Lambda$, which follow from the definition of $\mathcal{D}_\Lambda$:

$$\begin{aligned}(x.x) = 0 &\Leftrightarrow (e + \beta f + z\,.\,e + \beta f + z) = 0\\ &\Leftrightarrow 2\beta(e.f) + (z.z) = 0\\ &\Leftrightarrow \beta = \tfrac{1}{2}(z.z)\end{aligned}$$

and based on the above calculation:

$$(\operatorname{Im} z.\operatorname{Im} z) = \tfrac{1}{2}\left(e + \tfrac{1}{2}(z.z)f + z\,.\,\overline{e + \tfrac{1}{2}(z.z)f + z}\right) = \tfrac{1}{2}(x.\bar{x}) > 0. \qquad \square$$

In particular, for $n_+ = 2$ and $n_- = 1$, we obtain that $W_\mathbb{C}$ is isometric to $\mathbb{C}$ with the standard quadratic form. Consequently, we have $H_\Lambda = \{z \in \mathbb{C} : |\operatorname{Im} z|^2 > 0\} = \mathbb{H} \sqcup (-\mathbb{H})$, where $\mathbb{H} \coloneqq \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denotes the upper half plane.

Corollary 1.2.8. *For $n_+ = 2$ and $n_- = 1$, the map in Proposition 1.2.7 induces a biholomorphism $\mathbb{H} \to \mathcal{D}_\Lambda^+$.*

1.3 Autoequivalences and Bridgeland's conjecture

We took inspiration from the exposition in [Bri08, Sec. 1] and [Huy16, Ch. 16]. For a comprehensive and thorough account on the matter, the reader is encouraged to consult [Huy06].

Let X be a K3 surface.

Definition 1.3.1. Let $\mathrm{D^b}(X) \coloneqq \mathrm{D^b}(\mathrm{Coh}(X))$ denote the *bounded derived category of coherent sheaves on X*. It defines a $\mathbb{C}$-linear triangulated category.

In particular, $\mathrm{D^b}(X)$ comes equipped with a distinguished additive equivalence $\mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$: the shift functor $E^\bullet \mapsto E^\bullet[1]$.

Definition 1.3.2. Let $\mathrm{Aut}(\mathrm{D^b}(X))$ denote the group of $\mathbb{C}$-linear exact autoequivalences of $\mathrm{D^b}(X)$, taken up to isomorphism of functors. We refer to its elements simply as *autoequivalences* of $\mathrm{D^b}(X)$.

Besides the shift functor, natural examples of autoequivalences include tensoring with a line bundle $\mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$, $E \mapsto L \otimes E$ for $L \in \mathrm{Pic}(X)$, see [Huy16, Sec. 16.2.3]. More elaborate constructions, introduced by Mukai in [Muk87, Prop. 2.25], give rise to *spherical twist functors*, which play a central role in the structure of $\mathrm{Aut}(\mathrm{D^b}(X))$.

Definition 1.3.3. An object $E \in \mathrm{Ob}(\mathrm{D^b}(X))$ is called *spherical* if

$$\mathrm{Hom}(E, E[i]) \cong \begin{cases} \mathbb{C} & \text{if } i = 0, 2\\ 0 & \text{otherwise.}\end{cases}$$

For a spherical object $E \in \mathrm{Ob}(\mathrm{D}^{\mathrm{b}}(X))$, the *spherical twist functor* $T_E \colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X)$ is defined by the exact triangle

$$\mathrm{Hom}^{\bullet}(E, F) \otimes E \xrightarrow{\mathrm{ev}} F \longrightarrow T_E(F),$$

where ev denotes the natural evaluation map. Equivalently, T_E is the mapping cone of ev.

Indeed, Seidel and Thomas established in [ST01, Thm. 1.2] that T_E defines an autoequivalence of $\mathrm{D}^{\mathrm{b}}(X)$ whenever E is spherical.

A fundamental result of Orlov in [Orl97, Prop. 3.5], building on earlier work of Mukai in [Muk87, Thm. 4.9], asserts that every autoequivalence of $\mathrm{D}^{\mathrm{b}}(X)$ induces a Hodge isometry of $\widetilde{H}(X, \mathbb{Z})$. This gives rise to a representation

$$\varpi \colon \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z})).$$

Example 1.3.4. Let us revisit the autoequivalences introduced above, cf. [Huy16, Ex. 16.3.4].

(i) The shift functor [1] induces $-\mathrm{id}$ on $\widetilde{H}(X, \mathbb{Z})$.

(ii) Tensoring with a line bundle L of first Chern class $\ell \in H^2(X, \mathbb{Z})$ acts on $\widetilde{H}(X, \mathbb{Z})$ by multiplication with $(1, \ell, \frac{1}{2}\ell^2) \in \widetilde{H}^{1,1}(X) \cap \widetilde{H}(X, \mathbb{Z})$.

(iii) The spherical shift functor T_E associated with a spherical object $E \in \mathrm{Ob}(\mathrm{D}^{\mathrm{b}}(X))$ acts on cohomology as the reflection $\widetilde{H}(X, \mathbb{Z}) \to \widetilde{H}(X, \mathbb{Z})$, $\alpha \mapsto \alpha + (\alpha.v(E)) \cdot v(E)$ in the hyperplane orthogonal to the Mukai vector $v(E) \coloneqq \mathrm{ch}(E)\sqrt{\mathrm{td}(X)} \in \widetilde{H}(X, \mathbb{Z})$.

Huybrechts, Macrì and Stellari showed in [HMS09, Cor. 3] that the image of ϖ was precisely the subgroup $\mathrm{Aut}^+(\widetilde{H}(X, \mathbb{Z})) \subseteq \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}))$ of all orientation preserving Hodge isometries of $\widetilde{H}(X, \mathbb{Z})$.

Definition 1.3.5. Let Λ be a lattice with signature (n_+, n_-). An isometry $\varphi \in \mathrm{O}(\Lambda)$ is said to be *orientation preserving* if $\varphi_{\mathbb{R}}$ preserves the orientations of positive definite n_+-planes in $\Lambda_{\mathbb{R}}$.

Remark 1.3.6. Note that the orientations of two positive definite n_+-planes P and P' in $\Lambda_{\mathbb{R}}$ can be compared via orthogonal projection onto a fixed positive definite n_+-plane Q, see [Ray06, Appx. A]. This does not depend on the choice of Q.

Thus, the study of the group of autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$ comes down to understanding the kernel of the representation ϖ, which we shall denote $\mathrm{Aut}_0(\mathrm{D}^{\mathrm{b}}(X))$. In the general case of a K3 surface of arbitrary Picard number, the kernel $\mathrm{Aut}_0(\mathrm{D}^{\mathrm{b}}(X))$ remains, at present, largely unknown. Beyond the double shift [2], note that the square T_E^2 of spherical twist functor lies in the kernel of ϖ, as reflections are involutions. Furthermore, it is straightforward to check that $T_E(E) \cong E[-1]$, so we see that the element T_E^2 is of infinite order.

In [Bri08], Bridgeland develops the theory of stability conditions on K3 surfaces and proposes a conjectural description of the group structure of $\mathrm{Aut}_0(\mathrm{D}^{\mathrm{b}}(X))$. Before stating the conjecture, let us first introduce some notation.

Let $\mathcal{P}(X)$ denote the open subset of $\mathcal{N}(X)_{\mathbb{C}}$ given by

$$\mathcal{P}(X) \coloneqq \{x \in \mathcal{N}(X)_{\mathbb{C}} \colon \ \mathrm{Re}\, x \text{ and } \mathrm{Im}\, x \text{ span a positive definite plane } P_x \text{ in } \mathcal{N}(X)_{\mathbb{R}}\}.$$

Note that the real and imaginary parts of a given $x \in \mathcal{P}(X)$ determine the orientation of the induced plane $P_x \subseteq \mathcal{N}(X)_{\mathbb{R}}$. Therefore, $\mathcal{P}(X)$ decomposes as the disjoint union of its two connected components $\mathcal{P}^+(X)$ and $\mathcal{P}^-(X)$. Let $\mathcal{P}^+(X)$ denote the connected component containing $(1, \mathbf{i}\,\omega, -\frac{1}{2}\omega^2)$ for an ample class $\omega \in \mathrm{NS}(X)_{\mathbb{R}}$.

Consider now the root system $\Delta(X) \coloneqq \{\delta \in \mathcal{N}(X) : (\delta.\delta) = -2\}$ and, for each $\delta \in \Delta(X)$, let $\delta_{\mathbb{C}}^{\perp} \coloneqq \{x \in \mathcal{N}(X)_{\mathbb{C}} : (x.\delta) = 0\}$ denote the corresponding hyperplane complement in $\mathcal{N}(X)_{\mathbb{C}}$.

Definition 1.3.7. Let $\mathcal{P}_0^+(X)$ denote the open subset of $\mathcal{N}(X)_{\mathbb{C}}$ given by

$$\mathcal{P}_0^+(X) \coloneqq \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta_{\mathbb{C}}^{\perp}.$$

Bridgeland's conjecture reads now as follows:

Conjecture 1.3.8. (Bridgeland, [Bri08, Conj. 1.2]) *There is a short exact sequence of groups*

$$1 \longrightarrow \pi_1(\mathcal{P}_0^+(X)) \longrightarrow \mathrm{Aut}(\mathrm{D^b}(X)) \longrightarrow \mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z})) \longrightarrow 1.$$

1.4 Modular curves

Let $\mathrm{SL}_2(\mathbb{R}) \coloneqq \{M \in \mathrm{GL}_2(\mathbb{R}) : \det(M) = 1\}$ and define $\mathrm{PSL}_2(\mathbb{R}) \coloneqq \mathrm{SL}_2(\mathbb{R})/\{\pm I_2\}$. Note that $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{C} \cup \{\mathbf{i}\infty\}$ via Möbius transformations, i.e. for $z \in \mathbb{C} \cup \{\mathbf{i}\infty\}$ we define

$$\gamma.z \coloneqq \frac{az+b}{cz+d} \qquad \text{and} \qquad \gamma.\mathbf{i}\infty \coloneqq \frac{a}{c}, \qquad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}).$$

In this setting, one easily verifies that $\mathrm{Im}(\gamma.z) = \mathrm{Im}(z) \cdot |cz+d|^{-2}$, so we see that the group action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{C} \cup \{\mathbf{i}\infty\}$ fixes the upper half plane $\mathbb{H} \coloneqq \{\tau \in \mathbb{C} : \mathrm{Im}\,\tau > 0\}$. This, together with some straightforward computations shows that:

Proposition 1.4.1. *The map $\mathrm{PSL}_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{H}$, $(\gamma, \tau) \to \gamma.\tau$ yields a continuous and well-defined group action. Moreover, the group action is transitive and faithful.*

Usually, one is rather interested in the action of certain subgroups of $\mathrm{PSL}_2(\mathbb{R})$, such as congruence or Fuchsian subgroups:

Definition 1.4.2. For $n \in \mathbb{N}_{n \geqslant 1}$, define the *principal congruence subgroup $\Gamma(n)$ of level n* as

$$\Gamma(n) \coloneqq \{\gamma \in \mathrm{PSL}_2(\mathbb{Z}) : \gamma \equiv I_2 \bmod n\} \subseteq \mathrm{PSL}_2(\mathbb{Z}),$$

where the matrix congruence is interpreted entrywise. A subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$ is said to be a *congruence subgroup of level n* if $\Gamma(n) \subseteq \Gamma$ for some $n \in \mathbb{N}_{n \geqslant 1}$. Finally, a *Fuchsian group* denotes a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

Remark 1.4.3. Clearly, every congruence subgroup is Fuchsian, for $\mathrm{PSL}_2(\mathbb{Z})$ forms a discrete subset of $\mathrm{PSL}_2(\mathbb{R})$.

Henceforth, let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group.

Definition 1.4.4. A point $\tau \in \mathbb{H}$ is said to be an *elliptic point of order j* if the stabilizer $\Gamma_\tau \coloneqq \{\gamma \in \Gamma : \gamma.\tau = \tau\}$ of τ in Γ is finite of order $j > 1$.

Proposition 1.4.5. *The stabilizer $\Gamma_{\mathbf{i}}$ of $\mathbf{i}$ in Γ is finite.*

Proof. Let us first determine $\mathrm{PSL}_2(\mathbb{R})_{\mathbf{i}}$, i.e. solve

$$\gamma.\mathbf{i} = \frac{a\mathbf{i}+b}{c\mathbf{i}+d} = \mathbf{i} \qquad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}).$$

This is equivalent to $a\mathbf{i} + b = -c + d\mathbf{i}$, and since a, b, c, $d \in \mathbb{R}$, this can happen if and only if $a = d$ and $b = -c$. In particular, we see that

$$\mathrm{PSL}_2(\mathbb{R})_{\mathbf{i}} \coloneqq \{\gamma \in \mathrm{PSL}_2(\mathbb{R}) : \gamma.\mathbf{i} = \mathbf{i}\}$$

$$
\begin{aligned}
&= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_4(\mathbb{R}) : a^2 + b^2 = 1 \right\} / \pm I_2 \\
&\cong S^1 / \sim, \quad \text{where } x \sim -x,
\end{aligned}
$$

which is compact, as it is a quotient of S^1. We therefore see that $\varGamma_{\mathbf{i}} = \mathrm{PSL}_2(\mathbb{R})_{\mathbf{i}} \cap \varGamma$ is a discrete subset of a compact set and, hence, finite. □

Corollary 1.4.6. *Let τ_1, $\tau_2 \in \mathbb{H}$. Then, the set $\{\gamma \in \varGamma : \gamma.\tau_1 = \tau_2\}$ is finite. In particular, the stabilizer $\varGamma_\tau$ of any point $\tau \in \mathbb{H}$ is finite.*

Proof. Once again, we first solve $\gamma.\tau_1 = \tau_2$ in $\mathrm{PSL}_2(\mathbb{R})$. Since the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive, there exist γ_1, $\gamma_2 \in \mathrm{PSL}_2(\mathbb{R})$ with $\gamma_1.\mathbf{i} = \tau_1$ and $\gamma_2.\mathbf{i} = \tau_2$. Thus,

$$
\{\gamma \in \mathrm{PSL}_2(\mathbb{R}) : \gamma.\tau_1 = \tau_2\} = \{\gamma \in \mathrm{PSL}_2(\mathbb{R}) : (\gamma\gamma_1).\mathbf{i} = \gamma_2.\mathbf{i}\} = \gamma_2 \, \mathrm{PSL}_2(\mathbb{R})_{\mathbf{i}} \gamma_1^{-1}
$$

is also compact, for $\mathrm{PSL}_2(\mathbb{R})_{\mathbf{i}}$ is. As before, we conclude that $\{\gamma \in \varGamma : \gamma.\tau_1 = \tau_2\} = \{\gamma \in \mathrm{PSL}_2(\mathbb{R}) : \gamma.\tau_1 = \tau_2\} \cap \varGamma$ is a discrete subset of a compact set and, hence, finite. □

Another family of special points is given by the cusps:

Definition 1.4.7. A point $s \in \mathbb{R} \cup \{\mathbf{i}\infty\}$ is called a *cusp* of $\varGamma$ if there exists $\gamma \in \varGamma$ whose only fixed point on $\mathbb{R} \cup \{\mathbf{i}\infty\}$ is s. Let $\mathbb{H}^* := \mathbb{H} \cup \{s \in \mathbb{R} \cup \{\mathbf{i}\infty\} : s \text{ is a cusp of } \varGamma\}$ denote the *extended upper half plane* with respect to $\varGamma$.

Proposition 1.4.8. *Let $\tau \in \mathbb{H}^*$ and let $\tilde{\gamma} \in \varGamma$. Then, τ is an elliptic point of order j if and only if $\tilde{\gamma}.\tau$ is an elliptic point of order j. Similarly, τ is a cusp if and only if $\tilde{\gamma}.\tau$ is a cusp as well.*

Proof. Note that we have an isomorphism

$$
\begin{aligned}
\varGamma_\tau &\longrightarrow \varGamma_{\tilde{\gamma}.\tau} \\
\gamma &\longmapsto \tilde{\gamma}\gamma\tilde{\gamma}^{-1}.
\end{aligned}
$$

For the second claim, observe that for a point $\tau' \in \mathbb{R} \cup \{\mathbf{i}\infty\}$, it clearly holds that $\tilde{\gamma}.\tau' \in \mathbb{R} \cup \{\mathbf{i}\infty\}$. Thus, $\gamma \in \varGamma_\tau$ fixes another $\tau' \in (\mathbb{R} \cup \{\mathbf{i}\infty\}) \setminus \{\tau\}$ if and only if $\tilde{\gamma}\gamma\tilde{\gamma}^{-1}$ fixes $\tilde{\gamma}.\tau' \in (\mathbb{R} \cup \{\mathbf{i}\infty\}) \setminus \{\tilde{\gamma}.\tau\}$. □

Remark 1.4.9. For this reason, we will—by abuse of language—also refer to the orbit $\varGamma.\tau \in \mathbb{H}^*/\varGamma$ as an elliptic point of order j (resp. a cusp) whenever $\tau \in \mathbb{H}^*$ is an elliptic point of order j (resp. a cusp).

For future reference, let us introduce a topological property of the group action $\varGamma \times \mathbb{H} \to \mathbb{H}$.

Proposition 1.4.10. *The action of $\varGamma$ on $\mathbb{H}$ is properly discontinuous in the sense of [Thu80, Def. 8.2.1], i.e. for every compact set $K \subseteq \mathbb{H}$, it holds that*

$$
|\{\gamma \in \varGamma : \gamma.K \cap K \neq \varnothing\}| < \infty.
$$

Proof. By [Kap24, Thm. 11], it suffices to show that $\varGamma_\tau$ is finite for every $\tau \in \mathbb{H}$ and that for any two points τ, $\tau' \in \mathbb{H}$ with $\tau' \notin \varGamma.\tau$, there exist open neighborhoods U, $V \subseteq \mathbb{H}$ of τ resp. τ' such that $\gamma.U \cap V = \varnothing$ for every $\gamma \in \varGamma$. The former is Corollary 1.4.6 and the latter follows from [Shi94, Prop. 1.8]. □

Remark 1.4.11. Let $\mathbb{H}_0$ be an open subset of $\mathbb{H}$ such that $\varGamma$ fixes $\mathbb{H}_0$. Then, the proof shows that the induced action $\varGamma \times \mathbb{H}_0 \to \mathbb{H}_0$ is also properly discontinuous.

In what follows in this section, we also require that $\varGamma$ and $\mathrm{PSL}_2(\mathbb{Z})$ be commensurable:

Definition 1.4.12. Let $\mathit{\Gamma}_1$ and $\mathit{\Gamma}_2$ be subgroups of $\mathrm{PSL}_2(\mathbb{R})$. Then, $\mathit{\Gamma}_1$ and $\mathit{\Gamma}_2$ are said to be *(mutually) commensurable* if $\mathit{\Gamma}_1 \cap \mathit{\Gamma}_2$ is of finite index in $\mathit{\Gamma}_1$ and in $\mathit{\Gamma}_2$.

Proposition 1.4.13. *The cusps of $\mathit{\Gamma}$ are then given by $\mathbb{Q} \cup \{\mathbf{i}\infty\}$.*

Proof. By [Shi94, Prop. 1.30], the set of cusps of two mutually commensurable subgroups of $\mathrm{PSL}_2(\mathbb{R})$ coincide. Finally, the set of cusps of $\mathrm{PSL}_2(\mathbb{Z})$ is given by $\mathbb{Q} \cup \{\mathbf{i}\infty\}$, see [Shi94, Sec. 1.4]. □

Proposition 1.4.14. *Let $Y(\mathit{\Gamma})$ denote the quotient space of orbits under $\mathit{\Gamma}$, i.e. $Y(\mathit{\Gamma}) \coloneqq \mathbb{H}/\mathit{\Gamma}$. Then, one can put local holomorphic coordinates on $Y(\mathit{\Gamma})$ such that it becomes a Riemann surface.*

Similarly, define $X(\mathit{\Gamma}) \coloneqq \mathbb{H}^*/\mathit{\Gamma}$. Then, one can endow $\mathbb{H}^*$ with a topology extending the analytic topology of $\mathbb{H}$ so that $X(\mathit{\Gamma})$, equipped with the quotient topology, becomes a Hausdorff, connected and compact space. As before, one can introduce local holomorphic coordinates on $X(\mathit{\Gamma})$ extending those on $Y(\mathit{\Gamma})$, making $X(\mathit{\Gamma})$ into a Riemann surface as well.

Proof. See [Shi94, Sec. 1.3] for the construction of a suitable topology on $\mathbb{H}^*$ extending the analytic topology of $\mathbb{H}$. The resulting quotient space $X(\mathit{\Gamma})$ is then clearly connected. Moreover, it is Hausdorff by [Shi94, Thm. 1.28] and compact by [Shi94, Prop. 1.31], for $\mathit{\Gamma}$ and $\mathrm{PSL}_2(\mathbb{Z})$ are mutually commensurable and $X(\mathrm{PSL}_2(\mathbb{Z}))$ is compact, cf. [Shi94, Sec. 1.4]. Finally, see [Shi94, Sec. 1.5] for the construction of holomorphic charts on $X(\mathit{\Gamma})$ turning it into a Riemann surface. □

In this thesis, we will only consider two subgroups of $\mathrm{PSL}_2(\mathbb{R})$. Let us start with the congruence subgroup

$$\mathit{\Gamma}_0(n) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : c \equiv 0 \bmod n \right\}.$$

Then, $\mathit{\Gamma}_0(n)$ is clearly Fuchsian and it also holds that it is commensurable with $\mathrm{PSL}_2(\mathbb{Z})$, see [Shi94, Prop. 1.43]. The other subgroup of interest for this thesis is the *Fricke group* $\mathit{\Gamma}_0^+(n)$, given by

$$\mathit{\Gamma}_0^+(n) \coloneqq \langle \mathit{\Gamma}_0(n), w_n \rangle \subseteq \mathrm{PSL}_2(\mathbb{R}), \qquad \text{where } w_n \coloneqq \begin{pmatrix} 0 & -\frac{1}{\sqrt{n}} \\ \sqrt{n} & 0 \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}).$$

Note that the congruence subgroup $\mathit{\Gamma}_0(n)$ is normalized by the Fricke involution w_n. Indeed, a straightforward computation yields

$$w_n \gamma w_n^{-1} = \begin{pmatrix} -d & \frac{c}{n} \\ bn & -a \end{pmatrix} \in \mathit{\Gamma}_0(n), \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathit{\Gamma}_0(n).$$

Since w_n is an involution, we find that $\mathit{\Gamma}_0^+(n) = \mathit{\Gamma}_0(n) \cup \mathit{\Gamma}_0(n) w_n$. Therefore, $\mathit{\Gamma}_0^+(n)$ and $\mathit{\Gamma}_0(n)$ are commensurable, from which it follows that $\mathit{\Gamma}_0^+(n)$ and $\mathrm{PSL}_2(\mathbb{Z})$ are also commensurable. In particular, the results gathered in this section apply to both $\mathit{\Gamma}_0(n)$ and $\mathit{\Gamma}_0^+(n)$.

2 Two isomorphic orbifolds

Henceforth, let X be a K3 surface with Picard number $\rho(X) = 1$. Equivalently, its Néron–Severi group is of the form $\mathrm{NS}(X) \cong \mathbb{Z}h$, generated by an ample class h, and the degree of h is $(h.h) = 2n$ for some $n \in \mathbb{N}_{\geqslant 1}$. Throughout this chapter, we shall further assume $n \geqslant 2$. The special case $n = 1$ requires a separate treatment, to which we return in Section 2.5.

In [BB17, Thm. 1.3], Bayer and Bridgeland prove that Bridgeland's conjecture (see Conjecture 1.3.8) holds when $\rho(X) = 1$. Moreover, they show (see [BB17, Rem. 7.2]) that this leads to the isomorphism of groups

$$\pi_1^{\mathrm{orb}}\left(\left[\,\mathcal{Q}_0^+(X)\,\Big/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))\right]\right) \cong \operatorname{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2].$$

Thus, the study of the group of symplectic autoequivalences of $\mathrm{D^b}(X)$ up to even shifts comes down to the computation of an orbifold fundamental group. As we shall see in Section 2.5, this isomorphism requires a modification when $n = 1$.

The aim of this chapter is to show that the orbifold $[\mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))]$ is diffeomorphic to the modular curve $[\mathbb{H}_0/\Gamma_0^+(n)]$, whose structure is considerably more accessible. In Chapter 3 we will exploit this identification: By drawing on the rich structure and well-developed theory of classical modular curves, we will obtain a more transparent approach to the computation of the orbifold fundamental group of $[\mathbb{H}_0/\Gamma_0^+(n)]$.

The organization of this chapter is as follows. In Section 2.1, we introduce the biholomorphic spaces $\mathcal{Q}^+(X)$ and $\mathbb{H}$, together with the actions of the groups $\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ and $\Gamma_0^+(n)$. Furthermore, we show that the two group actions are compatible under the biholomorphism between $\mathcal{Q}^+(X)$ and $\mathbb{H}$. In Section 2.2, we restrict to the open subsets $\mathcal{Q}_0^+(X) \subseteq \mathcal{Q}^+(X)$ and $\mathbb{H}_0 \subseteq \mathbb{H}$, and provide a characterization of $\mathbb{H}_0$ which will prove useful in Chapter 3. Section 2.3 constitutes a short digression: as a byproduct of the analysis of $\mathbb{H}_0$, we obtain a structural description of the group $\operatorname{Aut}_0(\mathrm{D^b}(X))$ in the Picard number 1 case. Finally, Section 2.4 brings the arguments together, showing that the orbifolds $[\mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))]$ and $[\mathbb{H}_0/\Gamma_0^+(n)]$ are indeed diffeomorphic and that, as a result, their orbifold fundamental groups are isomorphic. Section 2.5 concludes by revisiting the case $n = 1$ and outlining the modifications required in that setting.

2.1 Compatible group actions

Recall that the numerical Grothendieck lattice $\mathcal{N}(X)$ of X was defined as $\mathcal{N}(X) \coloneqq H^0(X,\mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X,\mathbb{Z}) \subseteq \widetilde{H}(X,\mathbb{Z})$. Letting e_0 and e_4 be generators of $H^0(X,\mathbb{Z})$, $H^4(X,\mathbb{Z}) \cong \mathbb{Z}$, we find that $\mathcal{N}(X)$ is given by the lattice $\mathbb{Z}e_0 \oplus \mathbb{Z}h \oplus \mathbb{Z}e_4$ with signature $(2,1)$ and Gram matrix

$$G_{\mathcal{N}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2n & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Definition 2.1.1. Let $\mathcal{Q}(X)$ denote the period domain $\mathcal{D}_{\mathcal{N}(X)}$ associated with the numerical Grothendieck group $\mathcal{N}(X)$ of X, i.e.

$$\mathcal{Q}(X) \coloneqq \{[x] \in \mathbb{P}(\mathcal{N}(X)_{\mathbb{C}}) : (x.x) = 0, (x.\bar{x}) > 0\} \subseteq \mathbb{P}(\mathcal{N}(X)_{\mathbb{C}}).$$

As observed in Remark 1.2.4, $\mathcal{Q}(X)$ has two connected components $\mathcal{Q}^+(X)$ and $\mathcal{Q}^-(X)$. Let $\mathcal{Q}^+(X)$ denote the connected component of $\mathcal{Q}(X)$ containing the element $[e_0 + \mathbf{i}\,h - ne_4]$. We are now ready to define the period space that appears in Bayer-Bridgeland's result in [BB17, Rem. 7.2]:

Definition 2.1.2. Let $\mathcal{Q}_0^+(X)$ denote the open subset of $\mathcal{Q}^+(X)$ given by

$$\mathcal{Q}_0^+(X) := \mathcal{Q}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} (\delta_{\mathbb{C}}^{\perp} \cap \mathcal{Q}^+(X)).$$

Before analyzing the period space $\mathcal{Q}_0^+(X)$ and the action of the group $\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ on it, we will begin by studying the more accessible period domain $\mathcal{Q}^+(X)$. To this end, recall that

Proposition 2.1.3. *Let $\mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$ denote the upper half plane. Then, the following map defines a biholomorphism:*

$$\begin{aligned} \psi \colon \mathbb{H} &\longrightarrow \mathcal{Q}^+(X) \\ \tau &\longmapsto [e_0 + \tau h + n\tau^2 e_4]. \end{aligned}$$

Proof. This is a straightforward consequence of Proposition 1.2.7 and Corollary 1.2.8, by noting that, in this case, $\Lambda = \mathcal{N}(X)$ and $W_\Lambda = \mathbb{R}h$. □

Let us first study how $\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ acts on $\mathcal{Q}^+(X)$. To this end, take a Hodge isometry $\varphi \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ and note that, by definition, $\varphi_{\mathbb{C}}(\widetilde{H}^{1,1}(X)) = \widetilde{H}^{1,1}(X)$. As $\mathcal{N}(X) = \widetilde{H}^{1,1}(X) \cap \widetilde{H}(X,\mathbb{Z})$, φ restricts to an isometry $\varphi^{\mathcal{N}} \colon \mathcal{N}(X) \to \mathcal{N}(X)$, which, in turn, extends $\mathbb{C}$-linearly to an isometry $\varphi_{\mathbb{C}}^{\mathcal{N}} \colon \mathcal{N}(X)_{\mathbb{C}} \to \mathcal{N}(X)_{\mathbb{C}}$. As a result, it descends to a well-defined injective holomorphic map $\hat{\varphi} \colon \mathcal{Q}^+(X) \to \mathcal{Q}(X), [x] \mapsto [\varphi_{\mathbb{C}}^{\mathcal{N}}(x)]$.

Proposition 2.1.4. *The isometry $\varphi_{\mathbb{R}}^{\mathcal{N}} \colon \mathcal{N}(X)_{\mathbb{R}} \to \mathcal{N}(X)_{\mathbb{R}}$ preserves the orientation of positive definite 2-planes in $\mathcal{N}(X)_{\mathbb{R}}$.*

Proof. Let $U := \operatorname{span}(u_1, u_2) \subseteq \mathcal{N}(X)_{\mathbb{R}}$ be a positive definite 2-plane spanned by u_1, $u_2 \in \mathcal{N}(X)_{\mathbb{R}}$. Furthermore, let $\pi_U \colon \mathcal{N}(X)_{\mathbb{R}} \to U$ denote the projection homomorphism of $\mathcal{N}(X)_{\mathbb{R}}$ onto U along its orthogonal complement. Since $\varphi_{\mathbb{R}}^{\mathcal{N}}$ is an isometry, the 2-plane $\operatorname{span}(\varphi_{\mathbb{R}}^{\mathcal{N}}(u_1), \varphi_{\mathbb{R}}^{\mathcal{N}}(u_2))$ is also positive definite. Therefore, we see that $(\pi \circ \varphi_{\mathbb{R}}^{\mathcal{N}})(u_1)$ and $(\pi \circ \varphi_{\mathbb{R}}^{\mathcal{N}})(u_2)$ also span U—otherwise, there would be a $\lambda \in \mathbb{R}$ with $\varphi_{\mathbb{R}}^{\mathcal{N}}(u_1 - \lambda u_2) \in \ker(\pi_U) = U^{\perp}$, which is negative definite as $\mathcal{N}(X)_{\mathbb{R}}$ has signature $(2,1)$. Denote by $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$ the transition matrix from (u_1, u_2) to $(\pi \circ \varphi_{\mathbb{R}}^{\mathcal{N}}(u_1), \pi \circ \varphi_{\mathbb{R}}^{\mathcal{N}}(u_2))$. We want to show that $\det(A) > 0$.

Consider some non-zero $\sigma \in \widetilde{H}^{2,0}(X)$. Using Remark 1.1.3 together with the fact that $\Omega_X^{4,0} = 0$, one easily finds that $(\sigma.\sigma) = 0$ and that $(\sigma.\bar{\sigma}) > 0$. By the discussion preceding Proposition 1.2.3, the 2-plane $P_\sigma \subseteq \widetilde{H}(X,\mathbb{R})$ spanned by $\operatorname{Re}(\sigma), \operatorname{Im}(\sigma) \in \widetilde{H}(X,\mathbb{R})$ is positive definite. As a result, we observe that $V := \operatorname{span}(u_1, u_2, \operatorname{Re}(\sigma), \operatorname{Im}(\sigma))$ yields a positive definite 4-plane in $\widetilde{H}(X,\mathbb{R})$. Besides, since φ is a Hodge isometry, we infer that $\varphi_{\mathbb{C}}(\sigma) = \lambda\sigma$ for some $\lambda \in \mathbb{C}^{\times}$, from which $\varphi_{\mathbb{R}}(\operatorname{Re}(\sigma)) = \operatorname{Re}(\lambda\sigma)$ and $\varphi_{\mathbb{R}}(\operatorname{Im}(\sigma)) = \operatorname{Im}(\lambda\sigma)$ follow. Finally, recall that φ is orientation preserving, so the transition matrix B from $(u_1, u_2, \operatorname{Re}(\sigma), \operatorname{Im}(\sigma))$ to $\pi \circ \varphi_{\mathbb{R}}(u_1, u_2, \operatorname{Re}(\sigma), \operatorname{Im}(\sigma))$, where $\pi \colon \widetilde{H}(X,\mathbb{R}) \twoheadrightarrow \operatorname{span}((u_1, u_2, \operatorname{Re}(\sigma), \operatorname{Im}(\sigma)))$, given by

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & \operatorname{Re}\lambda & -\operatorname{Im}\lambda \\ 0 & 0 & \operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix},$$

has positive determinant. It follows that $\det(A) = \frac{1}{|\lambda|^2}\det(B) > 0$. □

Corollary 2.1.5. *The map $\hat{\varphi} \colon \mathcal{Q}^+(X) \to \mathcal{Q}(X)$ defines a biholomorphism $\mathcal{Q}^+(X) \to \mathcal{Q}^+(X)$.*

Proof. It suffices to show that $\hat{\varphi}(\mathcal{Q}^+(X)) \subseteq \mathcal{Q}^+(X)$. This is a direct consequence of Proposition 2.1.4 by using the description of $\mathcal{Q}(X)$ in terms of oriented positive definite planes, see Proposition 1.2.3. □

The action of $\varphi \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ on $\mathcal{Q}^+(X)$ is thus given by $\varphi.[x] = [\varphi_{\mathbb{C}}^{\mathcal{N}}(x)]$. Conversely, the action of $\Gamma_0^+(n)$ on the upper half plane $\mathbb{H}$ is given by Möbius transformations, see Section 1.4.

We will now move on to examine how the group actions of $\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ (resp. $\Gamma_0^+(n)$) on $\mathcal{Q}^+(X)$ (resp. $\mathbb{H}$) behave with respect to $\psi\colon \mathbb{H} \to \mathcal{Q}^+(X)$. We took inspiration from [Kaw14].

Let us introduce some convenient notation. First, let $\mathrm{O}^+(\mathcal{N}(X))$ denote the group of lattice isometries $\mathcal{N}(X) \xrightarrow{\sim} \mathcal{N}(X)$ which preserve the orientation of positive definite 2-planes in $\mathcal{N}(X)_{\mathbb{R}}$. Moreover, define $\mathrm{SO}^+(\mathcal{N}(X))$ to be the connected component of $\mathrm{O}^+(\mathcal{N}(X))$ containing those orientation-preserving isometries φ such that $\det(\varphi) = 1$ and, similarly, let $\mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}}) := \{\varphi \in \mathrm{O}^+(\mathcal{N}(X)_{\mathbb{R}}) : \det(\varphi) = 1\}$. We can naturally view $\mathrm{SO}^+(\mathcal{N}(X))$ as a subset of $\mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}})$ via $\varphi \mapsto \varphi_{\mathbb{R}}$.

Note that every element $f \in \mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}})$ acts on the period domain $\mathcal{Q}^+(X)$ via

$$\begin{aligned} \mathcal{Q}^+(X) &\longrightarrow \mathcal{Q}^+(X) \\ [x] &\longmapsto [f_{\mathbb{C}}(x)]. \end{aligned}$$

Indeed, since $f_{\mathbb{C}}$ is a complex linear isomorphism $\mathcal{N}(X)_{\mathbb{C}} \to \mathcal{N}(X)_{\mathbb{C}}$, it induces a holomorphic automorphism $\mathbb{P}(\mathcal{N}(X)_{\mathbb{C}}) \to \mathbb{P}(\mathcal{N}(X)_{\mathbb{C}})$. Because f is an orientation-preserving isometry, this automorphism preserves $\mathcal{Q}^+(X)$. Thus, the action of $\mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}})$ on $\mathcal{Q}^+(X)$ is given by biholomorphisms with respect to the usual complex manifold structure of $\mathcal{Q}^+(X)$, see Remark 1.2.2.

It is a standard result in complex analysis that the group of holomorphic automorphisms of the upper half plane is, precisely, $\mathrm{PSL}_2(\mathbb{R})$. Hence, the biholomorphism ψ induces, by conjugation, a group homomorphism $\mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}}) \to \mathrm{PSL}_2(\mathbb{R})$, which we will show to be an isomorphism below.

In fact, we show that ψ also induces an inverse group homomorphism $R\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}})$ by conjugation, i.e. with $(\psi \circ \gamma \circ \psi^{-1})([x]) = R(\gamma).[x]$ for every $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ and every $[x] \in \mathcal{Q}^+(X)$.

Note that if $R(\gamma) \in \mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}})$, then this last condition determines $R(\gamma)$ uniquely, as we obtain for instance $R(\gamma).[e_0 + \mathbf{i}h - ne_4]$, $R(\gamma).[e_0 + \frac{1}{2}\mathbf{i}h - \frac{1}{4}ne_4]$ and $R(\gamma).[e_0 + \frac{3}{2}\mathbf{i}h - \frac{9}{4}ne_4]$ in terms of γ, and $R(\gamma) \in \mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}})$ is a linear map with $\det(R(\gamma)) = 1$. We use this observation in order to describe $R(\gamma)$ via its transformation matrix M_γ with respect to the ordered basis $B = (e_0, h, e_4)$ of $\mathcal{N}(X)_{\mathbb{R}}$. In order to do so, denote by c_B the canonical coordinate isomorphism $\mathcal{N}(X)_{\mathbb{R}} \to \mathbb{R}^3$, mapping $e_0 \mapsto (1\ 0\ 0)^T$, $h \mapsto (0\ 1\ 0)^T$ and $e_4 \mapsto (0\ 0\ 1)^T$.

Proposition 2.1.6. *For $\gamma \in \mathrm{PSL}_2(\mathbb{R})$, define*

$$M_\gamma := \begin{pmatrix} d^2 & 2cd & \frac{1}{n}c^2 \\ bd & ad+bc & \frac{1}{n}ac \\ nb^2 & 2nab & a^2 \end{pmatrix}, \quad \textit{where} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, the following map defines a group isomorphism:

$$\begin{aligned} R\colon \mathrm{PSL}_2(\mathbb{R}) &\longrightarrow \mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}}) \\ \gamma &\longmapsto c_B \circ M_\gamma \circ c_B^{-1}. \end{aligned}$$

The group isomorphism R can be viewed as being induced by the biholomorphism ψ through conjugation. This means that for every $\gamma \in PSL_2(\mathbb{R})$ and every $[x] \in \mathcal{Q}^+(X)$, we have that

$$(\psi \circ \gamma \circ \psi^{-1})([x]) = R(\gamma).[x] = [R(\gamma)_{\mathbb{C}}(x)].$$

Conversely, $R^{-1}\colon \mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}}) \to \mathrm{PSL}_2(\mathbb{R})$ maps $f \in \mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}})$ to the automorphism $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ satisfying

$$\gamma.(\psi^{-1}([x])) = \psi^{-1}(f.[x]) = \psi^{-1}([f_{\mathbb{C}}(x)]).$$

Proof. For γ as given above, a straightforward computation confirms that

$$\det M_\gamma = \det \begin{pmatrix} d^2 & 2cd & \frac{1}{n}c^2 \\ bd & ad+bc & \frac{1}{n}ac \\ nb^2 & 2nab & a^2 \end{pmatrix} = (ad-bc)^3 = 1$$

and that

$$M_\gamma^T G_{\mathcal{N}} M_\gamma = G_{\mathcal{N}},$$

where we recall that $G_{\mathcal{N}}$ stands for the Gram matrix of $\mathcal{N}(X)_{\mathbb{R}}$ and the bilinear form $(\,.\,)_{\mathbb{R}}$ with respect to the basis B. We conclude that $R(\gamma)$ defines an isometry $\mathcal{N}(X)_{\mathbb{R}} \xrightarrow{\sim} \mathcal{N}(X)_{\mathbb{R}}$.

So far, this shows that $R(\gamma) \in \mathrm{SO}(\mathcal{N}(X)_{\mathbb{R}}) \cong \mathrm{SO}(2,1)$, which has two connected components, see [Hal03, Sec. 1.4]. The fact that $R(\gamma)$ is orientation preserving will follow with a topological argument. Indeed, it is a standard result from the theory of Lie Groups that $\mathrm{SL}_2(\mathbb{R})$ is connected, see for instance [Hal03, Sec. 1.4]. Since $\mathrm{PSL}_2(\mathbb{R})$ is a quotient of $\mathrm{SL}_2(\mathbb{R})$, it is also connected. Besides, R is continuous, for the entries of M_γ are polynomials in the entries of γ, so the image of $\mathrm{PSL}_2(\mathbb{R})$ under R is connected as well. Finally, note that $R(I_2) = \mathrm{id}_{\mathcal{N}(X)_{\mathbb{R}}}$, which is orientation preserving, thus showing that $R(\mathrm{PSL}_2(\mathbb{R}))$ lies completely within the connected component of $\mathrm{SO}(\mathcal{N}(X)_{\mathbb{R}})$ containing the identity. This is precisely $\mathrm{SO}^+(\mathcal{N}(X)_{\mathbb{R}})$. It follows that R is well-defined as a map of sets.

Before proving that R is in fact a group isomorphism, let us show that it is induced by ψ through conjugation. Let $[x] \in \mathcal{Q}^+(X)$ and $\gamma \in \mathrm{PSL}_2(\mathbb{R})$. By Proposition 2.1.3 we can assume that $x = e_0 + \tau h + n\tau^2 e_4$ for some $\tau \in \mathbb{H}$. Now we calculate

$$\begin{aligned}
R(\gamma).[x] =& \; [R(\gamma)_{\mathbb{C}}(x)] \\
=& \; [(c_{B,\mathbb{C}} \circ M_\gamma \circ c_{B^{-1},\mathbb{C}})(e_0 + \tau h + n\tau^2 e_4)] \\
=& \left[(c_{B,\mathbb{C}} \circ M_\gamma) \begin{pmatrix} 1 \\ \tau \\ n\tau^2 \end{pmatrix} \right] \\
=& \left[c_{B,\mathbb{C}} \begin{pmatrix} d^2 + 2cd\tau + c^2\tau^2 \\ bd + (ad+bc)\tau + ac\tau^2 \\ nb^2 + 2nab\tau + na^2\tau^2 \end{pmatrix} \right] \\
=& \left[c_{B,\mathbb{C}} \begin{pmatrix} (c\tau+d)^2 \\ (a\tau+b)(c\tau+d) \\ n(a\tau+b)^2 \end{pmatrix} \right] \\
=& \; [(c\tau+d)^2 e_0 + (a\tau+b)(c\tau+d)h + n(a\tau+b)^2 e_4] \\
=& \left[e_0 + \frac{a\tau+b}{c\tau+d} h + n \left(\frac{a\tau+b}{c\tau+d} \right)^2 e_4 \right] \\
=& \; \psi\left(\frac{a\tau+b}{c\tau+d} \right) \\
=& \; (\psi \circ \gamma \circ \psi^{-1})([x]).
\end{aligned}$$

It follows that $R(\gamma)$ is, indeed, induced by ψ via conjugation. As a result, we see that

$$\begin{aligned}
R(\gamma\tilde{\gamma}).[x] &= (\psi \circ \gamma\tilde{\gamma} \circ \psi^{-1})([x]) \\
&= (\psi \circ \gamma \circ \psi^{-1}) \circ (\psi \circ \tilde{\gamma} \circ \psi^{-1})([x]) \\
&= (R(\gamma) \circ R(\tilde{\gamma})).[x].
\end{aligned}$$

Since the action of $\mathrm{SO}^+(\mathcal{N}(X))$ on $\mathcal{Q}^+(X)$ is clearly faithful, we obtain that $R(\gamma\tilde{\gamma}) = R(\gamma) \circ R(\tilde{\gamma})$, so R defines indeed a group homomorphism. Similarly, $R(\gamma) = R(\tilde{\gamma})$ implies that $(\psi \circ \gamma \circ \psi^{-1}).[x] = (\psi \circ \tilde{\gamma} \circ \psi^{-1}).[x]$ for every $[x] \in \mathcal{Q}^+(X)$. This can only happen if $\gamma.\tau = \tilde{\gamma}.\tau$ for every $\tau \in \mathbb{H}$, as ψ is bijective. Since the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is faithful, this forces $\gamma = \tilde{\gamma}$, so R is injective. We are left to show that R is surjective.

Let $f \in \mathrm{SO}^+(\mathcal{N}(X)_\mathbb{R})$ and consider the map

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{H} \\ \tau &\longmapsto (\psi^{-1} \circ f \circ \psi)(\tau). \end{aligned}$$

Since $\mathrm{SO}^+(\mathcal{N}(X)_\mathbb{R})$ acts on $\mathcal{Q}^+(X)$ by biholomorphisms and $\psi\colon \mathbb{H} \to \mathcal{Q}^+(X)$ is a biholomorphism, it follows that the map defined above must be a biholomorphism as well. Thus, it must be given by a Möbius transformation $\gamma_f \in \mathrm{PSL}_2(\mathbb{R})$. As a result, we find that

$$(\psi^{-1} \circ f \circ \psi)(\tau) = \gamma_f.\tau$$

for every $\tau \in \mathbb{H}$. Equivalently, we obtain that

$$\begin{aligned} & \psi^{-1}(f.[x]) = (\gamma_f \circ \psi^{-1})([x]) \\ \Leftrightarrow \quad & f.[x] = (\psi \circ \gamma_f \circ \psi^{-1})([x]) \end{aligned}$$

for every $x \in \mathcal{Q}^+(X)$. Note that, as already shown, $R(\gamma_f)$ also fulfills $R(\gamma_f).[x] = (\psi \circ \gamma_f \circ \psi^{-1})([x])$, and since the action of $\mathrm{SO}^+(\mathcal{N}(X)_\mathbb{R})$ on $\mathcal{Q}^+(X)$ is faithful, we once again infer that $f = R(\gamma_f)$. Thus, R not only defines an injective but also a surjective group homomorphism. □

In [Kaw14, Sec. 2.4], Kawatani considers the group homomorphism

$$\begin{aligned} \mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z})) \xrightarrow{|_\mathcal{N}} \mathrm{O}^+(\mathcal{N}(X)) \longrightarrow \mathrm{O}^+(\mathcal{N}(X))/\pm\mathrm{id} \xrightarrow{\ \sim\ } \\ \xrightarrow{\ \sim\ } \mathrm{SO}^+(\mathcal{N}(X)) \hookrightarrow \mathrm{SO}^+(\mathcal{N}(X)_\mathbb{R}) \xrightarrow{R^{-1}} \mathrm{PSL}_2(\mathbb{R}) \end{aligned}$$

and shows, building on a result due to Dolgachev in [Dol96, Thm. 7.1], that its image is precisely the Fricke group $\mathit{\Gamma}_0^+(n)$.

Proposition 2.1.7. *The group homomorphism*

$$\begin{aligned} \Psi\colon \quad \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})) \xrightarrow{|_\mathcal{N}} \mathrm{O}^+(\mathcal{N}(X)) \longrightarrow \mathrm{O}^+(\mathcal{N}(X))/\pm\mathrm{id} \xrightarrow{\ \sim\ } \\ \xrightarrow{\ \sim\ } \mathrm{SO}^+(\mathcal{N}(X)) \hookrightarrow \mathrm{SO}^+(\mathcal{N}(X)_\mathbb{R}) \xrightarrow{R^{-1}} \mathrm{PSL}_2(\mathbb{R}) \end{aligned}$$

mapping

$$\begin{aligned} \varphi \longmapsto \varphi^\mathcal{N} \longmapsto [\varphi^\mathcal{N}] \longmapsto \\ \longmapsto \pm\varphi^\mathcal{N} \longmapsto \pm\varphi^\mathcal{N}_\mathbb{R} \longmapsto R^{-1}(\pm\varphi^\mathcal{N}_\mathbb{R}) \end{aligned}$$

surjects onto $\mathit{\Gamma}_0^+(n)$ and induces a group isomorphism $\Psi\colon \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})) \xrightarrow{\sim} \mathit{\Gamma}_0^+(n)$, $\varphi \mapsto R^{-1}(\pm\varphi^\mathcal{N}_\mathbb{R})$. The map $\mathrm{O}^+(\mathcal{N}(X))/\pm\mathrm{id} \to \mathrm{SO}^+(\mathcal{N}(X))$ takes $[\varphi^\mathcal{N}]$ to $\varphi^\mathcal{N}$ if $\det\varphi^\mathcal{N} = 1$ and to $-\varphi^\mathcal{N}$ otherwise.

Proof. Let us start by addressing the surjectivity onto $\mathit{\Gamma}_0^+(n)$. As $\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})) \subseteq \mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z}))$, [Kaw14, Prop. 2.9] implies that $\Psi(\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})))$ is contained in $\mathit{\Gamma}_0^+(n)$. The reverse inclusion follows by Kawatani's proof, which we will follow here. By virtue of [Dol96, Thm. 7.1], it suffices to show that every map in $\mathrm{O}^+(\mathcal{N}(X))^* \coloneqq \ker(\mathrm{O}(\mathcal{N}(X)) \to \mathrm{O}(A_{\mathcal{N}(X)})) \cap \mathrm{O}^+(\mathcal{N}(X))$ lifts to a map in $\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$—here, $\mathrm{O}(A_{\mathcal{N}(X)})$ denotes the group of automorphisms of the discriminant group $A_{\mathcal{N}(X)}$. Indeed, by [Huy16, Prop. 14.2.6], $\tilde{\varphi} \in \mathrm{O}^+(\mathcal{N}(X))^*$ can be lifted to a $\varphi \in \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))$ such that $\varphi|_{T(X)} = \mathrm{id}_{T(X)}$. It follows that $\varphi_\mathbb{C}$ restricts to the identity on $\widetilde{H}^{2,0}(X)$, so $\varphi \in \mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}))$. Moreover, φ preserves the orientation of positive definite 4-planes in $\widetilde{H}(X,\mathbb{R})$, as $\tilde{\varphi}$ preserves the orientation of positive definite 2-planes in $\mathcal{N}(X)_{\mathbb{R}}$, $\varphi_{\mathbb{R}}$ restricts to the identity on $T(X)_{\mathbb{R}}$ and $\widetilde{H}(X,\mathbb{R}) = \mathcal{N}(X)_{\mathbb{R}} \oplus T(X)_{\mathbb{R}}$. Thus, we see that $\varphi \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$.

We are left to show that Ψ is injective. We modify the proof by Kawatani to fit our setting. Note that the injectivity is equivalent to $\mathrm{id} \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ being the only map in $\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ that restricts to $\pm\mathrm{id}_{\mathcal{N}(X)}$ in $\mathrm{O}^+(\mathcal{N}(X))$, for the succeeding homomorphisms are injective. Thus, let $\varphi \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ such that $\varphi^{\mathcal{N}} \in \{\pm\mathrm{id}_{\mathcal{N}(X)}\}$. Since $\rho(X) = 1$, [Ogu02, Lem. 4.1] applies, and we obtain that $\varphi^T \coloneqq \varphi|_{T(X)} = \pm\mathrm{id}_{T(X)}$. It follows that $\varphi^T = \mathrm{id}_{T(X)}$, as φ is symplectic. If $\varphi^{\mathcal{N}} = \mathrm{id}_{\mathcal{N}(X)}$, we are done, since then $\varphi_{\mathbb{R}} = \varphi_{\mathbb{R}}^{\mathcal{N}} \oplus \varphi_{\mathbb{R}}^T = \mathrm{id}_{\widetilde{H}(X,\mathbb{R})}$, so $\varphi = \mathrm{id}_{\widetilde{H}(X,\mathbb{Z})}$. We still have to show that there is no $\varphi \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ such that $\varphi^{\mathcal{N}} = -\mathrm{id}_{\mathcal{N}(X)}$ and $\varphi^T = \mathrm{id}_{T(X)}$. For the sake of contradiction, suppose that such a $\varphi \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ exists. By [Huy16, Lem. 14.2.5], we obtain that $\mathrm{id}_{A(\mathcal{N}(X))} = -\mathrm{id}_{A(\mathcal{N}(X))}$, which requires every element in $A(\mathcal{N}(X))$ to be 2-torsion. Note that by [Huy16, Sec. 14.0.2 & Sec. 14.0.3 iv] together with Remark 1.1.7, we obtain

$$A(\mathcal{N}(X)) \cong A(U \oplus \mathbb{Z}(2n)) \cong A(U) \oplus A(\mathbb{Z}(2n)) \cong A(\mathbb{Z}(2n)) \cong \mathbb{Z}/2n\mathbb{Z}.$$

Finally, as $n \geqslant 2$, we find that $\mathbb{Z}/2n\mathbb{Z}$ has elements which are not 2-torsion, so $\mathrm{id}_{\mathbb{Z}/2n\mathbb{Z}} \neq -\mathrm{id}_{\mathbb{Z}/2n\mathbb{Z}}$. This contradicts the existence of a $\varphi \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ such that $\varphi^{\mathcal{N}} = -\mathrm{id}_{\mathcal{N}(X)}$ and $\varphi^T = \mathrm{id}_{T(X)}$. □

Remark 2.1.8. In Section 2.5, we will construct a $\varphi \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ such that $\varphi^{\mathcal{N}} = -\mathrm{id}_{\mathcal{N}(X)}$ and $\varphi^T = \mathrm{id}_{T(X)}$ for $n = 1$.

We find that the group isomorphism Ψ is compatible with the biholomorphism ψ:

Corollary 2.1.9. *The following diagram commutes for every $\varphi \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$:*

$$\begin{tikzcd} \mathcal{Q}^+(X) \arrow[r, "\psi^{-1}"] \arrow[d, "\varphi"'] & \mathbb{H} \arrow[d, "\Psi(\varphi)"] \\ \mathcal{Q}^+(X) \arrow[r, "\psi^{-1}"] & \mathbb{H}. \end{tikzcd}$$

Proof. Let $[x] \in \mathcal{Q}^+(X)$ and note that

$$\Psi(\varphi).(\psi^{-1}([x])) = R^{-1}(\pm\varphi_{\mathbb{R}}^{\mathcal{N}}).(\psi^{-1}([x])) = \psi^{-1}([\varphi_{\mathbb{C}}^{\mathcal{N}}(x)]) = \psi^{-1}(\varphi.[x]),$$

where the second equality follows from Proposition 2.1.6. □

Remark 2.1.10. Similarly, the following diagram commutes as well for every $\gamma \in \Gamma_0^+(n)$:

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\psi"] \arrow[d, "\gamma"'] & \mathcal{Q}^+(X) \arrow[d, "\Psi^{-1}(\gamma)"] \\ \mathbb{H} \arrow[r, "\psi"] & \mathcal{Q}^+(X). \end{tikzcd}$$

As a consequence of the compatibility of the group actions with the biholomorphism ψ, we find that the quotients $\mathbb{H}/\Gamma_0^+(n)$ and $\mathcal{Q}^+(X)/\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ are homeomorphic. Indeed, let $\pi\colon \mathbb{H} \twoheadrightarrow \mathbb{H}/\Gamma_0^+(n)$ and $\mathrm{pr}\colon \mathcal{Q}^+(X) \twoheadrightarrow \mathcal{Q}^+(X)/\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ denote the projection maps and consider the following commutative diagram:

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\cong", "\psi"'] \arrow[d, "\pi"', two heads] & \mathcal{Q}^+(X) \arrow[d, "\mathrm{pr}", two heads] \\ \mathbb{H}/\Gamma_0^+(n) \arrow[r, dashed, shift left, "\tilde{\psi}"] & \mathcal{Q}^+(X)/\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})) \arrow[l, dashed, shift left, "\eta"] \end{tikzcd} \qquad \begin{tikzcd} \tau \arrow[r, mapsto] \arrow[d, mapsto] & \psi(\tau) \arrow[d, mapsto] \\ \Gamma_0^+(n).\tau \arrow[r, dashed, shift left, mapsto] & \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})).\psi(\tau). \arrow[l, dashed, shift left, mapsto] \end{tikzcd}$$

Let τ, $\tau' \in \mathbb{H}$ and suppose that $\pi(\tau) = \pi(\tau')$, i.e. $\tau' = \gamma.\tau$ for some $\gamma \in \Gamma_0^+(n)$. We then find, by Remark 2.1.10, that

$$\begin{aligned}\mathrm{pr} \circ \psi(\tau') = \mathrm{pr} \circ \psi(\gamma.\tau) = \mathrm{pr} \circ (\Psi^{-1}(\gamma)).(\psi(\tau)) &= \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})).(\Psi^{-1}(\gamma).(\psi(\tau))) \\ &= \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})).(\psi(\tau)) \\ &= \mathrm{pr} \circ \psi(\tau).\end{aligned}$$

Therefore, we see that there exists a unique continuous map $\tilde{\psi} \colon \mathbb{H}/\Gamma_0^+(n) \to \mathcal{Q}^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ such that $\tilde{\psi} \circ \pi = \mathrm{pr} \circ \psi$. An analogous argument shows the existence of a unique continuous map $\eta \colon \mathcal{Q}^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})) \to \mathbb{H}/\Gamma_0^+(n)$ with $\eta \circ \mathrm{pr} = \pi \circ \psi^{-1}$. In particular, we observe that $\eta \circ \tilde{\psi} \circ \pi = \eta \circ \mathrm{pr} \circ \psi = \pi \circ \psi^{-1} \circ \psi = \pi$ and the uniqueness given by the universal property forces $\eta \circ \tilde{\psi} = \mathrm{id}_{\mathbb{H}/\Gamma_0^+(n)}$. The other direction follows analogously, so $\tilde{\psi}$ and η are inverse to each other, thus showing that:

Corollary 2.1.11. *The map*

$$\begin{aligned}\tilde{\psi} \colon \mathbb{H}/\Gamma_0^+(n) &\longrightarrow \mathcal{Q}^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})) \\ \Gamma_0^+(n).\tau &\longmapsto \operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})).(\psi(\tau))\end{aligned}$$

defines a homeomorphism.

So far, we have shown that the Riemann surfaces $\mathcal{Q}^+(X)$ and $\mathbb{H}$ are biholomorphic and that this biholomorphism ψ fulfills a “generalized equivariance” property with respect to the actions given by $\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ and $\Gamma_0^+(n)$ on $\mathcal{Q}^+(X)$ resp. $\mathbb{H}$, as established in the previous corollary.

Since $\mathbb{H}$ is a manifold and the action of $\Gamma_0^+(n)$ is properly discontinuous, cf. Proposition 1.4.10, we can endow the quotient space $\mathbb{H}/\Gamma_0^+(n)$ with a natural orbifold structure—see [Thu80, Prop. 13.2.1] for the construction of the orbifold atlas. In general, quotients of manifolds by properly discontinuous actions are therefore often referred to as *good* orbifolds.

Conversely, note that $\mathcal{Q}^+(X)$ naturally becomes a manifold via the biholomorphism $\psi^{-1} \colon \mathcal{Q}^+(X) \xrightarrow{\sim} \mathbb{H} \subseteq \mathbb{C} \cong \mathbb{R}^2$. Furthermore, the action of $\varphi \in \operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ on $\mathcal{Q}^+(X)$ induces an action on $\mathbb{H}$ via conjugation with ψ, which by virtue of Corollary 2.1.9 is precisely the action of $\Psi(\varphi)$. Thus, the natural orbifold structure on $\mathbb{H}/\Gamma_0^+$ carries over to the natural orbifold structure on $\mathcal{Q}^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$. Concretely, let

$$\mathcal{A} = \{(U_i \subseteq \mathbb{H} \subseteq \mathbb{R}^2,\, G_i \subseteq \Gamma_0^+(n),\, (\pi|_{U_i} \colon U_i \to \pi(U_i))) : i \in I\}$$

be the natural orbifold atlas on $\mathbb{H}/\Gamma_0^+$ and take a chart $(U_i, G_i, \pi|_{U_i})$, where $\pi|_{U_i}$ induces the homeomorphism $\eta_i \colon U_i/G_i \xrightarrow{\sim} \pi(U_i)$. As discussed above, we let $\Psi^{-1}(G_i) \subseteq \operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ act on $\mathbb{H}$ as G_i, so we obtain a homeomorphism $\tilde{\psi} \circ \eta_i \colon U_i/\Psi^{-1}(G_i) \xrightarrow{\sim} \tilde{\psi} \circ \pi(U_i)$ induced by $\tilde{\psi} \circ \pi|_{U_i}$. Thus,

$$\mathcal{B} := \{U_i \subseteq \mathbb{H} \subseteq \mathbb{R}^2,\, \Psi^{-1}(G_i) \subseteq \operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})),\, (\tilde{\psi} \circ \pi|_{U_i} \colon U_i \to \tilde{\psi} \circ \pi(U_i)) : i \in I\}$$

becomes the natural orbifold atlas on $\mathcal{Q}^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$. In order to distinguish between the orbifold and the underlying topological spaces, we will denote the orbifolds by $[\mathbb{H}/\Gamma_0^+(n)]$ and $[\mathcal{Q}^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))]$ and will continue to denote the underlying quotient spaces as we have done so far. It is straightforward to confirm that

Proposition 2.1.12. *The continuous map $\tilde{\psi}$ defines an orbifold diffeomorphism*

$$\left[\mathbb{H}\middle/\Gamma_0^+(n)\right] \xrightarrow{\ \sim\ } \left[\mathcal{Q}^+(X)\middle/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))\right]$$

as defined in [Car22, Sec. 1.4].

Let now $\mathbb{H}_0 \coloneqq \psi^{-1}(\mathcal{Q}_0^+(X))$ and recall that our aim in this chapter is to show that

$$\pi_1^{\mathrm{orb}}\left(\left[\,\mathcal{Q}_0^+(X) \Big/ \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))\right]\right) \cong \pi_1^{\mathrm{orb}}\left(\left[\,\mathbb{H}_0 \Big/ \mathit{\Gamma}_0^+(n)\right]\right).$$

To this end, we must first show that the punctured quotients remain diffeomorphic orbifolds. Before proceeding, however, we still need an explicit description of $\mathbb{H}_0$, which will additionally prove useful later in Chapter 3.

2.2 The punctured upper half plane $\mathbb{H}_0$

Let us start with the observation that

$$\mathbb{H} \setminus \mathbb{H}_0 = \psi^{-1}\left(\bigcup_{\delta \in \Delta(X)} \delta^\perp \cap \mathcal{Q}^+(X)\right).$$

Let now $\delta \in \Delta(X)$ and consider the isometry $s_\delta \colon \widetilde{H}(X,\mathbb{Z}) \to \widetilde{H}(X,\mathbb{Z})$ given by the reflection in the hyperplane $V_\delta \coloneqq \{x \in \widetilde{H}(X,\mathbb{Z}) : (x.\delta) = 0\}$:

$$\begin{aligned} s_\delta \colon \widetilde{H}(X,\mathbb{Z}) &\longrightarrow \widetilde{H}(X,\mathbb{Z}) \\ x &\longmapsto x - 2\frac{(x.\delta)}{(\delta.\delta)}\delta = x + (x.\delta)\delta. \end{aligned}$$

As $T(X) = \mathcal{N}(X)^\perp$ and $\delta \in \Delta(X) \subseteq \mathcal{N}(X)$, it follows that $T(X) \subseteq V_\delta$. Hence, the isometry s_δ restricts to the identity on $T(X)$ and since $H^{2,0}(X) \subseteq T(X)_\mathbb{C}$, we find that s_δ is a symplectic Hodge isometry, i.e. $s_\delta \in \mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}))$.

Proposition 2.2.1. *The isometry $s_{\delta,\mathbb{R}} \coloneqq s_\delta \otimes \mathbb{R} \colon \widetilde{H}(X,\mathbb{R}) \to \widetilde{H}(X,\mathbb{R})$ preserves the orientation of positive definite 4-planes.*

Proof. Let $P = \mathrm{span}(x_1,\dots,x_4) \subseteq \widetilde{H}(X,\mathbb{R})$ be a positive definite 4 plane, and let $x_i = t_i + n_i$ with $t_i \in T(X)_\mathbb{R}$ and $n_i \in \mathcal{N}(X)_\mathbb{R}$. We may assume that the ordered basis $(x_1,\dots,x_4)$ of P is orthonormal. Furthermore, let $\pi_P \colon \widetilde{H}(X,\mathbb{R}) \twoheadrightarrow P$ denote the orthogonal projection onto P. By the above discussion, we find that $s_{\delta,\mathbb{R}}(x_j) = x_j + (n_j.\delta)\delta$ and further, by a well-known result in linear algebra, that

$$\pi_P \circ s_{\delta,\mathbb{R}}(x_j) = \sum_{i=1}^4 (x_j + (n_j.\delta)\delta\,.\,x_i)x_i = x_j + \sum_{i=1}^4((n_j.\delta)(n_i.\delta))x_i.$$

Therefore, the change-of-basis matrix A is given by $A = I_4 + aa^T \in \mathbb{R}^{4,4}$ with $a = (a_1 \cdots a_4)^T \in \mathbb{R}^4$, $a_i \coloneqq (n_i.\delta)$. We claim that A is positive definite. Indeed, letting $r \coloneqq (r_1 \cdots r_4)^T \in \mathbb{R}^4 \setminus \{0\}$, a straightforward computation shows that

$$r^T A r = r^T r + r^T(aa^T)r = \|r\|^2 + \|r^T a\|^2 > 0.$$

It follows that $\det(A) > 0$. Note that for another (non-orthonormal) basis of P, the corresponding transition matrix A' is conjugate to A, so $\det(A') > 0$ as well. ☐

Hence, $s_\delta \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$. As we have seen before, its action on $\mathcal{Q}^+(X)$ yields a biholomorphism $\sigma_\delta \colon \mathcal{Q}^+(X) \xrightarrow{\sim} \mathcal{Q}^+(X)$, $[x] \mapsto [s^{\mathcal{N}}_{\delta,\mathbb{C}}(x)] = [x + (x.\delta)\delta]$. This gives the following characterization:

Proposition 2.2.2. *The set $\delta_\mathbb{C}^\perp \cap \mathcal{Q}^+(X) \subseteq \mathbb{P}(\mathcal{N}(X)_\mathbb{C})$ is, precisely, the set of fixed points of σ_δ.*

Proof. Indeed, if $[x] \in \delta_{\mathbb{C}}^{\perp} \cap \mathcal{Q}^+(X)$, it follows that $\sigma_\delta([x]) = [x + (x.\delta)\delta] = [x]$. Conversely, suppose that $[x] \in \mathcal{Q}^+(X)$ is a fixed point of σ_δ, so $x + (x.\delta)\delta = \lambda x$ for some $\lambda \in \mathbb{C}^\times$. As a result, we find that $0 = (\lambda - 1)^2(x.x) = (x.\delta)^2(\delta.\delta) = -2(x.\delta)^2$, from which $(x.\delta) = 0$ and, thus, $[x] \in \delta_{\mathbb{C}}^{\perp} \cap \mathcal{Q}^+(X)$ follow. ☐

The next proposition shows that we can also characterize fixed points of σ_δ (and, hence, points in $\delta^\perp \cap \mathcal{Q}^+(X)$) in terms of fixed points of $\Psi(s_\delta)$.

Proposition 2.2.3. *Let $[x] \in \mathcal{Q}^+(X)$ and $\delta \in \Delta(X)$. Then, $[x]$ is a fixed point of σ_δ if and only if $\psi^{-1}([x])$ is a fixed point of $\Psi(s_\delta)$.*

Proof. Recall that by Corollary 2.1.9 we have the following commutative diagram

$$\begin{tikzcd} \mathcal{Q}^+(X) \arrow[r, "\psi^{-1}"] \arrow[d, "s_\delta"] & \mathbb{H} \arrow[d, "\Psi(s_\delta)"] \\ \mathcal{Q}^+(X) \arrow[r, "\psi^{-1}"] & \mathbb{H}. \end{tikzcd}$$

Suppose that $s_\delta.([x]) = \sigma_\delta([x]) = [x]$ for some $\delta \in \Delta(X)$. Then, $\psi^{-1}([x]) = \psi^{-1}(s_\delta.([x])) = \Psi(s_\delta).(\psi^{-1}([x]))$, so $\psi^{-1}([x])$ is indeed a fixed point of $\Psi(s_\delta)$. Conversely, observe that if $\psi^{-1}([x])$ is a fixed point of $\Psi(s_\delta)$, it follows that $\sigma_\delta([x]) = s_\delta.([x]) = \psi(\Psi(s_\delta).(\psi^{-1}([x]))) = \psi(\psi^{-1}([x])) = [x]$.☐

We are left to establish how $\Psi(s_\delta)$ acts on $\mathbb{H}$.

Proposition 2.2.4. [FL23, Lem. 4.2] *Let $\delta = re_0 + dh + se_4 \in \Delta(X) \subseteq \mathcal{N}(X)$. The biholomorphism $\Psi(s_\delta)\colon \mathbb{H} \to \mathbb{H}$ is given by the action of the involution*

$$\begin{pmatrix} \sqrt{n}d & -\frac{s}{\sqrt{n}} \\ \sqrt{n}r & -\sqrt{n}d \end{pmatrix} = \begin{pmatrix} s & d \\ nd & r \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\sqrt{n}} \\ \sqrt{n} & 0 \end{pmatrix},$$

which lies in the coset $\Gamma_0(n)w_n \subseteq \Gamma_0(n)^+$. Conversely, every involution in $\Gamma_0(n)w_n$ has the above form and, thus, is equal to $\Psi(s_\delta)$ for some suitable $\delta \in \Delta(X)$.

Proof. Let us first show that $\Psi(s_\delta)$ is given, as stated, by the action of

$$\begin{pmatrix} \sqrt{n}d & -\frac{s}{\sqrt{n}} \\ \sqrt{n}r & -\sqrt{n}d \end{pmatrix}.$$

Let $\tau \in \mathbb{H}$ and note that by Corollary 2.1.9:

$$\begin{aligned} \Psi(s_\delta).\tau &= \psi^{-1}(s_\delta.(\psi(\tau))) \\ &= \psi^{-1}(s_\delta.([e_0 + \tau h + n\tau^2 e_4])) \\ &= \psi^{-1}([e_0 + \tau h + n\tau^2 e_4 + (e_0 + \tau h + n\tau^2 e_4 \,.\, re_0 + dh + se_4)(re_0 + dh + se_4)]) \\ &= \psi^{-1}([e_0 + \tau h + n\tau^2 e_4 + (-s + 2n\tau d - n\tau^2 r)(re_0 + dh + se_4)]) \\ &= \psi^{-1}([(-rs + 2n\tau dr - n\tau^2 r^2 + 1)e_0 + (-sd + 2n\tau d^2 - n\tau^2 rd + \tau)h \\ &\qquad + (-s^2 + 2n\tau ds - n\tau^2 rs + n\tau^2)e_4]). \end{aligned}$$

Recall that $\delta \in \Delta(X)$, so $-2 = (re_0 + dh + se_4 \,.\, re_0 + dh + se_4) = 2nd^2 - 2rs$ or, equivalently, $nd^2 - rs = -1$. Upon substitution, we obtain

$$\begin{aligned} \Psi(s_\delta).\tau = \psi^{-1}([&(-nd^2 + 2n\tau dr - n\tau^2 r^2)e_0 + (-sd + n\tau d^2 + rs\tau - n\tau^2 rd)h \\ &+ (-s^2 + 2n\tau ds - n\tau^2 nd^2)e_4]) \end{aligned}$$

$$
\begin{aligned}
&= \psi^{-1}([-n(r\tau - d)^2 e_0 - (r\tau - d)(nd\tau - s)h - (nd\tau - s)^2 e_4]) \\
&= \psi^{-1}\left(\left[e_0 + \frac{nd\tau - s}{n(r\tau - d)} h + \frac{(s - nd\tau)^2}{n(r\tau - d)^2} e_4\right]\right) \\
&= \frac{\sqrt{n} d\tau - \frac{s}{\sqrt{n}}}{\sqrt{n} r\tau - \sqrt{n} d}.
\end{aligned}
$$

The claim follows, as the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is faithful. A straightforward computation shows that

$$
\begin{pmatrix} \sqrt{n} d & -\frac{s}{\sqrt{n}} \\ \sqrt{n} r & -\sqrt{n} d \end{pmatrix} = \begin{pmatrix} s & d \\ nd & r \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\sqrt{n}} \\ \sqrt{n} & 0 \end{pmatrix},
$$

and, furthermore, we see by Corollary 2.1.9 that

$$
\Psi(s_\delta)(\Psi(s_\delta).\tau) = \Psi(s_\delta)(\psi^{-1} \circ s_\delta \circ \psi(\tau)) = \psi^{-1} \circ s_\delta \circ \psi(\psi^{-1} \circ s_\delta \circ \psi(\tau)) = \tau
$$

as s_δ is an involution. This shows that $\Psi(s_\delta)$ is an involution which lies in the coset $\mathit{\Gamma}_0(n) w_n \subseteq \mathit{\Gamma}_0^+(n)$.

Let us proceed with the reverse implication, i.e. that every involution in $\mathit{\Gamma}_0(n) w_n$ can be written in this form. Let

$$
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathit{\Gamma}_0(n) \subseteq \mathrm{PSL}_2(\mathbb{Z})
$$

and suppose that γw_n yields an involution. This implies, again due to the faithfulness of the action, that

$$
I_2 = (\gamma w_n)^2 = \begin{pmatrix} b\sqrt{n} & -\frac{a}{\sqrt{n}} \\ d\sqrt{n} & -\frac{c}{\sqrt{n}} \end{pmatrix}^2 = \begin{pmatrix} b^2 n - ad & -ab + \frac{ac}{n} \\ dbn - cd & -ad + \frac{c^2}{n} \end{pmatrix}.
$$

Our goal is to show that $c = nb$, so suppose for the sake of contradiction that this is not the case. Then, $-ab + \frac{ac}{n} = 0 = dbn - cd$ yields $a = d = 0$. Hence, it follows that $b^2 n = \frac{c^2}{n} = \pm 1$, but this is absurd unless $n = 1$ and $(b, c) = \pm(1, -1)$. This implies that $\gamma w_1 = I_2$, which we do not regard as an involution. The contradiction shows that, indeed, $c = nb$. It follows that $2nb^2 - 2ad = -2\det(\gamma) = -2$, so $de_0 + bh + ae_4 \in \Delta(X)$, hence completing the proof. □

Corollary 2.2.5. *It holds that $\tau \in \mathbb{H} \backslash \mathbb{H}_0$ if and only if τ is a fixed point of an involution in $\mathit{\Gamma}_0(n) w_n$. Thus, we obtain that*

$$
\mathbb{H}_0 = \mathbb{H} \setminus \{\text{fixed points of involutions in } \mathit{\Gamma}_0(n) w_n\}.
$$

Proof. As noted at the beginning, we have that

$$
\tau \in \mathbb{H} \setminus \mathbb{H}_0 \Leftrightarrow \psi(\tau) \in \delta^\perp \cap \mathcal{Q}^+(X)
$$

for some $\delta \in \Delta(X)$. In turn, we found this to be equivalent to $\psi(\tau)$ being a fixed point of s_δ. By Proposition 2.2.3 and Proposition 2.2.4 this is the case if and only if τ is a fixed point of an involution in $\mathit{\Gamma}_0(n) w_n$. □

Before we move on, we have yet to show that $\mathbb{H}_0 \subseteq \mathbb{H}$ is an open subspace, in order to define a suborbifold structure on $\mathbb{H}_0 / \mathit{\Gamma}_0^+(n)$. Let us first introduce $\Delta^+(X) := \{\delta = re_0 + dh + se_4 \in \Delta(X) : r \in \mathbb{Z}_{>0}\}$ and, similarly, let $\Delta^-(X) := \{\delta = re_0 + dh + se_4 \in \Delta(X) : r \in \mathbb{Z}_{<0}\}$. Note that for $\delta = re_0 + dh + se_4 \in \Delta(X)$, the fact that $(\delta.\delta) = 2nd^2 - 2rs = -2 < 0$ forces $r \neq 0$, so we obtain a disjoint union $\Delta(X) = \Delta^+(X) \sqcup \Delta^-(X)$. We start with a simple observation:

Proposition 2.2.6. *Let $\delta = re_0 + dh + se_4 \in \Delta(X)$. Then, $P_\delta := \frac{d}{r} + \mathbf{i}\frac{1}{|r|\sqrt{n}} \in \mathbb{H}$ is the unique fixed point of the involution $\Psi(s_\delta)\colon \mathbb{H} \to \mathbb{H}$.*

Proof. We have to solve the equation $\Psi(s_\delta).\tau = \tau$ over $\mathbb{H}$. Note that Proposition 2.2.4 gives the Möbius transformation by which $\Psi(s_\delta)$ acts on $\mathbb{H}$. A straightforward computation shows that

$$\Psi(s_\delta).\tau = \tau \Leftrightarrow 0 = nr\tau^2 - 2nd\tau + s, \quad \text{where } \delta = re_0 + dh + se_4 \in \Delta(X),$$

and that $P_\delta \in \mathbb{H}$ is one of the solutions, whereas the other one—its complex conjugate—does not lie in $\mathbb{H}$. $\square$

Corollary 2.2.7. *We have that*

$$\mathbb{H}_0 = \mathbb{H} \setminus \{P_\delta : \delta \in \Delta(X)\} = \mathbb{H} \setminus \{P_\delta : \delta \in \Delta^+(X)\}$$
$$= \mathbb{H} \setminus \left\{\frac{d}{r} + \mathbf{i}\,\frac{1}{r\sqrt{n}} : r \in \mathbb{Z}_{>0},\, d \in \mathbb{Z}\right\}.$$

Proof. Corollary 2.2.5 yields the characterization $\mathbb{H}_0 = \mathbb{H} \setminus \{\text{fixed points of involutions in } \Gamma_0(n)w_n\}$. By Proposition 2.2.3, we know that every involution in $\Gamma_0(n)w_n$ is given by $\Psi(s_\delta)$ for some $\delta \in \Delta(X)$ and Proposition 2.2.6 shows that P_δ is the only fixed point of $\Psi(s_d)$ in $\mathbb{H}$. The first equality follows. The second one comes down to the observation that $P_\delta = P_{-\delta}$, so it suffices to remove those P_δ with $\delta \in \Delta^+(X)$. The third equality follows from Proposition 2.2.6 as well. $\square$

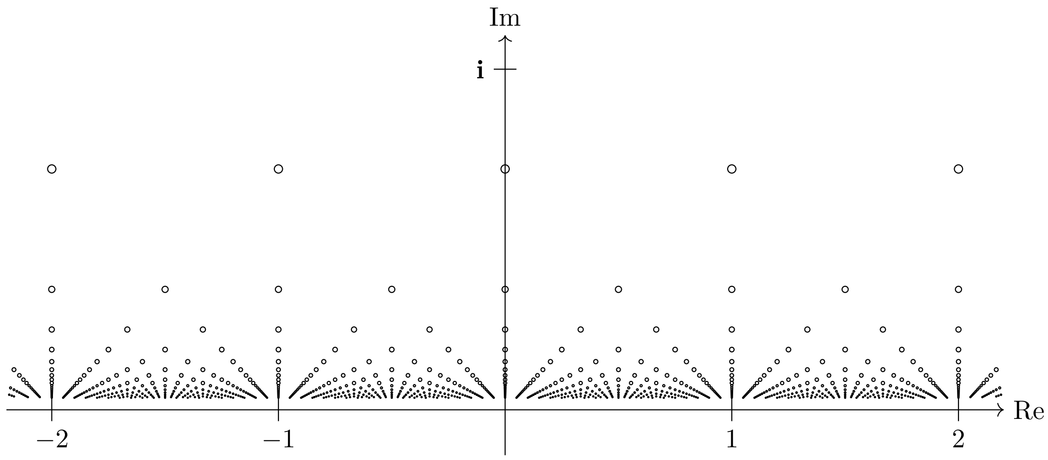

Figure 2.1: The punctured upper half plane $\mathbb{H}_0$ for $n = 2$ and $(r, d) \in [1, 20] \times [-40, 40]$.

This description of the punctured upper half plane $\mathbb{H}_0$ allows us to establish its openness in $\mathbb{H}$ (with respect to the usual topology) via the sequential criterion:

Proposition 2.2.8. *The set* $\mathbb{H}_0$ *is an open subset of* $\mathbb{H}$.

Proof. We will prove the equivalent statement that

$$\mathbb{H} \setminus \mathbb{H}_0 = \left\{\frac{d}{r} + \mathbf{i}\,\frac{1}{r\sqrt{n}} : r \in \mathbb{Z}_{>0},\, d \in \mathbb{Z}\right\}$$

(see Corollary 2.2.7) is a closed subset of $\mathbb{H}$ using the sequential criterion. Hence, let $(r_k)_{k\in\mathbb{N}} \subseteq \mathbb{Z}_{>0}$ and $(d_k)_{k\in\mathbb{N}} \subseteq \mathbb{Z}$ be sequences such that

$$\tau_k := \left(\frac{d_k}{r_k} + \mathbf{i}\,\frac{1}{r_k\sqrt{n}}\right) \xrightarrow{k\to\infty} \tau \in \mathbb{H}.$$

Then $\operatorname{Re}(\tau_k) = \frac{d_k}{r_k} \to \operatorname{Re}(\tau)$ and $\operatorname{Im}(\tau_k) = \frac{1}{r_k\sqrt{n}} \to \operatorname{Im}(\tau)$ as $k \to \infty$. The latter implies that r_k converges as well (possibly to ∞); let r denote its limit. Since $\mathbb{Z}_{>0}$ is a discrete subspace of $\mathbb{R}$, it follows that either $r = \infty$ or $r \in \mathbb{Z}_{>0}$. Observe that the former is not possible, as it would yield $\operatorname{Im}(\tau) = 0$, contrary to $\tau \in \mathbb{H}$, so it must hold that $r \in \mathbb{Z} > 0$. It follows that $d_k = \operatorname{Re}(\tau_k)r_k$ converges to $\operatorname{Re}(\tau)r =: d$ as $k \to \infty$, and since $\mathbb{Z}$ is discrete in $\mathbb{R}$, we conclude that $d \in \mathbb{Z}$. Putting the pieces together, we find that $\tau_k \to \frac{d}{r} + \mathbf{i}\,\frac{1}{r\sqrt{n}} \in \mathbb{H} \setminus \mathbb{H}_0$ as $k \to \infty$. □

As we will see in Section 2.4, the openness of $\mathbb{H}_0$ in $\mathbb{H}$ is one of the crucial properties that allows the orbifold structure of $[\mathbb{H}/\Gamma_0^+(n)]$ to carry over to $\mathbb{H}_0/\Gamma_0^+(n)$. We will revisit this observation shortly, but before doing so, let us briefly step away from the main discussion to further explore the topological properties of $\mathbb{H}_0$. As observed by Kawatani in [Kaw13], a more refined analysis leads to the group structure of the kernel $\operatorname{Aut}_0(\mathrm{D^b}(X))$ of the representation $\varpi\colon \operatorname{Aut}(\mathrm{D^b}(X)) \to \operatorname{Aut}(\widetilde{H}(X,\mathbb{Z}))$.

2.3 Detour: The kernel of $\varpi\colon \operatorname{Aut}(\mathrm{D^b}(X)) \to \operatorname{Aut}(\widetilde{H}(X,\mathbb{Z}))$

By virtue of [BB17, Thm. 1.3], when X is a K3 surface of Picard number 1, one has an isomorphism $\operatorname{Aut}_0(\mathrm{D^b}(X)) \cong \pi_1(\mathcal{P}_0^+(X))$. To compute this group, we first revisit the geometry of this period space. Recall that $\mathcal{P}^+(X)$ is one of the two connected components of

$$\mathcal{P}(X) := \{x \in \mathcal{N}(X)_{\mathbb{C}} : \operatorname{Re} x \text{ and } \operatorname{Im} x \text{ span a positive definite plane } P_x \text{ in } \mathcal{N}(X)_{\mathbb{R}}\}.$$

Given $x \in \mathcal{P}(X)$, the real and imaginary parts form an oriented basis of a positive definite plane $P_x \subset \mathcal{N}(X)_{\mathbb{R}}$. In fact, as we will see in Proposition 2.3.2, one can view $\mathcal{P}(X)$ as parametrizing oriented bases of such planes.

Note that $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathcal{P}^+(X)$ upon identifying $\mathcal{N}(X)_{\mathbb{C}} \cong \mathcal{N}(X) \otimes \mathbb{R}^2$. This action becomes more transparent by passing to a matrix model. Recall that $\mathcal{N}(X)_{\mathbb{R}}$ has signature $(2,1)$. Hence, there exists a basis $C = (c_1, c_2, c_3)$ of $\mathcal{N}(X)_{\mathbb{R}}$ with Gram matrix $G_C = \operatorname{diag}(1,1,-1)$. We can then identify $\mathcal{N}(X)_{\mathbb{C}}$ with the additive group $\mathbb{R}^{2,3}$ of real 2×3 matrices endowed with corresponding bilinear form via the group isomorphism

$$\begin{aligned} c_C\colon \mathcal{N}(X)_{\mathbb{C}} &\longrightarrow \mathbb{R}^{2,3} \\ a_1c_1 + a_2c_2 + a_3c_3 + \mathbf{i}\,(b_1c_1 + b_2c_2 + b_3c_3) &\longmapsto \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \end{aligned}$$

Remark 2.3.1. A direct computation shows that for $x, y \in \mathcal{N}(X)_{\mathbb{C}}$

$$c_C(x)G_Cc_C(y)^T = \begin{pmatrix} (\operatorname{Re} x . \operatorname{Re} y) & (\operatorname{Re} x . \operatorname{Im} y) \\ (\operatorname{Im} x . \operatorname{Re} y) & (\operatorname{Im} x . \operatorname{Im} y) \end{pmatrix}.$$

Hence, the 2-plane $P_x := \operatorname{span}(\operatorname{Re} x, \operatorname{Im} x) \subseteq \mathcal{N}(X)_{\mathbb{R}}$ is positive definite if and only if $c_C(x)G_Cc_C(x)^T$ is positive definite. Thus, we find that $c_C(\mathcal{P}(X)) = \{M \in \mathbb{R}^{2,3} : MG_CM^T \text{ is positive definite}\}$.

In this model, the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{P}^+(X)$ is simply left multiplication of matrices: $c_C(A.x) = Ac_C(x)$ for $A \in \mathrm{GL}_2^+(\mathbb{R})$ and $x \in \mathcal{P}^+(X)$. Indeed, for every non-zero $v \in \mathbb{R}^2$, we have that $v^TAc_C(x)G_Cc_C(x)^TA^Tv = (A^Tv)^Tc_C(x)G_Cc_C(x)^T(A^Tv) > 0$ as $c_C(x) \in c_C(\mathcal{P}(X))$. Therefore, $Ac_C(x) \in c_C(\mathcal{P}(X))$ as well. It is now clear that the action is free. For $x \in \mathcal{P}(X)$, the matrix $c_C(x)$ has rank 2, so its columns span $\mathbb{R}^2$. If $Ac_C(x) = c_C(x)$, then A fixes this spanning set, and hence acts trivially on all of $\mathbb{R}^2$, so $A = I_2$.

Proposition 2.3.2. *The smooth map*

$$\begin{array}{ccccc} \mathcal{P}^+(X) & \xrightarrow{\;p\;} & \mathrm{Gr}^{\mathrm{po}}(2,\mathcal{N}(X)_\mathbb{R}) & \xrightarrow{\eta^{-1}_{\mathcal{N}(X)_\mathbb{R}}} & \mathcal{Q}(X) \\ x & \longmapsto & P_x := \mathrm{span}(\operatorname{Re} x, \operatorname{Im} x) & \longmapsto & \eta^{-1}_{\mathcal{N}(X)_\mathbb{R}}(P_x) \end{array}$$

induces a diffeomorphism $\mathcal{P}^+(X)/\operatorname{GL}_2^+(\mathbb{R}) \cong \mathcal{Q}^+(X)$.

Proof. Consider the natural map $p\colon \mathcal{P}(X) \to \mathrm{Gr}^{\mathrm{po}}(2,\mathcal{N}(X)_\mathbb{R})$, $x \mapsto P_x$. For $x \in \mathcal{P}(X)$ and $M \in \operatorname{GL}_2^+(\mathbb{R})$, we see that x and $M.x$ induce the same 2-plane in $\mathrm{Gr}^{\mathrm{po}}(2,\mathcal{N}(X)_\mathbb{R})$. Furthermore, since $M \in \operatorname{GL}_2^+(\mathbb{R})$, the orientations of $p(x)$ and $p(M.x)$ are coincident, so p factors through the quotient $\mathcal{P}(X)/\operatorname{GL}_2^+(\mathbb{R})$. Conversely, consider the map $\mathrm{Gr}^{\mathrm{po}}(2,\mathcal{N}(X)_\mathbb{R}) \to \mathcal{P}(X)/\operatorname{GL}_2^+(\mathbb{R})$, $P_x \mapsto \operatorname{GL}_2^+(\mathbb{R}).x$. Since $\operatorname{GL}_2^+(\mathbb{R}).x$ contains precisely the bases of P_x with the same orientation as x, this map is well-defined and inverse to $\mathcal{P}(X)/\operatorname{GL}_2^+(\mathbb{R}) \to \mathrm{Gr}^{\mathrm{po}}(2,\mathcal{N}(X)_\mathbb{R})$. Thus, we obtain a diffeomorphism $\mathcal{P}(X)/\operatorname{GL}_2^+(\mathbb{R}) \xrightarrow{\sim} \mathrm{Gr}^{\mathrm{po}}(2,\mathcal{N}(X)_\mathbb{R})$, $\operatorname{GL}_2^+(\mathbb{R}).x \mapsto P_x$, which yields an embedding $\mathcal{P}^+(X)/\operatorname{GL}_2^+(\mathbb{R}) \hookrightarrow \mathrm{Gr}^{\mathrm{po}}(2,\mathcal{N}(X)_\mathbb{R})$ onto the connected component of $\mathrm{Gr}^{\mathrm{po}}(2,\mathcal{N}(X)_\mathbb{R})$ containing the plane P_x spanned by $x = e_0 + \mathbf{i}\,h - ne_4$. Recall that by Proposition 1.2.3, $\eta^{-1}_{\mathcal{N}(X)_\mathbb{R}}$ maps this connected component diffeomorphically to $\mathcal{Q}^+(X)$. □

Note that the holomorphic map $\mathbb{H} \to \mathcal{P}^+(X)$, $\tau \mapsto e_0 + \tau h + n\tau^2 e_4$ yields a global section of the projection $\mathcal{P}^+(X) \twoheadrightarrow \mathcal{P}^+(X)/\operatorname{GL}_2^+(\mathbb{R})$ upon identifying $\mathcal{P}^+(X)/\operatorname{GL}_2^+(\mathbb{R}) \cong \mathcal{Q}^+(X) \cong \mathbb{H}$.

Proposition 2.3.3. *The projection* $\mathcal{P}^+(X) \twoheadrightarrow \mathcal{P}^+(X)/\operatorname{GL}_2^+(\mathbb{R})$ *defines a principal* $\operatorname{GL}_2^+(\mathbb{R})$*-bundle.*

Proof. Since the action of $\operatorname{GL}_2^+(\mathbb{R})$ on $P^+(X)$ is free and there exists a global section of the projection $\mathcal{P}^+(X) \to \mathcal{P}^+(X)/\operatorname{GL}_2^+(\mathbb{R})$, by [Tom08, Prop. 14.1.8] it suffices to show that the map

$$\begin{aligned} \mathcal{P}^+(X) \times \operatorname{GL}_2^+(\mathbb{R}) &\longrightarrow \mathcal{P}^+(X) \times \mathcal{P}^+(X) \\ (x, M) &\longmapsto (x, M.x) \end{aligned}$$

defines a topological embedding. We will prove the equivalent statement—recall that $c_C(\mathcal{P}^+(X))$ and $\mathcal{P}^+(X)$ are diffeomorphic—that the map

$$\begin{aligned} E\colon c_C(\mathcal{P}^+(X)) \times \operatorname{GL}_2^+(\mathbb{R}) &\longrightarrow c_C(\mathcal{P}^+(X)) \times c_C(\mathcal{P}^+(X)) \\ (x, M) &\longmapsto (x, Mx) \end{aligned}$$

defines a topological embedding. Indeed, we claim that the map

$$\begin{aligned} E\left(c_C(\mathcal{P}^+(X)) \times \operatorname{GL}_2^+(\mathbb{R})\right) &\longrightarrow c_C(\mathcal{P}^+(X)) \times \operatorname{GL}_2^+(\mathbb{R}) \\ (x, y) &\longmapsto (x, y\, G_C\, x^T (x\, G_C\, x^T)^{-1}) \end{aligned}$$

defines an inverse map. Let $(x, y) \in E(c_C(\mathcal{P}^+(X)) \times \operatorname{GL}_2^+(\mathbb{R}))$, i.e. $x \in c_C(\mathcal{P}^+(X))$ and $y = Mx$ for some $M \in \operatorname{GL}_2^+(\mathbb{R})$. Since $\operatorname{GL}_2^+(\mathbb{R})$ acts freely on $\mathcal{P}^+(X)$, this M is unique. By Remark 2.3.1, $x\, G_C\, x^T$ is positive definite and is, thus, invertible. It follows that $y\, G_C\, x^T (x\, G_C\, x^T)^{-1} = M \in \operatorname{GL}_2^+(\mathbb{R})$, so the map is well-defined as a map of sets. Furthermore, note that it is also clearly smooth, as the image is polynomial in the entries. Finally, we see that

$$y\, G_C\, x^T (x\, G_C\, x^T)^{-1} x = Mx = y,$$

hence showing that the constructed map is indeed inverse to E. □

As we will see, we have an analogous result for $\mathcal{P}_0^+(X)$, yet its proof requires some modifications. Letting $\delta \in \Delta(X)$ and $x \in \mathcal{P}^+(X)$, Remark 2.3.1 yields the following equivalent propositions:

$$\begin{aligned}(x.\delta) = 0 \Leftrightarrow c_C(x)G_C c_C(\delta)^T = 0 &\Leftrightarrow \forall M \in \mathrm{GL}_2^+(\mathbb{R}) : Mc_C(x)G_C c_C(\delta)^T = 0 \\ &\Leftrightarrow \forall M \in \mathrm{GL}_2^+(\mathbb{R}) : (Mx.\delta) = 0.\end{aligned}$$

Thus, we infer that $\mathcal{P}_0^+(X)$ is closed with respect to the action of $\mathrm{GL}_2^+(\mathbb{R})$.

Proposition 2.3.4. *There exists a diffeomorphism $\mathcal{P}_0^+(X)/\mathrm{GL}_2^+(\mathbb{R}) \cong \mathcal{Q}_0^+(X)$. Moreover, the projection $\mathcal{P}_0^+(X) \twoheadrightarrow \mathcal{P}_0^+(X)/\mathrm{GL}_2^+(\mathbb{R})$ defines a principal $\mathrm{GL}_2^+(\mathbb{R})$-bundle.*

Proof. Let $\delta \in \Delta(X)$ and take $[x] \in \mathcal{Q}^+(X)$ with $(x.\delta) = 0$. Then, $x \in \mathcal{P}^+(X) \cap \delta^\perp$ and $\eta^{-1}_{\mathcal{N}(X)_\mathbb{R}}(P_x) = [x]$. Conversely, for $x \in \mathcal{P}^+(X) \cap \delta^\perp$, we see that $\eta^{-1}_{\mathcal{N}(X)_\mathbb{R}}(P_x) = [Mx]$ for some $M \in \mathrm{GL}_2^+(x)$. As $(x.\delta) = 0$, the above discussion implies that $(Mx.\delta) = 0$ as well, so $\eta^{-1}_{\mathcal{N}(X)_\mathbb{R}}(P_x) \in \mathcal{Q}^+(X) \cap \delta^\perp$. This shows that the map in Proposition 2.3.2 sends $\mathcal{P}^+(X) \cap \delta^\perp$ surjectively onto $\mathcal{Q}^+(X) \cap \delta^\perp$. Since $\mathcal{P}_0^+(X)$ and $\mathcal{Q}_0^+(X)$ are open in $\mathcal{P}^+(X)$ resp. $\mathcal{Q}^+(X)$, and since $\mathcal{P}_0^+(X)$ is closed with respect to the action of $\mathrm{GL}_2^+(\mathbb{R})$, it follows that the smooth map in Proposition 2.3.2 induces a diffeomorphism $\mathcal{P}_0^+(X)/\mathrm{GL}_2^+(\mathbb{R}) \cong \mathcal{Q}_0^+(X)$.

As before, note that the holomorphic map $\mathbb{H}_0 \to \mathcal{P}_0^+(X)$, $\tau \mapsto e_0 + \tau h + n\tau^2 e_4$ yields as well a global section of $\mathcal{P}_0^+(X) \twoheadrightarrow \mathcal{P}_0^+(X)/\mathrm{GL}_2^+(\mathbb{R})$ by identifying $\mathcal{P}_0^+(X)/\mathrm{GL}_2^+(\mathbb{R}) \cong \mathcal{Q}_0^+(X) \cong \mathbb{H}_0$.

We are left to show that the projection $\mathcal{P}_0^+(X) \twoheadrightarrow \mathcal{P}_0^+(X)/\mathrm{GL}_2^+(\mathbb{R})$ defines a principal $\mathrm{GL}_2^+(\mathbb{R})$-bundle. Since $\mathcal{P}_0^+(X)$ is closed under the action of $\mathrm{GL}_2^+(\mathbb{R})$, and using that $(x.\delta) = 0 \Leftrightarrow \forall M \in \mathrm{GL}_2^+(\mathbb{R}) : Mc_C(x)G_C c_C(\delta)^\perp = 0$, it is straightforward to verify that the proof given in Proposition 2.3.3 carries over to this case upon substituting $\mathcal{P}_0^+(X)$ for $\mathcal{P}^+(X)$. ☐

Corollary 2.3.5. *It holds that $\mathcal{P}_0^+(X) \cong \mathcal{Q}_0^+(X) \times \mathrm{GL}_2^+(\mathbb{R})$.*

Proof. Since the principal $\mathrm{GL}_2^+(\mathbb{R})$-bundle $\mathcal{P}_0^+(X) \twoheadrightarrow \mathcal{P}_0^+(X)/\mathrm{GL}_2^+(\mathbb{R})$ admits a global section, it is trivial, i.e. $\mathcal{P}_0^+(X)$ is diffeomorphic to $(\mathcal{P}_0^+(X)/\mathrm{GL}_2^+(\mathbb{R})) \times \mathrm{GL}_2^+(\mathbb{R}) \cong \mathcal{Q}_0^+(X) \times \mathrm{GL}_2^+(\mathbb{R})$. ☐

In particular, we find that computing the fundamental group of $\mathcal{P}_0^+(X)$ comes down to computing the fundamental group of the space $\mathbb{H}_0 \cong \mathcal{Q}_0^+(X)$, which we have already studied intensively. We only need one more result:

Proposition 2.3.6. [Kaw13, Lem. 2.12] *The set of punctures $\mathbb{H} \setminus \mathbb{H}_0$ is a discrete subset of $\mathbb{H}$.*

Proof. Take a $P_\delta \in \mathbb{H} \setminus \mathbb{H}_0$ with $\delta = re_0 + dh + se_4 \in \Delta^+(X)$ and consider an open neighborhood (in the usual metric)

$$U_\delta := B_{\varepsilon(\delta)}(P_\delta) \qquad \text{with } \varepsilon(\delta) := \frac{1}{2\sqrt{n}}\left(\frac{1}{r} - \frac{1}{r+1}\right) > 0.$$

We claim that $U_\delta \cap \mathbb{H} \setminus \mathbb{H}_0 = \{P_\delta\}$. Indeed, let $P_{\delta'} \in \mathbb{H} \setminus \mathbb{H}_0$ be different from P_δ, i.e. such that $\delta' = r'e_0 + d'h + s'e_4 \in \Delta^+(X)$ satisfies $(r', d') \neq (r, d)$. If $r \neq r'$, a straightforward computation shows that

$$|P_\delta - P_{\delta'}| \geqslant |\operatorname{Im} P_\delta - \operatorname{Im} P_{\delta'}| \geqslant \frac{1}{\sqrt{n}}\left(\frac{1}{r} - \frac{1}{r+1}\right) > \varepsilon(\delta).$$

Otherwise, suppose that $r' = r$. We then see that

$$|P_\delta - P_{\delta'}| \geqslant |\operatorname{Re} P_\delta - \operatorname{Re} P_{\delta'}| = \left|\frac{d-d'}{r}\right| \geqslant \frac{1}{r} \geqslant \frac{1}{r} - \frac{1}{r+1} > \varepsilon(\delta).$$

Thus, we conclude that $U_\delta \cap (\mathbb{H} \setminus \mathbb{H}_0) = \{P_\delta\}$. The claim follows. ☐

As an immediate corollary, we find that every compact subset of $\mathbb{H}_0$ contains only finitely many punctures. Furthermore, under some mild requirements to be specified shortly, the interior of such a compact subset is homeomorphic to $\mathbb{R}^2$ with a finite set of points removed—and its fundamental group is well-known to be the free product of copies of $\mathbb{Z}$, indexed by the punctures. The next result combines this local analysis with a limiting argument over an exhaustion of $\mathbb{H}$ to determine $\pi_1(\mathbb{H}_0, p_0)$. Since the considered subsets are path-connected, and conjugation by paths does not affect the results we use along the proof, we will suppress the base point in the notation.

Proposition 2.3.7. *The fundamental group of $\mathbb{H}_0$ is*

$$\pi_1(\mathbb{H}_0) \cong \mathop{\Huge\ast}_{\delta\in\Delta^+(X)} \mathbb{Z}.$$

Proof. Choose a transcendental $\zeta \in (0,1)$ and define, for each $j \in \mathbb{N}$, the open connected sets

$$V_j := \left\{\tau \in \mathbb{H} : |\operatorname{Re}\tau| < j+\zeta,\ \operatorname{Im}\tau > \frac{1}{j+\zeta}\right\} \subseteq \mathbb{H}.$$

By Corollary 2.2.7, the points of $\mathbb{H}\setminus\mathbb{H}_0$ have algebraic real and imaginary parts and, hence, do not lie on the boundary ∂V_j for any $j \in \mathbb{N}$. Moreover, since $\operatorname{Im}(P_\delta) = \frac{1}{r\sqrt{n}} < 2$ for any $\delta \in \Delta^+(X)$, the set of punctures inside V_j is given by

$$\begin{aligned} K_j &= \operatorname{cl}(V_j) \cap \{\tau \in \mathbb{H} : \operatorname{Im}\tau \leqslant 2\} \cap (\mathbb{H}\setminus\mathbb{H}_0) \\ &= \left\{\tau \in \mathbb{H} : |\operatorname{Re}\tau| \leqslant j+\zeta,\ 2 \geqslant \operatorname{Im}\tau \geqslant \frac{1}{j+\zeta}\right\} \cap (\mathbb{H}\setminus\mathbb{H}_0). \end{aligned}$$

This is a finite set, as it is the intersection of a discrete set with a compact one, see Proposition 2.3.6.

We will now proceed by induction on j to show that $\pi_1(V_j\setminus K_j) \cong \ast_{k\in K_j}\mathbb{Z}$. Furthermore, we will establish that, under this isomorphism, the group homomorphisms $\iota_j^*\colon \pi_1(V_j\setminus K_j) \to \pi_1(V_{j+1}\setminus K_{j+1})$, induced by the inclusions $\iota_j\colon V_j\setminus K_j \hookrightarrow V_{j+1}\setminus K_{j+1}$, send generators to the corresponding generators.

The base case $j = 0$ is clear, as $V_0 \subseteq \mathbb{H}$ is simply connected and is, thus, homeomorphic to $\mathbb{R}^2$. Under this homeomorphism, $V_0\setminus K_0$ is mapped to $\mathbb{R}^2$ with $|K_0|$ many punctures, whose fundamental group is well-known to be isomorphic to $\ast_{k\in K_0}\mathbb{Z}$.

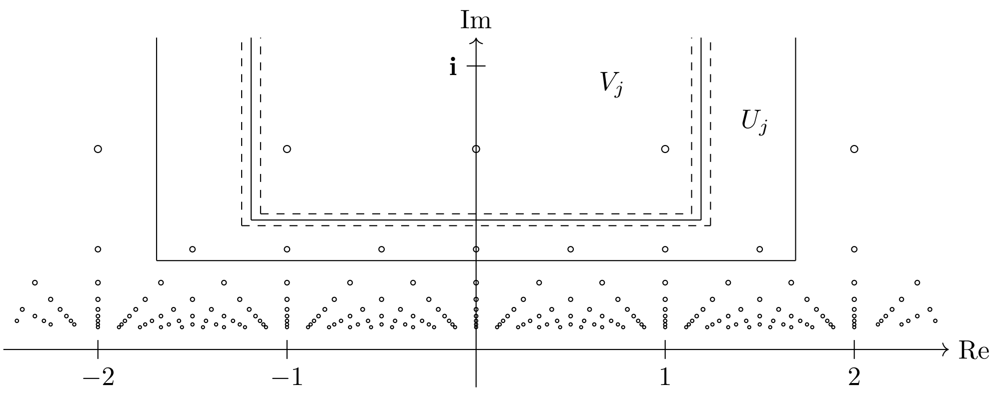

Figure 2.2: Open cover of V_{j+1} and thickening along ∂V_j.

Let us turn to the induction step. By hypothesis, we have $\pi_1(V_j) \cong \ast_{k\in K_j}\mathbb{Z}$. Let now U_j denote the interior of $V_{j+1}\setminus V_j$. Upon thickening V_j and U_j along the boundary ∂V_j, the Seifert-van Kampen theorem yields the following pushout of groups:

$$\begin{tikzcd}
\pi_1(\{*\}) \arrow[r] \arrow[d] & \pi_1(V_j \setminus K_j) \arrow[d, "\iota_j^*"] \\
\pi_1(U_j \setminus K_{j+1}) \arrow[r] & \pi_1(V_{j+1} \setminus K_{j+1}),
\end{tikzcd}$$

where we have used that ∂V_j is contractible. As U_j is simply connected, we find that $U_j \setminus K_{j+1} = U_j \setminus (K_{j+1} \setminus K_j)$ is homeomorphic to $\mathbb{R}^2$ with $|K_{j+1}| - |K_j|$ punctures. Thus, we conclude that

$$\pi_1(V_{j+1} \setminus K_{j+1}) \cong \pi_1(U_j \setminus K_{j+1}) * \pi_1(V_j \setminus K_j) \cong \Big(\mathop{\ast}_{k \in K_{j+1} \setminus K_j} \mathbb{Z}\Big) * \Big(\mathop{\ast}_{k \in K_j} \mathbb{Z}\Big) \cong \mathop{\ast}_{k \in K_{j+1}} \mathbb{Z},$$

where, by construction, $\iota_j^*\colon \pi_1(V_j \setminus K_j) \to \pi_1(V_{j+1} \setminus K_{j+1})$ maps generators to the corresponding generators. This completes the induction argument.

Finally, observe that, by definition, $V_j \subseteq V_{j+1}$ and $\bigcup_{j \in \mathbb{N}} V_j = \mathbb{H}$. Hence, for $V_j^0 := V_j \cap \mathbb{H}_0 = V_j \setminus K_j \subseteq \mathbb{H}_0$, we obtain that $V_j^0 = V_j \cap \mathbb{H}_0 \subseteq V_{j+1} \cap \mathbb{H}_0 = V_{j+1}^0$ and that $\bigcup_{j \in \mathbb{N}} V_j^0 = \bigcup_{j \in \mathbb{N}} (V_j \cap \mathbb{H}_0) = \mathbb{H} \cap \mathbb{H}_0 = \mathbb{H}_0$, so the inclusions induce the following commutative diagram:

$$\begin{tikzcd}
\pi_1(V_0^0) \arrow[r, "\iota_0^*"] \arrow[dr] & \pi_1(V_1^0) \arrow[r, "\iota_1^*"] \arrow[d] & \pi_1(V_2^0) \arrow[r, "\iota_2^*"] \arrow[dl] & \dots \arrow[dll] \\
& \pi_1(\mathbb{H}_0). & &
\end{tikzcd}$$

Consequently, we obtain a group homomorphism $\Omega\colon \varinjlim \pi_1(V_j^0) \to \pi_1(\mathbb{H}_0)$, which we claim to be an isomorphism. Let us first show that Ω is surjective. For a based loop $\ell\colon S^1 \to \mathbb{H}_0$, we see that $\ell(S^1) \subseteq \mathbb{H}_0 = \bigcup_{j \in \mathbb{N}} V_j^0$ is compact, and thus eventually contained in every V_j^0 for j large enough. In particular, the homotopy class $[\ell]$ of ℓ in V_j^0 is eventually contained in every $\pi_1(V_j^0)$. The surjectivity of Ω follows. As for the injectivity, suppose that two based loops $\ell, \ell'\colon S^1 \to \mathbb{H}_0$ are based homotopic in $\mathbb{H}_0$, i.e. there exists a continuous $H\colon S^1 \times [0,1] \to \mathbb{H}_0$ such that $H(-,0) = \ell$, $H(-,1) = \ell'$ and H is constant on the base point. Again, since $S^1 \times [0,1]$ is compact, so is its image $H(S^1 \times [0,1]) \subseteq \mathbb{H}_0 = \bigcup_{j \in \mathbb{N}} V_j^0$. Thus, the homotopy H eventually takes values within V_j^0 for j large enough, so there exists a $k \in \mathbb{N}$ such that $[\ell] = [\ell']$ in $\pi_1(V_j^0)$ for every $j \geqslant k$, thus showing that the homotopy classes also agree on $\varinjlim \pi_1(V_j^0)$. This confirms that Ω is, indeed, an isomorphism of groups. Finally, taking into account the isomorphisms $\pi_1(V_j^0) \cong \mathop{\ast}_{k \in K_j} \mathbb{Z}$ along with the fact that $\iota_j^*\colon \pi_1(V_j^0) \to \pi_1(V_{j+1}^0)$ sends generators to the corresponding generators, one readily verifies that $\mathop{\ast}_{k \cup K_j} \mathbb{Z} = \mathop{\ast}_{\delta \in \Delta^+(X)} \mathbb{Z}$ is a colimit of the system $\pi_1(V_0^0) \xrightarrow{\iota_0^*} \pi_1(V_1^0) \xrightarrow{\iota_1^*} \dots$ and thus

$$\mathop{\ast}_{\delta \in \Delta^+(X)} \mathbb{Z} \cong \varinjlim \pi_1(V_j^0) \cong \pi_1(\mathbb{H}_0).$$

□

Theorem 2.3.8. *We have the following group isomorphism:*

$$\mathrm{Aut}_0(\mathrm{D}^{\mathrm{b}}(X)) \cong \Big(\mathop{\ast}_{\delta \in \Delta^+(X)} \mathbb{Z}\Big) \times \mathbb{Z}.$$

Proof. The affirmative answer to Bridgeland's conjecture in the case of K3 surfaces of Picard number 1 implies that $\mathrm{Aut}_0(\mathrm{D}^{\mathrm{b}}(X)) \cong \pi_1(\mathcal{P}_0^+(X))$. By Corollary 2.3.5, we have that $\mathcal{P}_0^+(X) \cong \mathcal{Q}_0^+(X) \times \mathrm{GL}_2^+(\mathbb{R})$, which combined with Proposition 2.3.7 yields

$$\mathrm{Aut}_0(\mathrm{D}^{\mathrm{b}}(X)) \cong \pi_1(\mathcal{P}_0^+(X)) \cong \pi_1(\mathcal{Q}_0^+(X)) \times \pi_1(\mathrm{GL}_2^+(\mathbb{R})) \cong \Big(\mathop{\ast}_{\delta \in \Delta^+(X)} \mathbb{Z}\Big) \times \mathbb{Z},$$

where we have used the well-known fact that $\pi_1(\mathrm{GL}_2^+(\mathbb{R})) \cong \mathbb{Z}$, cf. [Hal03, Sec. 1.5]. □

2.4 Towards the orbifold $[\mathbb{H}_0/\Gamma_0^+(n)]$

We established in Proposition 2.2.8 that $\mathbb{H}_0$ is an open subset of $\mathbb{H}$, so $\mathcal{Q}_0^+(X) = \psi(\mathbb{H}_0)$ is also open in $\mathcal{Q}^+(X) = \psi(\mathbb{H})$. Therefore, $\mathbb{H}_0$ and $\mathcal{Q}_0^+(X)$ become submanifolds of $\mathbb{H}$ resp. $\mathcal{Q}^+(X)$, thus showing that the quotient spaces $\mathbb{H}_0/\Gamma_0^+$ and $\mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ can, as long as they are well-defined, also be endowed with a natural good orbifold structure. In this section, we will see that the restriction of $\tilde{\psi}$ to $\mathbb{H}_0/\Gamma_0^+$ defines a diffeomorphism between the orbifolds $[\mathbb{H}_0/\Gamma_0^+(n)]$ and $[\mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))]$ and, building on this, we will finally show that their orbifold fundamental groups are isomorphic.

Proposition 2.4.1. *The punctured upper half plane $\mathbb{H}_0$ is closed under the action of $\Gamma_0^+(n)$. Similarly, $\mathcal{Q}_0^+(X)$ is closed under the action of $\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$.*

Proof. Clearly, the second assertion follows from the first one by Corollary 2.1.9. Hence, it suffices to show that $\mathbb{H}\setminus\mathbb{H}_0$ is closed under the action of $\Gamma_0^+(n)$. This follows from the fact that $\mathbb{H}\setminus\mathbb{H}_0$ is the set of fixed points of an involution in $\Gamma_0(n)w_n$, see Corollary 2.2.5. Indeed, let $\tau\in\mathbb{H}\setminus\mathbb{H}_0$ and let $\hat{\gamma}\in\Gamma_0(n)$ such that $\hat{\gamma}w_n$ is an involution and $\hat{\gamma}w_n.\tau=\tau$. We then see that for every $\gamma\in\Gamma_0^+(n)$, the point $\gamma.\tau$ is a fixed point of the involution $\gamma\hat{\gamma}w_n\gamma^{-1}\in\Gamma_0(n)w_n$, from which $\gamma.\tau\in\mathbb{H}\setminus\mathbb{H}_0$ follows. □

In particular, $\mathbb{H}_0/\Gamma_0^+$ and $\mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ are well-defined. The restriction of $\tilde{\psi}$ to $\mathbb{H}_0/\Gamma_0^+$ yields an embedding from $\mathbb{H}_0/\Gamma_0^+$ onto $\{\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})).(\psi(\tau)) : \tau\in\mathbb{H}_0\}\subseteq\mathcal{Q}^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$. Since $\psi(\mathbb{H}_0)=\mathcal{Q}_0^+(X)$, we obtain:

Corollary 2.4.2. *The map*

$$\begin{aligned}\tilde{\psi}_0\colon \mathbb{H}_0/\Gamma_0^+ &\longrightarrow \mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))\\ \Gamma_0^+(n).\tau &\longmapsto \operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})).(\psi(\tau))\end{aligned}$$

defines a homeomorphism.

Let us now turn to the orbifold structures. Since $\mathbb{H}_0$ is a manifold and $\Gamma_0^+(n)$ acts properly discontinuously on it, see Remark 1.4.11, the quotient $\mathbb{H}_0/\Gamma_0^+(n)$ can be endowed with a good orbifold structure. As in Proposition 2.1.12, building on this, we can also endow $\mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ with a good orbifold structure, and we find that

Corollary 2.4.3. *The continuous map $\tilde{\psi}_0$ defines an orbifold diffeomorphism*

$$\tilde{\psi}_0\colon \left[\mathbb{H}_0\Big/\Gamma_0^+(n)\right] \xrightarrow{\ \sim\ } \left[\mathcal{Q}_0^+(X)\Big/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))\right].$$

We are left to show that:

Proposition 2.4.4. *There is an isomorphism $\pi_1^{\mathrm{orb}}(\mathbb{H}_0/\Gamma_0^+(n))\cong\pi_1^{\mathrm{orb}}(\mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})))$.*

Proof. This follows essentially from the argument one could invoke with the usual fundamental group. We let $\rho_1\colon\hat{\mathcal{O}}_1\to[\mathbb{H}_0/\Gamma_0^+(n)]$ and $\rho_2\colon\hat{\mathcal{O}}_2\to[\mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))]$ be orbifold universal covers, which exist due to $\mathbb{H}_0$ and $\mathcal{Q}_0^+(X)$ being connected, see [Car22, Thm. 2.3.4]. Since $\tilde{\psi}_0$ is a diffeomorphism, we easily find that $(\hat{\mathcal{O}}_1,\tilde{\psi}_0\circ\rho_1)$ is a covering of $[\mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))]$ and that $(\hat{\mathcal{O}}_2,\tilde{\psi}_0^{-1}\circ\rho_2)$ is also a covering of $[\mathbb{H}_0/\Gamma_0^+(n)]$. The definition of universal covers yields the existence of smooth maps $\mu_1\colon\hat{\mathcal{O}}_1\to\hat{\mathcal{O}}_2$ and $\mu_2\colon\hat{\mathcal{O}}_2\to\hat{\mathcal{O}}_1$ such that the following diagram commutes:

$$\begin{tikzcd}
\hat{\mathcal{O}}_1 \arrow[rr, "\rho_1"] \arrow[d, dashed, shift right, "\mu_1"'] & & {[\mathbb{H}_0/\Gamma_0^+(n)]} \arrow[d, "\tilde{\psi}_0"', "\cong"] \\
\hat{\mathcal{O}}_2 \arrow[u, dashed, shift right, "\mu_2"'] \arrow[r, "\rho_2"] & {[\mathcal{Q}_0^+(X)/\operatorname{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))].} &
\end{tikzcd}$$

Once again, the uniqueness given by the universal property implies that μ_1 and μ_2 are inverse to each other, thus showing that $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$ are diffeomorphic as orbifolds. It follows that the groups of deck transformations $\mathrm{Deck}(\rho_1)$ and $\mathrm{Deck}(\rho_2)$ are isomorphic by $\Theta\colon \mathrm{Deck}(\rho_1) \to \mathrm{Deck}(\rho_2)$, $f \mapsto \mu_1 \circ f \circ \mu_2$. Indeed, by the definitions of μ_1 and μ_2, we see that for $f \in \mathrm{Deck}(\rho_1)$:

$$\rho_2 \circ \Theta(f) = \rho_2 \circ \mu_1 \circ f \circ \mu_2 = \tilde{\psi}_0 \circ \rho_1 \circ f \circ \mu_2 = \tilde{\psi}_0 \circ \rho_1 \circ \mu_2 = \tilde{\psi}_0 \circ \tilde{\psi}_0^{-1} \circ \rho_2 = \rho_2,$$

so Θ is a well defined group homomorphism. One easily confirms that the inverse map $\mathrm{Deck}(\rho_2) \to \mathrm{Deck}(\rho_1)$, $g \mapsto \mu_2 \circ g \circ \mu_1$ is also a group homomorphism, thus showing that Θ defines an isomorphism.

Finally, by virtue of [Car22, Prop. 2.3.5] along with $\mathrm{Deck}(\rho_1) \cong \mathrm{Deck}(\rho_2)$, we obtain that

$$\pi_1^{\mathrm{orb}}\left(\left[\mathbb{H}_0 \middle/ \Gamma_0^+(n)\right]\right) \cong \mathrm{Deck}(\rho_1) \cong \mathrm{Deck}(\rho_2) \cong \pi_1^{\mathrm{orb}}\left(\left[\mathcal{Q}_0^+(X) \middle/ \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))\right]\right). \qquad \square$$

2.5 The special case $n = 1$

Since most of the theory developed thus far carries over to the case $n = 1$ with minor adjustments, we will forgo a fully rigorous and detailed treatment of the matter and will instead outline these modifications and their consequences.

When $n = 1$, i.e. when X has degree 2, Saint-Donat proved in [Sai74, 5.1] that X arises as a double cover of $\mathbb{P}^2$ branched along a curve $C \subseteq \mathbb{P}^2$ of degree six. The covering involution of this double cover defines an automorphism $\iota\colon X \xrightarrow{\sim} X$, which —as established by Nikulin in [Nik83, Cor. 10.1.3]— is the only non-trivial automorphism a K3 surface X of Picard number $\rho(X) = 1$ may have. The existence of ι is, hence, precisely what distinguishes the case $n = 1$.

The induced Hodge isometry $\iota_H^*\colon \widetilde{H}(X,\mathbb{Z}) \to \widetilde{H}(X,\mathbb{Z})$ is not symplectic (see [Huy16, Sec. 15.4.3]), so by [Ogu02, Lem. 4.1], it must be anti-symplectic. One can further show that ι_H^* restricts to the identity on $\mathcal{N}(X)$. Consequently, we have that $-\iota_H^*|_{\mathcal{N}(X)} = -\mathrm{id}_{\mathcal{N}(X)}$ and $-\iota_H^*|_{T(X)} = \mathrm{id}_{T(X)}$, which is precisely the counterexample we alluded to in Remark 2.1.8. In particular, we see that the injectivity result of Proposition 2.1.7 fails for $n = 1$: The element $-\iota_H^* \in \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ acts trivially on the period domain $\mathcal{Q}^+(X)$, so the action of $\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ on $\mathcal{Q}^+(X)$ is no longer faithful. Faithfulness is restored after passing to the quotient by $\langle -\iota_H^* \rangle$.

From the perspective of derived categories, the automorphism $\iota \in \mathrm{Aut}(X)$ induces an anti-symplectic autoequivalence $\iota^* \in \mathrm{Aut}(\mathrm{D^b}(X))$ by pullback. As shown in [BK22, Thm. 8.1], ι^* lies in the center of $\mathrm{Aut}(\mathrm{D^b}(X))$. Composing the shift functor yields the central and symplectic autoequivalence $\iota^*[1]$, whose square is the double shift functor: $(\iota^*[1])^2 = \iota^*\iota^*[1][1] = [2]$. The image of ι^* under $\varpi\colon \mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))$ coincides with ι_H^*.

Building on these findings, in [FL23, Sec. 4.3] Fan and Lei give the following corrected result for $n = 1$:

$$\pi_1^{\mathrm{orb}}\left(\left[\mathcal{Q}_0^+(X) \middle/ \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))/\langle -\iota_H^* \rangle\right]\right) \cong \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}(\iota^*[1]).$$

Thus, throughout this chapter, when $n = 1$ every occurrence of $\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))$ should be replaced by the factor group $\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))/\langle -\iota_H^* \rangle$ and the corresponding adjustments should be made. In particular, adapting the proof of Proposition 2.1.7 under this convention yields an isomorphism

$$\Psi\colon \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))/\langle -\iota_H^* \rangle \longrightarrow \Gamma_0^+(1)$$

induced by the biholomorphism ψ via conjugation. A parallel analysis of the orbifolds, analogous to that at the end of Section 2.1 and in Section 2.4 finally gives:

Corollary 2.5.1. *When $n = 1$, it holds that*

$$\begin{aligned}\pi_1^{\mathrm{orb}}\left(\left[\mathbb{H}_0 \middle/ \Gamma_0^+(1)\right]\right) &\cong \pi_1^{\mathrm{orb}}\left(\left[\mathcal{Q}_0^+(X) \middle/ \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))/\langle -\iota_H^* \rangle\right]\right) \\ &\cong \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}(\iota^*[1]).\end{aligned}$$

3 The fundamental group of the period space and further classifications

This chapter is dedicated to the computation of the group of autoequivalences and their classification, up to even shifts. We begin in Section 3.1 by drawing on the theory of the modular curve $\mathbb{H}/\mathit{\Gamma}_0(n)$ to analyze the properties of $\mathbb{H}/\mathit{\Gamma}_0^+(n)$ relevant for the computation of its orbifold fundamental group. This will lead to a description of $\pi_1^{\mathrm{orb}}([\mathbb{H}_0/\mathit{\Gamma}_0^+(n)])$, which, in turn, yields the structure of the group of symplectic autoequivalences modulo even shifts. In Section 3.2, we leverage these results to accomplish the classification of finite subgroups of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$.

3.1 Computing $\pi_1^{\mathrm{orb}}(\mathbb{H}_0/\mathit{\Gamma}_0^+(n))$

Building on Corollary 2.2.5, which provides an explicit description of the structure of $\mathbb{H}_0$, we are prepared to examine the orbifold fundamental group of $[\mathbb{H}_0/\mathit{\Gamma}_0^+(n)]$. To that end, we introduce the modular curves

$$Y_0(n) \coloneqq Y(\mathit{\Gamma}_0(n)) \coloneqq \mathbb{H}/\mathit{\Gamma}_0(n) \qquad \text{and} \qquad Y_0^+(n) \coloneqq Y(\mathit{\Gamma}_0^+(n)) \coloneqq \mathbb{H}/\mathit{\Gamma}_0^+(n).$$

The first modular curve is a classical and extensively studied object, playing a central role in the classification of elliptic curves along with additional data on their n torsion subgroups, see [DS05, Thm. 1.5.1]. It thus provides a natural starting point for deductions concerning the second curve. Since our ultimate goal is to compute the fundamental group of the orbifold $Y_0^+(n)$, we must first understand its topology and identify the points with non-trivial stabilizers, i.e. the elliptic points. Finally, we shall examine the orbits of points $\tau \in \mathbb{H} \setminus \mathbb{H}_0$ in order to discard them.

Rather than working directly with $Y_0(n)$ and $Y_0^+(n)$, we will compactify them by adjoining the corresponding cusps. The reason for doing so is that we can equip the compactifications $X_0(n) \coloneqq X(\mathit{\Gamma}_0(n)) = \overline{Y_0(n)}$ and $X_0^+(n) \coloneqq X(\mathit{\Gamma}_0^+(n)) = \overline{Y_0^+(n)}$ with local holomorphic coordinates in such a way that they become compact Riemann surfaces, cf. Proposition 1.4.14. This is convenient because every compact Riemann surface is homeomorphic to a g-holed torus for some $g \geqslant 0$. In return, we must keep track of the cusps, which will eventually need to be removed. Hence, denote:

$$\begin{aligned}
\nu_i(n) &\coloneqq \text{ number of elliptic points of order } i \text{ in } Y_0(n),\\
\nu_\infty(n) &\coloneqq \text{ number of cusps in } Y_0(n) \text{, i.e. points in } X_0(n) \setminus Y_0(n),\\
g(n) &\coloneqq \text{ genus of } X_0(n).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nu_i^+(n) &\coloneqq \text{ number of elliptic points of order } i \text{ in } Y_0^+(n),\\
\nu_\infty^+(n) &\coloneqq \text{ number of cusps in } Y_0^+(n) \text{, i.e. points in } X_0^+(n) \setminus Y_0^+(n),\\
g^+(n) &\coloneqq \text{ genus of } X_0^+(n).
\end{aligned}$$

Remark 3.1.1. There are only finitely many elliptic points in $X_0(n)$, cf. [DS05, Cor. 2.3.5], and likewise, only finitely many cusps exist, see [DS05, Lem. 2.4.1]. Besides, we have $\nu_i(n) \neq 0$ only for $i \in \{2, 3, \infty\}$, see [DS05, Cor. 2.3.5]. Explicit formulae for these values and the genus of the curves can be found in [DS05, Cor. 3.7.2, Sec. 3.8 & Thm. 3.1.1].

Note that for $n = 1$ the Fricke involution w_1 is contained in $\Gamma_0(1)$, so we see that $X_0^+(1) \cong X_0(1)$. For $n \geqslant 2$ this is not the case anymore and the fact that $\Gamma_0(n) \subseteq \Gamma_0^+(n) = \Gamma_0(n) \cup \Gamma_0(n)w_n$ is an index 2 subgroup induces the following ramified double covering map:

$$\begin{aligned} \pi_n \colon X_0(n) &\longrightarrow X_0^+(n) \\ \Gamma_0(n).\tau &\longmapsto \Gamma_0^+(n).\tau, \\ \Gamma_0(n).(w_n.\tau) &\longmapsto \Gamma_0^+(n).\tau. \end{aligned}$$

Fricke classified the ramification points of this map in his book [Fri28] which we recount here. First, let $h(D)$ denote the class number of primitive integral quadratic forms of discriminant D and define

$$\xi_n := \begin{cases} h(-4n) & \text{if } n \not\equiv 3 \bmod 4, \\ h(-n) + h(-4n) & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

For $D < 0$, the class number $h(D)$ can be computed algorithmically, cf. [Coh93, Alg. 5.3.5]. Thus, ξ_n can also be computed algorithmically.

Proposition 3.1.2. [Fri28, II, 4, §3] *For $n \geqslant 2$, the covering map π_n ramifies at ξ_n many ordinary points. Furthermore, for $n \geqslant 5$, only ordinary points occur as ramification points.*

The following proposition helps us gain the full picture about the distribution of ordinary, elliptic and cusps points in $X_0^+(n)$ with respect to $X_0(n)$:

Proposition 3.1.3. *Let $n \geqslant 2$. Away from the ramification locus, π_n maps an ordinary point to an ordinary point, an elliptic point to an elliptic point of the same order, and a cusp to a cusp. Along the ramification locus, it maps an ordinary point to an elliptic point of order 2, an elliptic point of order i to an elliptic point of order $2i$, and a cusp to a cusp.*

Proof. The case of cusps follows from the fact that the set of cusps $\mathbb{Q} \cup \{\mathbf{i}\infty\}$ is closed under the action of $\Gamma_0^+(n)$, see Proposition 1.4.8.

Now, let $\tau \in \mathbb{H}$ and $\gamma \in \Gamma_0(n)$. Then, clearly, $\gamma.(w_n.\tau) = \tau$ if and only if $w_n.\tau = \gamma^{-1}.\tau$. In particular, τ can only have non-trivial stabilizers in the index two subgroup $\Gamma_0(n)w_n \subseteq \Gamma_0^+(n)$ if it lies on the ramification locus of π_n. This solves the case away from the ramification locus. For the second case, suppose that $\Gamma_0(n).(w_n.\tau) = \Gamma_0(n).\tau$. We then have $w_n.\tau = \tilde{\gamma}.\tau$ for some $\tilde{\gamma} \in \Gamma_0(n)$. Recall that $\Gamma_0(n)_\tau \subset \Gamma_0(n)$ denotes the stabilizer subgroup of τ in $\Gamma_0(n)$ and observe that, for $\gamma \in \Gamma_0(n)$, the following holds:

$$\gamma.(w_n.\tau) = \tau \Leftrightarrow (\gamma\tilde{\gamma}).\tau = \tau \Leftrightarrow \gamma \in \Gamma_0(n)_\tau \tilde{\gamma}^{-1}.$$

We conclude that the stabilizer of τ in the subgroup $\Gamma_0(n)w_n$ is given by $\Gamma_0(n)_\tau\tilde{\gamma}^{-1}$, which has the same order as $\Gamma_0(n)_\tau$. Hence, along the ramification locus of π_n, the order of any point doubles. This completes the proof. $\square$

The following figure summarizes the situation:

Unramified:

$$\begin{tikzcd}[column sep=tiny] \Gamma_0(n).\tau \arrow[dr, mapsto] & & \Gamma_0(n).(w_n.\tau) \arrow[dl, mapsto] & \text{ordinary} \arrow[dr, mapsto] & & \text{ordinary} \arrow[dl, mapsto] & \text{elliptic } i \arrow[dr, mapsto] & & \text{elliptic } i \arrow[dl, mapsto] & \text{cusp} \arrow[dr, mapsto] & & \text{cusp} \arrow[dl, mapsto] \\ & \Gamma_0^+(n).\tau & & & \text{ordinary} & & & \text{elliptic } i & & & \text{cusp} & \end{tikzcd}$$

Ramified:

$$\begin{tikzcd} \Gamma_0(n).\tau = \Gamma_0(n).(w_n.\tau) \arrow[d, mapsto] & \text{ordinary} \arrow[d, mapsto] & \text{elliptic } i \arrow[d, mapsto] & \text{cusp} \arrow[d, mapsto] \\ \Gamma_0^+(n).\tau & \text{elliptic } 2 & \text{elliptic } 2i & \text{cusp} \end{tikzcd}$$

Figure 3.1: Fibers of π_n along and away from the ramification locus.

We are now ready to determine ν_i^+ as well as g^+.

Proposition 3.1.4. [FL23, Lem. 4.4] *When $n = 1, 2, 3, 4$, we have that $g^+(n) = 0$, and the other non-zero invariants along with the branch loci of the covering π_n are as follows:*

- $(n = 1)$ $\nu_2^+(1) = \nu_3^+(1) = \nu_\infty^+(1) = 1$, *no branch locus in this case.*
- $(n = 2)$ $\nu_2^+(2) = \nu_4^+(2) = \nu_\infty^+(2) = 1$, *where two elliptic points form the branch locus.*
- $(n = 3)$ $\nu_2^+(3) = \nu_6^+(3) = \nu_\infty^+(3) = 1$, *where two elliptic points form the branch locus.*
- $(n = 4)$ $\nu_2^+(4) = 1$, $\nu_\infty^+(4) = 2$, *where the elliptic point and a cusp form the branch locus.*

For $n \geqslant 5$, only $\nu_2^+(n)$, $\nu_3^+(n)$, $\nu_\infty^+(n)$ and $g^+(n)$ are possibly non-zero, and they are given by the following formulae:

$$\nu_2^+(n) = \frac{\nu_2(n)}{2} + \xi_n, \qquad \nu_3^+(n) = \frac{\nu_3(n)}{2}, \qquad \nu_\infty^+(n) = \frac{\nu_\infty(n)}{2}, \qquad g^+(n) = \frac{g(n)+1}{2} - \frac{\xi_n}{4}.$$

For $n \geqslant 5$, the branch locus consists of ξ_n many elliptic points of order 2.

Proof. Let us first start with $n \geqslant 5$. By Proposition 3.1.2, the covering π_n ramifies at exactly ξ_n ordinary points in $X_0(n)$, thus producing ξ_n elliptic points of order 2 in $X_0^+(n)$, see Proposition 3.1.3. Away from the ramification locus, Proposition 3.1.3 implies that the covering π_n reduces the number of each type of existing elliptic and cusps points in $X_0(n)$ by half. These observations together with Remark 3.1.1 yield the formulae and the statement about the branch locus. Finally, the formula for the genus $g^+(n)$ is a straightforward consequence of the Riemann–Hurwitz formula.

The case $n = 1$ follows from the fact that $Y_0(1) = Y_0^+(1)$ and from the corresponding statement for $Y_0(1)$.

Thus, let us turn to $n \in \{2, 3, 4\}$. In this case, we have that $g(n) = 0$. By the the Riemann–Hurwitz formula, we obtain

$$2g^+(n) = 2g(n) + 1 - \frac{1}{2}\sum_{P \in X_0(n)} (e_P - 1) = 1 - \frac{1}{2}\sum_{P \in X_0(n)} (e_P - 1) \leqslant 1,$$

so we see that $g^+(n) = 0$ and, therefore, that π_n ramifies at precisely two points. By Proposition 3.1.2, one of them is ordinary. The curve $Y_0(2)$ (resp. $Y_0(3)$) has one elliptic point of order 2 (resp. 3) and two cusps. Furthermore, one can easily verify that π_2 (resp. π_3) also ramifies at the elliptic point—along with the previously mentioned ordinary point. The formulae and the assertion about the branch locus follow by Proposition 3.1.3. Finally, let us address the case $n = 4$. The curve $Y_0(4)$ has no elliptic points and three cusps, and the covering π_4 ramifies at one of the cusps besides the mentioned ordinary point. Once again, Proposition 3.1.3 yields the formulae and the distribution of the branch locus. $\square$

Thus, we now have a complete description of the distribution of cusps and elliptic points on $X_0^+(n)$. Nevertheless, in order to compute the orbifold fundamental group of $[\mathbb{H}_0/\Gamma_0^+(n)]$, it remains to examine the orbits of the points in $\mathbb{H} \setminus \mathbb{H}_0$, as they must likewise be removed. The following results provide precisely this information:

Proposition 3.1.5. [FL23, Lem. 4.5] *Let $n \geqslant 2$. Then the set of orbits $\Gamma_0^+(n).\tau$ for $\tau \in \mathbb{H} \setminus \mathbb{H}_0$ is precisely the set of orbits $\Gamma_0^+(n).\tau$ on the branch locus of π_n, for which $\Gamma_0(n).\tau$ is either ordinary or elliptic of odd order.*

Proof. First, let $\tau \in \mathbb{H} \setminus \mathbb{H}_0$. So in particular, $\Gamma_0(n).\tau$ is not a cusp. By virtue of Corollary 2.2.5, there exists a $\gamma \in \Gamma_0(n)$ such that γw_n is an involution and $\gamma w_n.\tau = \tau$. It follows that $\Gamma_0(n).\tau$ lies on the ramification locus of π_n. Suppose now that $\Gamma_0(n).\tau$ is elliptic of even order, i.e. that $\Gamma_0(n)_\tau$ has an even order. Since $\Gamma_0(n)_\tau$ is a group, this can only happen if it contains an involution $\tilde{\gamma}$. We claim that $\gamma w_n = \tilde{\gamma} \in \Gamma_0(n)$, contradicting $w_n \notin \Gamma_0(n)$. This then shows that $\Gamma_0(n).\tau$ is either ordinary or elliptic of odd order. Indeed, recall that the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive, so let $f \in \mathrm{PSL}_2(\mathbb{R})$ be a Möbius transformation such that $f.\tau = \mathbf{i}$. Then, both $f\gamma w_n f^{-1}$ and $f\tilde{\gamma}f^{-1}$ are involutions in $\mathrm{PSL}_2(\mathbb{R})$ fixing $\mathbf{i}$. By Proposition 1.4.5, we see that for every $g \in \mathrm{PSL}_2(\mathbb{R})_{\mathbf{i}}$ with $g \neq I_2$, it holds that

$$g^2 = \begin{pmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{pmatrix} = I_2 \Leftrightarrow (a,b) = (0,\pm 1), \qquad \text{where } g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Since $(a,b) = (0,\pm 1)$ yield the same g in $\mathrm{PSL}_2(\mathbb{R})$, we conclude that there is only one involution in $\mathrm{PSL}_2(\mathbb{R})_{\mathbf{i}}$, from which $f\gamma w_n f^{-1} = f\tilde{\gamma}f^{-1}$ and, thus, $\gamma w_n = \tilde{\gamma}$ follows. This concludes the forward inclusion.

Conversely, let $\tau \in \mathbb{H}$ and suppose that the orbit $\Gamma_0^+(n).\tau$ lies on the branch locus of π_n and that $\Gamma_0(n).\tau$ is either ordinary or elliptic of odd order. By Proposition 3.1.3, it follows that $\Gamma_0(n)_\tau$ has an odd order i and $\Gamma_0^+(n)_\tau \supseteq \Gamma_0(n)_\tau$ has an even order $2i$. Thus, there exists an involution in $\Gamma_0^+(n) \setminus \Gamma_0(n) = \Gamma_0(n)w_n$ fixing τ, which by Corollary 2.2.5 implies that $\tau \in \mathbb{H} \setminus \mathbb{H}_0$. □

In the $n = 1$ case, recall that $\Gamma_0(1) = \Gamma_0^+(1)$. Letting $\tau \in \mathbb{H} \setminus \mathbb{H}_0$, Corollary 2.2.5 yields the existence of an element $\gamma \in \Gamma_0(1)$ such that γw_1 is an involution fixing τ. In particular, $\Gamma_0(1).\tau$ has a non-trivial stabilizer which, by Proposition 3.1.4, is either of order 2 or of order 3. In the latter case, there would be another involution in $\Gamma_0(n)$ fixing τ, which we have shown not to be possible. Consequently, $\Gamma_0^+(n).\tau = \Gamma_0(n).\tau$ is the elliptic point of order 2.

Combining this and the previous proposition with the description of the branch loci given in Proposition 3.1.4, we find that:

Corollary 3.1.6. *The set of orbits $\Gamma_0^+(n).\tau$ for $\tau \in \mathbb{H} \setminus \mathbb{H}_0$ consists of*

- $(n = 1, 2, 4)$ *the only elliptic point of order* 2*,*
- $(n = 3)$ *the two elliptic points, one of order* 2 *and the other of order* 6*,*
- $(n \geqslant 5)$ *ξ_n-many elliptic points of order* 2*.*

Finally, we need an elementary result on the fundamental group of certain orbifolds, which we will later make use of when computing $\pi_1^{\mathrm{orb}}([\mathbb{H}_0/\Gamma_0^+(n)])$.

Proposition 3.1.7. *Let $\mathcal{O}$ be the orbifold whose underlying topological space is the open disk $D \subseteq \mathbb{R}^2$, endowed with k punctures and, for each $j \geqslant 2$, with a_j cone points of order j, where $\sum_{j\geqslant 2} a_j < \infty$. Here, a cone point of order j is a marked point with local model $\mathbb{R}^2/(\mathbb{Z}/j\mathbb{Z})$, where $\mathbb{Z}/j\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotations around* 0*, cf. [Car22, Ex. 1.1.6]. Then*

$$\pi_1^{\mathrm{orb}}(\mathcal{O}) \cong \mathbb{Z}^{*k} * \left(\mathop{\ast}_{j\geqslant 2} (\mathbb{Z}/j\mathbb{Z})^{*a_j} \right).$$

Proof. We proceed by induction on the data $(k, a_2, a_3, \ldots)$.

Let us start with the base case. If $k = 1$ and all $a_j = 0$, then $\mathcal{O}$ is just the punctured disk, which deformation retracts onto S^1. Hence, by [Car22, Ex. 2.2.1], we have $\pi_1^{\mathrm{orb}}(\mathcal{O}) = \pi_1(\mathcal{O}) \cong \pi_1(S^1) = \mathbb{Z}$. If $k = 0$ and $a_j = 1$ for some j (with all other $a_i = 0$), then $\mathcal{O}$ is a disk with a single cone point of order j. By [Car22, Ex. 2.2.2] one obtains $\pi_1^{\mathrm{orb}}(\mathcal{O}) \cong \mathbb{Z}/j\mathbb{Z}$.

Let us turn to the induction step. Suppose the statement holds for some orbifold $\mathcal{O}$ as in the proposition. Let $\mathcal{O}'$ be obtained from $\mathcal{O}$ by adjoining an additional puncture (resp. an additional cone point of order j). Choose open connected subsets $U, V \subseteq \mathcal{O}'$ with $\mathcal{O}' = U \cup V$ such that $U \cong \mathcal{O}$, V is homeomorphic to D with a puncture (resp. a single cone point of order j) and $U \cap V$ is contractible. By virtue of the Seifert-van Kampen theorem for orbifolds—see [Car22, Thm. 2.2.3]—, we obtain

$$\pi_1^{\mathrm{orb}}(\mathcal{O}') \cong \pi_1^{\mathrm{orb}}(U) *_{\pi_1^{\mathrm{orb}}(\{*\})} \pi_1^{\mathrm{orb}}(V) \cong \pi_1^{\mathrm{orb}}(\mathcal{O}) * \pi_1^{\mathrm{orb}}(V).$$

By the base cases, we know that $\pi_1^{\mathrm{orb}}(V) \cong \mathbb{Z}$ (resp. $\cong \mathbb{Z}/j\mathbb{Z}$). The desired formula follows from the induction hypothesis on $\mathcal{O}$. □

Remark 3.1.8. In our setting, elliptic points of order j are modeled locally as cone points of order j. Indeed, let $\tau \in \mathbb{H}_0$ be an elliptic point. In the good orbifold atlas of $\mathbb{H}_0/\Gamma_0^+(n)$, a neighborhood of the image of $\Gamma_0^+(n).\tau$ has local model $U_\tau/\Gamma_0^+(n)_\tau$, where $U_\tau \subseteq \mathbb{H}_0$ is simply connected and $\Gamma_0^+(n)\tau$ denotes the stabilizer of τ in $\Gamma_0^+(n)$; see [Thu80, Prop. 13.2.1]. By [Shi94, Prop. 1.16], $\Gamma_0^+(n)_\tau$ is cyclic, generated by some $\gamma \in \Gamma_0^+(n)$. Furthermore, we note that γ has no more than two fixed points. Consequently, by [Thu80, Prop. 13.3.1], the local orbifold chart around τ is isomorphic to the standard cone-point model of order j.

We now turn to the computation of the orbifold fundamental group of $[\mathbb{H}_0/\Gamma_0^+(n)]$, leveraging the results obtained thus far concerning the closely related modular curve $X_0^+(n)$. This latter space features a cusp given by the $\Gamma_0^+(n)$-orbit of $\mathbf{i}\infty$ for each value of n. Throughout, a cusp *not* given by $\Gamma_0^+(n).\mathbf{i}\infty$ will be called a *real cusp.*

Proposition 3.1.9. [FL23, Prop. 4.6] *We have that*

$$\pi_1^{\text{orb}}\left(\left[\mathbb{H}_0 \middle/ \Gamma_0^+(n)\right]\right) \cong \begin{cases} \mathring{\mathbb{Z}} * (\mathbb{Z}/3\mathbb{Z}) & \text{if } n = 1 \\ \mathring{\mathbb{Z}} * (\mathbb{Z}/4\mathbb{Z}) & \text{if } n = 2 \\ \mathring{\mathbb{Z}} * \mathring{\mathbb{Z}} & \text{if } n = 3 \\ \mathring{\mathbb{Z}} * \check{\mathbb{Z}} & \text{if } n = 4 \\ (\mathbb{Z}/2\mathbb{Z})^{*\frac{\nu_2}{2}} * (\mathbb{Z}/3\mathbb{Z})^{*\frac{\nu_3}{2}} * \mathring{\mathbb{Z}}^{*\xi_n} * \check{\mathbb{Z}}^{*\left(\frac{\nu_\infty}{2}-1\right)} * \mathbb{Z}^{*\left(g+1-\frac{\xi_n}{2}\right)} & \text{if } n \geqslant 5, \end{cases}$$

where, for notational simplicity, we have suppressed the n in the notation of ν_i and g. Furthermore, a copy of $\mathbb{Z}$ is denoted by $\mathring{\mathbb{Z}}$ (resp. $\check{\mathbb{Z}}$) if it is generated by a loop around the $\Gamma_0^+(n)$ orbit of a point in $\mathbb{H} \setminus \mathbb{H}_0$ (resp. a real cusp).

Proof. Let us first address $n = 1, 2$. By Proposition 3.1.4 and Corollary 3.1.6, we find that $[\mathbb{H}_0/\Gamma_0^+(1)]$ (resp. $[\mathbb{H}_0/\Gamma_0^+(2)]$) has a unique elliptic point of order 3 (resp. of order 4). In both cases, the underlying topological space consists of a sphere with two holes: one arising from the orbit of $\mathbb{H} \setminus \mathbb{H}_0$ and the other one from the cusp $\mathbf{i}\infty$. The formulae for their orbifold fundamental group follow now from Proposition 3.1.7 and Remark 3.1.8 upon noticing that a sphere with one puncture is homeomorphic to an open disk.

We turn now to $n = 3, 4$. Again, by Proposition 3.1.4 and Corollary 3.1.6, neither $[\mathbb{H}_0/\Gamma_0^+(3)]$ nor $[\mathbb{H}_0/\Gamma_0^+(4)]$ have elliptic points. The underlying space is, in both cases, a sphere with three punctures: For $n = 3$, two come from the orbits of $\mathbb{H} \setminus \mathbb{H}_0$ and one from the cusp $\mathbf{i}\infty$; for $n = 4$, one corresponds to the orbit of $\mathbb{H} \setminus \mathbb{H}_0$, one to the cusp $\mathbf{i}\infty$, and one to a real cusp. Again, Proposition 3.1.7 and Remark 3.1.8 yield the desired result.

Finally, let $n \geqslant 5$. Proposition 3.1.4 and Corollary 3.1.6 show that $[\mathbb{H}_0/\Gamma_0^+(n)]$ has $\frac{1}{2}\nu_2(n)$ elliptic points of order 2 and $\frac{1}{2}\nu_3(n)$ elliptic points of order 3. Furthermore, the underlying topological space is a genus-$g^+(n)$ surface with $\frac{1}{2}\nu_\infty(n) + \xi_n$ punctures, of which ξ_n correspond to orbits of $\mathbb{H} \setminus \mathbb{H}_0$ and $\frac{1}{2}\nu_\infty(n)$ to cusps.

We decompose $[\mathbb{H}_0/\Gamma_0^+(n)] = S \cup D$, where S is a connected oriented genus-$g^+(n)$ surface with one puncture, D is an open disk that contains all the elliptic points and punctures, and $S \cap D$ is an annulus. The Seifert-van Kampen theorem for orbifolds together with [Car22, Ex. 2.2.1] yield

$$\pi_1^{\text{orb}}\left(\left[\mathbb{H}_0 \middle/ \Gamma_0^+(n)\right]\right) \cong \pi_1(S) *_{\pi_1(S\cap D)} \pi_1^{\text{orb}}(D).$$

We proceed to analyze the factors separately. Let us start with $\pi_1(S)$, where we will show by induction on the genus $g(S)$ of S that $\pi_1(S) \cong \mathbb{Z}^{*2g(S)}$, such that the counterclockwise loop around the puncture generates the last copy of $\mathbb{Z}$ in the free product. For $g(S) = 0$, S is homeomorphic to a sphere with a puncture, which is contractible. Hence, its fundamental group is trivial.

We now move on to the induction step. Thus, consider a connected oriented surface S of genus $g := g(S) > 0$ with one puncture and suppose the claim holds for genus $g-1$. Choose open connected subsets $S', U \subseteq S$ with $S = S' \cup U$ such that S' is homeomorphic to a connected oriented surface of genus $g-1$ with one puncture, U is homeomorphic to a torus $T \setminus \{p_1, p_2\}$ with two punctures and $S' \cap U$ is an annulus. Using the description of T as a quotient of $[0,1]^2$, note that $T \setminus \{p_1, p_2\}$ deformation retracts onto the union of three copies of S^1 touching pairwise, which is homeomorphic to $S^1 \vee S^1 \vee S^1$. Therefore, its fundamental group is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Moreover, by the induction hypothesis, we find that $\pi_1(S') \cong \mathbb{Z}^{*(2g-2)}$, where the counterclockwise loop ω around the puncture of S' generates the last copy of $\mathbb{Z}$ in the free product. Note that ω also generates $\pi_1(S' \cap U)$, for we glue S' to U along the puncture of S'. Thus, the factor generated by ω in $\pi_1(S' \cap D) \cong \mathbb{Z}[\omega]$ gets absorbed into $\pi_1(U)$ in the amalgamated product. In other words, the Seifert-van Kampen theorem gives

$$\pi_1(S) = \pi_1(S') *_{\pi_1(S' \cap U)} \pi_1(U) = \mathbb{Z}^{*(2g-3)} * \mathbb{Z}^{*3} = \mathbb{Z}^{*2g}.$$

Note that the counterclockwise loop ω' around the puncture of S satisfies $[\omega] = ab^{-1}$, where a, b generate the last two $\mathbb{Z}$ factors of $\pi_1(S)$. Upon noticing that $\langle a, b\rangle \cong \langle b, ab^{-1}\rangle$, we can write $\pi_1(S) \cong \mathbb{Z}^{*2g(S)}$ in such a way that ω' generates the last $\mathbb{Z}$ factor. This concludes the induction.

By Proposition 3.1.7, we obtain that $\pi_1^{\mathrm{orb}}(D) \cong (\mathbb{Z}/2\mathbb{Z})^{*\frac{\nu_2}{2}} * (\mathbb{Z}/3\mathbb{Z})^{*\frac{\nu_3}{2}} * \mathring{\mathbb{Z}}^{*\xi_n} * \check{\mathbb{Z}}^{*\left(\frac{\nu_\infty}{2}-1\right)} * \mathbb{Z}$, where the last copy of $\mathbb{Z}$ is generated by a loop around $\mathbf{i}\infty$.

We now return to the amalgamated product $\pi_1(S) *_{\pi_1(S\cap D)} \pi_1^{\mathrm{orb}}(D)$. Note that $\pi_1(S \cap D)$ is generated by a loop around the puncture of $\pi_1(S)$ which, by the discussion above, generates the last $\mathbb{Z}$ factor of $\pi_1(S) \cong \mathbb{Z}^{*2g^+(n)}$. As before, this factor gets absorbed into $\pi_1^{\mathrm{orb}}(D)$ in the amalgamated product. Consequently, we have that

$$\begin{aligned}
\pi_1^{\mathrm{orb}}\left(\left[\mathbb{H}_0 \big/ \varGamma_0^+(n)\right]\right) &\cong \pi_1(S) *_{\pi_1(S\cap D)} \pi_1^{\mathrm{orb}}(D) \\
&\cong \mathbb{Z}^{*2g^+} *_{\mathbb{Z}} \left((\mathbb{Z}/2\mathbb{Z})^{*\frac{\nu_2}{2}} * (\mathbb{Z}/3\mathbb{Z})^{*\frac{\nu_3}{2}} * \mathring{\mathbb{Z}}^{*\xi_n} * \check{\mathbb{Z}}^{*\left(\frac{\nu_\infty}{2}-1\right)} * \mathbb{Z}\right) \\
&\cong \mathbb{Z}^{*(2g^+-1)} * \left((\mathbb{Z}/2\mathbb{Z})^{*\frac{\nu_2}{2}} * (\mathbb{Z}/3\mathbb{Z})^{*\frac{\nu_3}{2}} * \mathring{\mathbb{Z}}^{*\xi_n} * \check{\mathbb{Z}}^{*\left(\frac{\nu_\infty}{2}-1\right)} * \mathbb{Z}\right) \\
&\cong \mathbb{Z}^{*(2g^+)} * \left((\mathbb{Z}/2\mathbb{Z})^{*\frac{\nu_2}{2}} * (\mathbb{Z}/3\mathbb{Z})^{*\frac{\nu_3}{2}} * \mathring{\mathbb{Z}}^{*\xi_n} * \check{\mathbb{Z}}^{*\left(\frac{\nu_\infty}{2}-1\right)}\right) \\
&\cong (\mathbb{Z}/2\mathbb{Z})^{*\frac{\nu_2}{2}} * (\mathbb{Z}/3\mathbb{Z})^{*\frac{\nu_3}{2}} * \mathring{\mathbb{Z}}^{*\xi_n} * \check{\mathbb{Z}}^{*\left(\frac{\nu_\infty}{2}-1\right)} * \mathbb{Z}^{*\left(g+1-\frac{\xi_n}{2}\right)},
\end{aligned}$$

where the last isomorphism follows from $2g^+(n) = g(n) + 1 - \frac{1}{2}\xi_n$, see Proposition 3.1.4. □

3.2 Classification of finite subgroups of $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$

Having determined the orbifold fundamental group of $[\mathbb{H}_0/\varGamma_0^+(n)]$, we can turn to the ultimate goal of this thesis: The classification of finite groups of $\mathrm{Aut}(\mathrm{D^b}(X))$ modulo even shifts. Let us start by combining previous results:

Proposition 3.2.1. *Let $n \geqslant 2$. It then holds that*

$$\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2] \cong \begin{cases} \mathbb{Z} * (\mathbb{Z}/4\mathbb{Z}) & \text{if } n = 2 \\ \mathbb{Z} * \mathbb{Z} & \text{if } n = 3, 4 \\ (\mathbb{Z}/2\mathbb{Z})^{*\frac{\nu_2}{2}} * (\mathbb{Z}/3\mathbb{Z})^{*\frac{\nu_3}{2}} * \mathbb{Z}^{*\left(g + \frac{1}{2}(\nu_\infty + \xi_n)\right)} & \text{if } n \geqslant 5. \end{cases}$$

For $n = 1$, we have that

$$\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}(\iota^*[1]) \cong \mathbb{Z} * (\mathbb{Z}/3\mathbb{Z}).$$

Proof. By virtue of [BB17, Rem. 7.2 & Prop. 7.3], when $n \geqslant 2$ we find that

$$\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2] \cong \pi_1^{\mathrm{orb}}\left(\left[\mathcal{Q}_0^+(X) \big/ \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))\right]\right).$$

At the same time, Proposition 2.4.4 yields the isomorphism

$$\pi_1^{\mathrm{orb}}\left(\left[\,\mathcal{Q}_0^+(X)\,\middle/\,\mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z}))\right]\right)\cong\pi_1^{\mathrm{orb}}\left(\left[\,\mathbb{H}_0\,\middle/\,\Gamma_0^+(n)\right]\right).$$

The result follows now by Proposition 3.1.9.

For $n=1$, we similarly obtain the result by combining Corollary 2.5.1 and Proposition 3.1.9. □

With these formulae, we are now ready to classify finite subgroups of symplectic autoequivalences up to even shifts. In order to do so, we first need a technical result on finite subgroups of the free product of groups:

Proposition 3.2.2. *Let $G := \mathop{\ast}_{i\in I} G_i$ be the free product of some groups G_i. Then, the finite subgroups of G are precisely those of the form gHg^{-1}, where $g\in G$ and H is a finite subgroup of some G_i.*

Proof. By the Kurosh subgroup theorem, any subgroup $H\subseteq G$ can be written as

$$H=F_S\ast\left(\mathop{\ast}_{j\in J} g_jH_jg_j^{-1}\right)$$

for some subset $S\subseteq G$ and some index set J with $g_j\in G$ and such that every H_j is a subgroup of some G_i. Here, F_S denotes the free group on the generating set S.

Furthermore, we easily see that the free product $A\ast B$ of two non-trivial groups A, B is always infinite. Indeed, letting $a\in A\setminus\{1_A\}$ and $b\in B\setminus\{1_B\}$, we find that the words $w_n=(ab)^n=ab\cdots ab\in A\ast B$ with length $2n$ cannot be further reduced, as the letters alternate between A and B and none of them are the identity. Since the lengths are strictly increasing, the words w_n are pairwise distinct, so we obtain an infinite cyclic subgroup in $A\ast B$ generated by $w_1=ab$.

This, along with the fact that non-trivial subgroups of free groups are free, so in particular of infinite order, shows that $H\subseteq G$ as given above is finite if and only if $S=\varnothing$, $|J|=1$ and H_j is a finite subgroup of some G_i. □

Our attention now turns to maximal subgroups, as they encapsulate all the information required for the classification.

Proposition 3.2.3. [FL23, Lem. 4.8] *Every maximal finite subgroup $G_s\subseteq \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$ is of the form*

$$G_s\cong\begin{cases}\mathbb{Z}/6\mathbb{Z} & \text{if } n=1\\ \mathbb{Z}/4\mathbb{Z} & \text{if } n=2\\ 1 & \text{if } n=3,4\\ 1,\mathbb{Z}/2\mathbb{Z} \text{ or } \mathbb{Z}/3\mathbb{Z} & \text{if } n\geqslant 5.\end{cases}$$

When $n=1,2$, there is precisely one such group up to conjugation. When $n\geqslant 5$, there exist precisely $\frac{\nu_2(n)}{2}$ (resp. $\frac{\nu_3(n)}{2}$) many such subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (resp. $\mathbb{Z}/3\mathbb{Z}$) up to conjugation.

Proof. For $n\geqslant 2$, the statement follows by combining Proposition 3.2.1 and Proposition 3.2.2, along with, again, the fact that non-trivial subgroups of free groups are free and, therefore, of infinite order.

Thus, let us address the case $n=1$. In this setting, we have the following chain of inclusions of subgroups: $\mathbb{Z}[2]\subseteq\mathbb{Z}(\iota^*[1])\subseteq\mathrm{Aut}_s(\mathrm{D^b}(X))$. By [BK22, Thm. 8.1], $\mathbb{Z}(\iota^*[1])$ lies in the center of $\mathrm{Aut}_s(\mathrm{D^b}(X))$. Consequently, each inclusion is, in fact, an inclusion of normal subgroups: $\mathbb{Z}[2]\trianglelefteq\mathbb{Z}(\iota^*[1])\trianglelefteq\mathrm{Aut}_s(\mathrm{D^b}(X))$. The third isomorphism theorem yields the canonical projection

$$f\colon\ \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]\twoheadrightarrow\frac{\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]}{\mathbb{Z}(\iota^*[1])/\mathbb{Z}[2]}\cong\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}(\iota^*[1])$$

with kernel $\mathbb{Z}(\iota^*[1])/\mathbb{Z}[2]$. We obtain the following short exact sequence of groups:

$$1 \longrightarrow \mathbb{Z}(\iota^*[1])/\mathbb{Z}[2] \hookrightarrow \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2] \xrightarrow{f} \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}(\iota^*[1]) \longrightarrow 1.$$

Let $C_3 \cong \mathbb{Z}/3\mathbb{Z}$ denote the subgroup of $\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}(\iota^*[1])$ generating its $(\mathbb{Z}/3\mathbb{Z})$ factor in Proposition 3.2.1. Observe that C_3 fits into the new short exact sequence:

$$1 \longrightarrow \mathbb{Z}(\iota^*[1])/\mathbb{Z}[2] \hookrightarrow f^{-1}(C_3) \twoheadrightarrow C_3 \longrightarrow 1.$$

Since $\mathbb{Z}(\iota^*[1])/\mathbb{Z}[2]$ has order 2 and $C_3 \cong \mathbb{Z}/3\mathbb{Z}$ has order 3, we see that $f^{-1}(C_3)$ has order 6 and is, thus, isomorphic to either $\mathbb{Z}/6\mathbb{Z}$ or to the symmetric group S_3. Note, however, that the latter is not possible, as $\mathbb{Z}(\iota^*[1])/\mathbb{Z}[2] \subseteq f^{-1}(C_3)$ and S_3 has a trivial center. It follows that $f^{-1}(C_3) \cong \mathbb{Z}/6\mathbb{Z}$. We claim that $f^{-1}(C_3)$ is, up to conjugation, the only maximal finite subgroup subgroup of $\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$.

Indeed, take a finite group $H \subseteq \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$ and note that $f(H) \subseteq \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}(\iota^*[1])$ is finite as well. By Proposition 3.2.1 and Proposition 3.2.2, we find that $f(H)$ is either trivial or equal to $f(\alpha)C_3f(\alpha)^{-1}$ for some $\alpha \in \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$. Regardless, we have that $H \subseteq \alpha f^{-1}(C_3)\alpha^{-1}$. The claim follows. ☐

Thus far, we have solved the classification problem up to even shifts in the symplectic setting. As the following proposition will demonstrate, this turns out to be no substantial limitation.

Proposition 3.2.4. [FL23, Lem. 4.12] *It holds that*

$$\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2] = \langle \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2], \mathbb{Z}[1]/\mathbb{Z}[2] \rangle \cong \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2] \times \mathbb{Z}/2\mathbb{Z}$$

As sets, we have that $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2] = (\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]) \sqcup (([1] \cdot \mathrm{Aut}_s(\mathrm{D^b}(X)))/\mathbb{Z}[2])$.

Proof. Since $[1]/\mathbb{Z}[2] \notin \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$ and it commutes with every element of $\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$, the group-theoretic description implies the set-theoretic one. We now turn to proving the former.

Recall that $T(X) \subseteq \widetilde{H}(X,\mathbb{Z})$ denotes the transcendental lattice of X. By [Ogu02, Lem. 4.1], when $\rho(X) = 1$, every Hodge isometry of $\widetilde{H}(X,\mathbb{Z})$ restricts to $\pm\mathrm{id}$ on $T(X)$. Furthermore, we note that $\varpi([2]) = \mathrm{id}_{\widetilde{H}(X,\mathbb{Z})}$, see Example 1.3.4, so the representation $\varpi\colon \mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))$ factors through $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$. Thus, we obtain a group homomorphism from $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$ to $\mathrm{O}(T(X)) = \{\pm\mathrm{id}\} \cong \mathbb{Z}/2\mathbb{Z}$ by restricting the action of every $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$ on $\widetilde{H}(X,\mathbb{Z})$ to $T(X)$:

$$g\colon \mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2] \longrightarrow \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z})) \xrightarrow{|_{T(X)}} \mathrm{O}(T(X)) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}[1]/\mathbb{Z}[2].$$

By Proposition 1.1.18, we see that the kernel of this group homomorphism is $\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$, so we obtain the following short exact sequence:

$$1 \longrightarrow \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2] \hookrightarrow \mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2] \xrightarrow{g} \mathbb{Z}[1]/\mathbb{Z}[2] \longrightarrow 1,$$

which right-splits, as the surjection g has a section given by $[1] \mapsto [1]$. It follows that $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$ splits as a semidirect product

$$\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2] = \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2] \rtimes \mathbb{Z}[1]/\mathbb{Z}[2].$$

This yields

$$\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2] = \langle \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2], \mathbb{Z}[1]/\mathbb{Z}[2] \rangle.$$

Moreover, since $\mathbb{Z}[1]/\mathbb{Z}[2]$ lives in the center of $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$, the semidirect product becomes a direct product, thus showing that

$$\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2] \cong \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2] \times \mathbb{Z}/2\mathbb{Z}.$$ ☐

Remark 3.2.5. In fact, the statement in Proposition 3.2.4 holds for every K3 surface S with odd Picard number, as the only Hodge isometries of the transcendental lattice $T(S)$ are still $\pm\mathrm{id}$, see [Huy16, Cor. 3.3.5]. Hence, the proof extends to this more general setting.

In particular, we obtain the following

Corollary 3.2.6. *Let G be a subgroup of $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$ and let $G_s \coloneqq G \cap \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$ denote its subgroup of symplectic elements. Then,*

$$G = \begin{cases} G_s \\ G_s \sqcup [1]/\mathbb{Z}[2] \cdot G_s \cong G_s \times \mathbb{Z}/2\mathbb{Z}. \end{cases}$$

Theorem 3.2.7. [FL23, Thm. 1.2] *Let X be a K3 surface of Picard number 1 and degree $2n$ and let G be a maximal finite subgroup of $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$. Then, G is isomorphic to one of the following:*

$$G \cong \begin{cases} \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2 \\ \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \text{ or } \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } n \geqslant 3. \end{cases}$$

For $n = 1, 2$, there is a unique such group up to conjugation. For $n \geqslant 3$, the number of conjugacy classes of subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is given by

$$\begin{cases} 2^{k-1} & \text{if } n = 2^\ell p_1^{\alpha_1} \cdots p_k^{\alpha_k} \text{ with } \ell \in \{0,1\};\ k, \alpha_i \in \mathbb{N}_{\geqslant 1};\ \text{and } p_i \equiv 1 \bmod 4 \text{ is a prime number,} \\ 0 & \text{if } n \text{ is divisible by } 4 \text{ or by a prime } p \equiv 3 \bmod 4. \end{cases}$$

Similarly, the number of conjugacy classes of subgroups isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is

$$\begin{cases} 2^{k-1} & \text{if } n = 3^\ell p_1^{\alpha_1} \cdots p_k^{\alpha_k} \text{ with } \ell \in \{0,1\};\ k, \alpha_i \in \mathbb{N}_{\geqslant 1};\ \text{and } p_i \equiv 1 \bmod 3 \text{ is a prime number,} \\ 0 & \text{if } n \text{ is divisible by } 9 \text{ or by a prime } p \equiv 2 \bmod 3. \end{cases}$$

Proof. By Corollary 3.2.6, every finite subgroup G of $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$ with subgroup of symplectic elements $G_s \coloneqq G \cap \mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$ is either identical to G_s or equal to $G_s \sqcup [1]/\mathbb{Z}[2] \cdot G_s \cong G_s \times \mathbb{Z}/2\mathbb{Z}$. If G is a maximal finite subgroup, the latter must hold and Proposition 3.2.3 yields the description of G_s together with the number of such subgroups up to conjugacy in terms of $\nu_2(n)$ and $\nu_3(n)$. The explicit formulae follow from [DS05, Cor. 3.7.2]. $\square$

Remark 3.2.8. In general, constructing autoequivalences of finite order modulo $\mathbb{Z}[2]$ is a non-trivial task. For completeness, let us record two examples:

(i) For $n = 1$, we have already seen that the autoequivalence ι^* induced by the covering involution $\iota\colon X \xrightarrow{\sim} X$ has order 2 both in $\mathrm{Aut}(\mathrm{D^b}(X))$ and in $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$.

(ii) When $n = 2$, consider the autoequivalence $\Theta \coloneqq \big(- \otimes \mathcal{O}_X(1)\big) \circ T_{\mathcal{O}_X}$, where $T_{\mathcal{O}_X}$ denotes the spherical twist along $\mathcal{O}_X$. It then holds that $\Theta^4 \cong [2]$, see [FL23, Ex. 4.9].

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