COMPLEX AND REAL MULTIPLICATION FOR K3 SURFACES

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These are notes of a talk at GAEL 2008. The aim was to provide an elementary introduction to complex multiplication for K3 surfaces. The second cohomology $H^2(X,\mathbb{Q})$ of an algebraic K3 surface contains the transcendental part $T(X)$ as an irreducible sub-Hodge structure. By definition, it is the orthogonal complement (with respect to the intersection pairing) of the Picard group. The endomorphism ring of this weight two Hodge structure $K = \mathrm{End}_{\mathrm{Hdg}}(T(X))$ is a number field which is either totally real (RM) or has complex multiplication (CM). For CM fields one can show that every $a \in K$ can be written as a linear combination of Hodge isometries.

We are only using linear algebra and some elementary number theory. Hodge groups, important for any deeper understanding, or references to Albert's classification are avoided altogether.

The main reference is of course Zarhin's paper [2]. The existence of the generating isometries in the CM case is taken from Borcea's article [1].

1. Hodge structures

Let H be a finite-dimensional vector space over $\mathbb{Q}$. A Hodge structure of weight n on H consists of a decomposition of its complexification $H_\mathbb{C} := H \otimes_\mathbb{Q} \mathbb{C}$:

$$H_\mathbb{C} = \bigoplus_{p+q=n} H^{p,q},$$

where $H^{p,q} \subset H_\mathbb{C}$ are complex linear subspaces such that complex conjugation identifies $H^{p,q}$ with $H^{q,p}$. Usually we will assume that $p, q \geq 0$.

Example 1.1. i) A weight one Hodge structure on H is simply a decomposition $H_\mathbb{C} = H^{1,0} \oplus H^{0,1}$ with $\overline{H^{1,0}} = H^{0,1}$. This gives rise to a complex structure I on the real vector space $H_\mathbb{R}$ by $I(v^{1,0} \oplus v^{0,1}) := iv^{1,0} \oplus (-iv^{0,1})$. For $v = v^{1,0} \oplus v^{0,1} \in H_\mathbb{R}$ one easily checks that $I(v) \in H_\mathbb{R}$.

ii) A weight two Hodge structure on H is given by a decomposition into $\mathbb{C}$-linear subspaces $H_\mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ such that complex conjugation identifies $H^{2,0}$ with $H^{0,2}$ and leaves $H^{1,1}$ invariant (as a subspace). In particular, $H^{1,1}$ and $H^{2,0} \oplus H^{0,2}$ are both defined over $\mathbb{R}$, i.e. $H^{1,1} = (H^{1,1} \cap H_\mathbb{R})_\mathbb{C}$ and similarly for $H^{2,0} \oplus H^{0,2}$.

Geometrically Hodge structures occur as additional structures on the rational singular cohomology of smooth projective complex varieties or compact Kähler manifolds. To be more precise, consider $H^n(X,\mathbb{Q})$ and its complexification $H^n(X,\mathbb{C})$. The quasi-isomorphism $\underline{\mathbb{C}} \xrightarrow{\sim} \Omega_X^\bullet$ yields a spectral sequence $E_1^{p,q} = H^q(X,\Omega_X^p) \Rightarrow H^{p+q}(X,\mathbb{C})$. One either uses Hodge theory with respect to a Kähler metric on X or the completely algebraic arguments of Deligne and Illusie to prove that this spectral sequence degenerates.

Example 1.2. Weight one and two Hodge structures can geometrically be realized as follows.

i) If $A = \mathbb{C}^g/\Gamma$ is an abelian variety or a complex torus, then the weight one Hodge structure $H^1(A,\mathbb{C}) = H^{1,0}(A) \oplus H^{0,1}(A)$ determines A completely. In fact, the projection $H^1(A,\mathbb{C}) \twoheadrightarrow H^{0,1}(A)$ allows one to view $H^1(A,\mathbb{Z})$ as a lattice in $H^{0,1}(A)$. The induced torus $H^{0,1}(A)/H^1(A,\mathbb{Z})$ is nothing but the dual abelian variety $\hat{A}$. Using the dual Hodge structure, gives back A more directly, i.e. $A \simeq H^{1,0}(A)^*/H_1(A,\mathbb{Z})$.

ii) For a K3 surface, i.e. a connected compact complex surface with trivial canonical bundle $K_X = \Omega_X^2 \simeq \mathcal{O}_X$ and $H^1(X,\mathcal{O}_X) = 0$, the natural weight two Hodge structure on $H^2(X,\mathbb{Q})$ has one-dimensional $H^{2,0}(X)$ spanned by any trivializing section of K_X (a non-degenerate holomorphic volume form $\sigma \in H^0(X,\Omega_X^2)$). The space of rational classes $H^{1,1}(X) \cap H^2(X,\mathbb{Q})$ is naturally isomorphic to $\mathrm{Pic}(X)_\mathbb{Q}$. To have a concrete example in mind, any abelian surface $A = \mathbb{C}^2/\Gamma$ gives rise to a K3 surface X by taking the minimal resolution of the quotient $A/\pm$, where $\pm$ is the standard involution $z \longmapsto -z$. The quotient $A/\pm$ has 16 ordinary double points which give rise to 16 exceptional curves C_i, $i = 1,\ldots,16$. Thus $H^2(X,\mathbb{Q}) = H^2(A/\pm,\mathbb{Q}) \oplus \bigoplus [C_i]\mathbb{Q} = H^2(A,\mathbb{Q}) \oplus \bigoplus [C_i]\mathbb{Q}$, which of dimension 22. This K3 surface X is called the Kummer surface associated to A.

Definition 1.3. A *sub-Hodge structure* of a weight n Hodge structure on H is given by a $\mathbb{Q}$-linear subspace $H' \subset H$ such that the Hodge structure on H induces a Hodge structure on H', i.e. $H'_\mathbb{C} = \bigoplus (H'_\mathbb{C} \cap H^{p,q})$. Any Hodge structure that does not contain any non-trivial proper Hodge structure is called *irreducible*.

Any rational $(1,1)$-class in a weight two Hodge structure spans a sub-Hodge structure. In particular, the Hodge structure $H^2(X,\mathbb{Q})$ of a K3 surface is not irreducible if $\mathrm{Pic}(X) \neq 0$.

2. The transcendental lattice

Since the natural weight two Hodge structure $H^2(X,\mathbb{Q})$ for an algebraic K3 surfaces is never irreducible, it is usually replaced by the transcendental part $T(X) \subset H^2(X,\mathbb{Q})$ which turns out to be irreducible. We will make use of the intersection pairing $\langle\ ,\ \rangle$ on $H^2(X,\mathbb{Q})$. Its $\mathbb{C}$-bilinear extension will also be denoted by $\langle\ ,\ \rangle$. It is easy to see that $\alpha \in H^2(X,\mathbb{R})$ is of type $(1,1)$ if and only if $\langle \alpha,\sigma\rangle = 0$. As before, σ denotes a non-trivial holomorphic two-form on X.

Definition 2.1. For an algebraic K3 surface X we define the *transcendental part* $T(X) \subset H^2(X,\mathbb{C})$ as the $\mathbb{Q}$-linear subspace

$$T(X) := \{\alpha \in H^2(X,\mathbb{Q}) \mid \langle \alpha, \mathrm{c}_1(L)\rangle = 0 \ \forall\ L \in \mathrm{Pic}(X)\}.$$

Remark 2.2. i) The Hodge index theorem shows that the intersection pairing $\langle\ ,\ \rangle$ restricted to $\mathrm{Pic}(X)$ is non-degenerate. Hence $T(X) \cap \mathrm{Pic}(X) = 0$. Thus,

$$H^2(X,\mathbb{Q}) = \mathrm{Pic}(X)_\mathbb{Q} \oplus T(X)$$

and $\langle\ ,\ \rangle$ restricted to $T(X)$ is also non-degenerate.

ii) Clearly, the transcendental part can be defined for any weight two Hodge structure endowed with a non-degenerate symmetric bilinear form. In i) we would simply replace the Hodge index theorem by the assumption that the symmetric form is non-degenerate on the subspace of rational $(1,1)$-classes.

Proposition 2.3. *If X is an algebraic K3 surface, then $T(X) \subset H^2(X,\mathbb{Q})$ is a sub-Hodge structure. The weight two Hodge structure $T(X)$ is irreducible.*

Proof. It is easy to see that $T(X)_{\mathbb{C}}$ is the set of all classes $\alpha \in H^2(X,\mathbb{C})$ orthogonal to $\mathrm{Pic}(X)$ with respect to the ($\mathbb{C}$-linear extension of) the intersection product $\langle\ ,\ \rangle$.

Next, for any $v \in T(X)_{\mathbb{C}}$ consider its Hodge decomposition (in $H^2(X,\mathbb{C})$) written as $v = v^{2,0} \oplus v^{1,1} \oplus v^{0,2}$. Since $H^{2,0} \oplus H^{0,2}$ is orthogonal to $\mathrm{Pic}(X)$, $v^{2,0}, v^{0,2} \in T(X)_{\mathbb{C}}$. But then also $v^{1,1} \in T(X)_{\mathbb{C}}$. This proves that $T(X) \subset H^2(X,\mathbb{Q})$ really is â sub-Hodge structure.

Next consider a sub-Hodge structure $T' \subset T(X)$. The intersection of T' with $H^{1,1}$ is trivial, for $\mathrm{Pic}(X) \cap T(X) = 0$. Thus $T'_{\mathbb{C}}$ must contain $T^{2,0}$. If $T' \neq T(X)$, then the orthogonal complement of T' cannot be trivial and, since it is orthogonal to $T^{2,0}$, must be contained in $T(X)^{1,1} \cap T(X)$. Contradiction. □

Remark 2.4. The transcendental part $T(X)$ can equivalently be defined as the smallest sub-Hodge structure $T \subset H^2(X,\mathbb{Q})$ such that $H^{2,0}(X) \subset T_{\mathbb{C}}$. This definition works well also for non-algebraic K3 surfaces and in fact for any weight two Hodge structure. Moreover, the irreducibility is (almost) immediate, where we have to assume that $H^{2,0}$ is one-dimensional for the case of an arbitrary weight two Hodge structure.

Example 2.5. The above definition makes also sense for any abelian surface. For a Kummer surface $X \longrightarrow A/\pm$ the transcendental parts $T(X)$ and $T(A)$ are isomorphic as weight two Hodge structures. If the quadratic form is included, this becomes $(T(X), \langle\ ,\ \rangle_X) \simeq (T(A), 2\langle\ ,\ \rangle_A)$.

3. End(T)

Let us consider an arbitrary irreducible weight two Hodge structure T with $\dim_{\mathbb{C}} T^{2,0} = 1$. We will be interested in its algebra of endomorphisms. More precisely, let $K := K(T) := \mathrm{End}_{\mathrm{Hdg}}(T)$ be the $\mathbb{Q}$-algebra of all $\mathbb{Q}$-linear maps $a : T \longrightarrow T$ such that its $\mathbb{C}$-linear extension $a_{\mathbb{C}}$ preserves the $T^{p,q}$. So by definition K is a finite-dimensional $\mathbb{Q}$-algebra, which, as we will see shortly, is in fact commutative.

The geometric motivation for studying K is explained by the following considerations.

Let us come back to the general situation of an irreducible wieght two Hodge structure T with one dimensional $T^{2,0}$ and consider the $\mathbb{Q}$-algebra homomorphism

$$\varepsilon : K \longrightarrow \mathbb{C},$$

which is defined by $a|_{T^{2,0}} = \varepsilon(a) \cdot \mathrm{id}$.

Proposition 3.1. *The map ε is injective. Hence, K is commutative and, more precisely, a number field.*

Proof. Suppose $a \in K$ such that $\varepsilon(a) = 0$. Then let T' be the kernel of a. Clearly, $T^{2,0} \subset T'_{\mathbb{C}}$ and hence the $(1,1)$-part of any class $v \in T'_{\mathbb{C}}$ is again contained in $T'_{\mathbb{C}}$. In other words, $T' \subset T(X)$ is a sub-Hodge structure. Since $T(X)$ is irreducible, either $T' = 0$, which contradicts $\varepsilon(a) = 0$, or $T' = T(X)$, which yields $a = 0$. □

What kind of algebraic number fields does one encounter as the endomorphism rings of irreducible weight two Hodge structures? As we shall explain, only two types do occur.

Let us consider the embeddings $K \hookrightarrow \mathbb{C}$. We denote the real embeddings by $\rho_1, \ldots, \rho_r : K \hookrightarrow \mathbb{R} \subset \mathbb{C}$ and the complex ones by $\sigma_1, \bar{\sigma}_1, \ldots, \sigma_s, \bar{\sigma}_s : K \hookrightarrow \mathbb{C}$. In particular, $[K : \mathbb{Q}] = r+2s$.

Recall that the trace $\mathrm{Tr}_{K/\mathbb{Q}}$ of any $a \in K$ can be written as $\mathrm{Tr}_{K/\mathbb{Q}}(a) = \sum_1^r \rho_i(a) + \sum_1^s \sigma_j(a) + \sum_j^s \bar{\sigma}_j(a) = \sum_1^r \rho_i(a) + 2\sum_1^s \mathrm{Re}(\sigma_j(a))$. The Hodge structure T is not only a vector space over $\mathbb{Q}$, but also over K. If $\dim_K(T) = n$, then $\mathrm{Tr}_{T/\mathbb{Q}}(a) = n \cdot \mathrm{Tr}_{K/\mathbb{Q}}(a)$ for all $a \in K$.

A number field K is called *totally real* if $s = 0$, i.e. all embeddings of $K \hookrightarrow \mathbb{C}$ are real. A number field K is a *CM field* if K contains a subfield $K_0 \subset K$ such that K_0 is totally real and K/K_0 is a purely imaginary quadratic extension, i.e. there exists an element α such that $\rho_i(\alpha) \in \mathbb{R}_{<0}$ for all embeddings $\rho : K_0 \hookrightarrow \mathbb{R}$ and $K = K_0(\sqrt{-\alpha})$. If K is a CM field, one says that T has complex multiplication. Similarly, T has real multiplication, if K is totally real.

In a first step, we will identify a totally real field $K_0 \subset K$ and then show that either $K_0 = K$ or that K/K_0 is purely imaginary quadratic. In order to define K_0, we will have to assume that T is endowed with a natural quadratic form. In the geometric situation this is just the intersection pairing. So, let us assume that $\langle\ ,\ \rangle$ is non-degenerate symmetric bilinear form on T of signature $(2, m)$ such that its $\mathbb{R}$-linear extension is positive definite on $(T^{2,0} \oplus T^{0,2}) \cap T_{\mathbb{R}}$ and such that the decomposition $T_{\mathbb{R}} = (T^{1,1} \cap T_{\mathbb{R}}) \oplus ((T^{2,0} \oplus T^{0,2}) \cap T_{\mathbb{R}})$ is orthogonal with respect to $\langle\ ,\ \rangle$. In particular, the $\mathbb{R}$-linear extension of $\langle\ ,\ \rangle$ is negative definite on $(T^{1,1} \cap T_{\mathbb{R}})$.

Definition 3.2. If $\langle\ ,\ \rangle$ as above exists on T, then the involution $K \longrightarrow K$, $a \longmapsto a'$ is defined by the condition

$$\langle av, w\rangle = \langle v, a'w\rangle$$

for all $v, w \in T$. In other words, a' is the formal adjoint of a with respect to $\langle\ ,\ \rangle$.

The definition tacitly claims that with $a \in K$ also $a' \in K$. This is granted by the following easy lemma.

Lemma 3.3. *If $a \in K$, then $a' \in K$, i.e. with a also a' preserves the Hodge structure.*

Proof. Suppose $v \in T^{1,1}$. Then $\langle v, a(w)\rangle = 0$ for all $w \in T^{2,0} \oplus T^{0,2}$, for $a(w)$ is again of type $(2,0) + (0,2)$. But this shows that $a'(v)$ is orthogonal to $T^{2,0} \oplus T^{0,2}$, i.e. also $a'(v)$ is of type $(1,1)$. The proof that a' preserves $T^{2,0}$ and $T^{0,2}$ is similar. □

Remark 3.4. Clearly, $(ab)' = a'b'$, i.e. $a \longmapsto a'$ is an automorphism of K. Also observe that for $a \in K$ and all $v, w \in T$ one has $\langle av, aw\rangle = \langle a'av, w\rangle$. Hence $aa' = 1$ if and only if a is an isometry.

Definition 3.5. Denote by $K_0 \subset K$ the subfield of all $a \in K$ with $a' = a$.

Thus, since $a \longmapsto a'$ is an automorphism of K of order two, its fixed field $K_0 = K'$ satisfies $[K : K_0] \leq 2$.

To study K_0, it is more convenient to work with a positive definite symmetric bilinear form, which however is only defined over $\mathbb{R}$. Let us define $(\ ,\ )$ on $T_{\mathbb{R}}$ by setting

$$(\ ,\ ) = \langle\ ,\ \rangle \text{ on } (T^{2,0} \oplus T^{0,2}) \cap T_{\mathbb{R}} \text{ and } (\ ,\ ) = -\langle\ ,\ \rangle \text{ on } T^{1,1} \cap T_{\mathbb{R}}.$$

Remark 3.6. As it turns out, for any $a \in K$, the formal adjoint a' with respect to $\langle\ ,\ \rangle$ is also the formal adjoint of the $\mathbb{R}$-linear extension of a with respect to $(\ ,\ )$. Indeed, For any $0 \neq a \in K$ one considers $\xi_a := a'a = aa' \in K$. Then ξ_a satisfies:

i) $(\xi_a v, w) = (v, \xi_a w)$ for all $v, w \in T_{\mathbb{R}}$, i.e. ξ_a is self-adjoint.

ii) $(\xi_a v, v) = (av, av) > 0$ for all $0 \neq v \in T_{\mathbb{R}}$.

In particular, all eigenvalues of ξ_a are positive and, therefore, $\mathrm{Tr}_{T/\mathbb{Q}}(\xi_a) > 0$ and also $\mathrm{Tr}_{K/\mathbb{Q}}(\xi_a) > 0$. This will be crucial on the proof of the next proposition.

Proposition 3.7. *Any number field L satisfying $\mathrm{Tr}_{L/\mathbb{Q}}(a^2) > 0$ for all $0 \neq a \in L$ is totally real.*

Proof. As before, we denote by ρ_i, $i = 1, \ldots, r$ the real embeddings of L and by $\sigma_1, \bar{\sigma}_1, \ldots, \sigma_s, \bar{\sigma}_s$ the complex ones. We have to show that $s = 0$. This will be derived from a contradiction as follows.

First observe $L \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^r \oplus \mathbb{C}^s$. Clearly, there exists $a \in L$ such that $\rho_i(a) \sim 0$ for i, $\sigma_j(a) \sim 0$ for $j < s$, and $\sigma_s(a) \sim i$.

By assumption $0 < \mathrm{Tr}_{L/\mathbb{Q}}(a^2) = \sum_i \rho_i(a^2) + 2\sum_{j<s} \mathrm{Re}(\sigma_j(a^2)) + 2\mathrm{Re}(\sigma_s(a^2))$. On the other hand, by construction $\rho_i(a^2) = \rho_i(a)^2$ and $\sigma_j(a^2) = \sigma_j(a)^2$ for $j < s$ are all close to zero, whereas $\sigma_s(a^2) \sim -1$. This yields the contradiction $0 < \mathrm{Tr}_{L/\mathbb{Q}}(a^2) < 0$. □

Corollary 3.8. *For any irreducible weight two Hodge structure T with one-dimensional $T^{2,0}$ and endowed with a symmetric bilinear form of signature $(2, m)$ positive definite on $(T^{2,0} \oplus T^{0,2}) \cap T_{\mathbb{R}}$, the fixed field K_0 of the adjoint $a \longmapsto a'$ is totally real.* □

4. The CM case

Proposition 4.1. *If $K_0 \neq K$, then K/K_0 is a purely imaginary quadratic extension. In particular, K is a CM field.*

Proof. As observed earlier, if $K_0 \neq K$, then $[K : K_0] = 2$ and, therefore, we can write $K = K_0(\sqrt{\alpha})$ for some $\alpha \in K_0$.

Fix one real embedding $K_0 \subset \mathbb{R}$ and suppose $\alpha \in \mathbb{R}_{>0}$. The natural inclusion $K_0(\sqrt{\alpha}) \subset \mathbb{R}$ yields one real embedding $\rho_1 : K \longrightarrow \mathbb{R}$ and a second one is given by ρ_2 which is the identity on K_0 and which sends $\sqrt{\alpha}$ to $-\sqrt{\alpha}$, i.e. $\rho_2 = \rho_1 \circ (\ )'$.

Let us denote the remaining embeddings of K by $\rho_3, \ldots, \rho_d$ (which may be real or complex).

Similar to the argument used in the proof of Proposition 3.7 we choose $a \in K$ such that $\rho_1(a) \sim -1$, $\rho(a) \sim 1$ and $\rho_i(a) \sim 0$ for $i \geq 3$. Then on the one hand, $0 < \mathrm{Tr}_{K/\mathbb{Q}}(\xi_a) = \mathrm{Tr}_{K/\mathbb{Q}}(aa')$ and on the other hand $\mathrm{Tr}_{K/\mathbb{Q}}(aa') = \rho_1(aa') + \rho_2(aa') + \sum_{i \geq 3} \rho_i(aa') \sim \rho_1(aa') + \rho_2(aa') = \rho_1(a)\rho_1(a') + \rho_2(a)\rho_2(a') = 2\rho_1(a)\rho_2(a) \sim -2$. Contradiction. □

Remark 4.2. If $\dim_{\mathbb{Q}} T \equiv 1(2)$, then $K_0 = K$, i.e. K is RM. Indeed, $[K : \mathbb{Q}]$ divides $\dim_{\mathbb{Q}} T$ and $[K : \mathbb{Q}]$ is even for a CM field.

If $\dim_K T = 1$, i.e. $[K : \mathbb{Q}] = \dim_{\mathbb{Q}} T$, then K is a CM field. A proof not using the Hodge group goes as follows: $T \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus T_\rho$, where the sum runs over all $\rho : K \longrightarrow \mathbb{C}$ and T_ρ is the $\mathbb{C}$-subspace on which the elements $\alpha \in K$ act by multiplication with $\rho(\alpha)$. Clearly $\dim_{\mathbb{C}}(T \otimes_{\mathbb{Q}} \mathbb{C}) = \dim_{\mathbb{Q}} T = \dim_{\mathbb{Q}} K = [K : \mathbb{Q}]$ and hence $\dim_{\mathbb{C}} T_\rho = 1$. Suppose K were totally real. Then $K = \mathbb{Q}(\alpha)$ with $\varepsilon(\alpha) \in \mathbb{R}$. Hence σ and $\bar{\sigma}$ are both contained in T_ε, which contradicts $\dim_{\mathbb{C}} T_\varepsilon = 1$.[1]

[1]Thanks to U. Schlickewei for his help with the argument.

Remark 4.3. For the record, in the case of complex multiplication $a \longmapsto a'$ is given by complex conjugation (for all complex embeddings).

As mentioned above, $a \in K$ is an isometry if and only if $\xi_a = aa' = 1$. For $a \in K_0$, this is only possible if $a = \pm 1$. Thus, in the case of real multiplication, there exist very few Hodge isometries of T. For the CM case, the situation is completely different.

Proposition 4.4. *Let T be a weight two Hodge structure with one-dimensional $T^{2,0}$ and a symmetric bilinear form $\langle\ ,\ \rangle$ of signature $(2,m)$ which is positive definite on $(T^{2,0} \oplus T^{0,2}) \cap T_{\mathbb{R}}$. If T has complex multiplication, then K is spanned as a $\mathbb{Q}$-vector space by Hodge isometries.*

We will in fact show the stronger statement that $K = \mathbb{Q}(\alpha)$ for a certain Hodge isometry α.

Proof. Let us fix the following notation. As before, K_0 is the fixed field of the involution $a \longmapsto a'$. Then write $K = K_0(\sqrt{-D})$ with $D \in K_0$ positive under each embedding. Also, fix a primitive element $\beta \in K_0$, i.e. $K_0 = \mathbb{Q}(\beta)$.

We start out with some elementary number theory and show that for any $\gamma \in \mathbb{Q}$ one has $\mathbb{Q}(\beta D + \gamma D) = K_0$. To see this, denote $M_\gamma := \mathbb{Q}(\beta D + \gamma D)$, which is a subfield $\mathbb{Q} \subset M_\gamma \subset K_0$. Whenever $M_{\gamma'} \subset M_\gamma$ for two rational numbers $\gamma' \neq \gamma$, then $M_\gamma = K_0$. Indeed, the inclusion implies $(\beta D + \gamma D) - (\beta D + \gamma' D) \in M_\gamma$ and hence $D \in M_\gamma$. The latter yields $\beta \in M_\gamma$, i.e. $M = K_0$. Since K_0 only has finitely many subfields, one finds that for all but finitely many $\gamma \in \mathbb{Q}$ one has $M_{\gamma/2} = K_0$.

Similarly, one defines for $\gamma \in \mathbb{Q}$ the subfield $L_\gamma := \mathbb{Q}(D(\beta + \gamma)^2) \subset K_0$. For an infinite set $S \subset \mathbb{Q}$ the field L_γ will be the same for all $\gamma \in S$. Among the infinite number of sums $\gamma + \gamma'$ with $\gamma, \gamma' \in S$ pick one for which $M_{(\gamma+\gamma')/2} = K_0$. Then use $D(2\beta + \gamma + \gamma')(\gamma - \gamma') = D(\beta + \gamma)^2 - D(\beta + \gamma')^2$ to deduce that $K_0 = \mathbb{Q}(D(2\beta + \gamma + \gamma')) = \mathbb{Q}(D(2\beta + \gamma + \gamma')(\gamma - \gamma')) \subset \mathbb{Q}(D(\beta + \gamma)^2, D(\beta + \gamma')^2) = \mathbb{Q}(D(\beta + \gamma)^2) \subset K_0$. Hence, $\mathbb{Q}(D(\beta + \gamma)^2) = K_0$.

From the above discussion we will only need that there exists a primitive element of the form D/ξ^2, i.e. $K_0 = \mathbb{Q}(D/\xi^2)$, with $\xi \in K_0$.

Then let $\alpha := (D - \xi^2)/(D + \xi^2) + 2\xi\sqrt{-D}/(D + \xi^2) \in K$. Easy calculations reveal that $\alpha\alpha' = \alpha\bar{\alpha} = 1$, i.e. α is a Hodge isometry, and $\alpha + \bar{\alpha} = 2(1 - 2(D/\xi^2 + 1)^{-1})$. The latter shows $D/\xi^2 \in \mathbb{Q}(\alpha)$. Since also $\sqrt{-D} \in \mathbb{Q}(\alpha)$, this suffices to conclude $K = \mathbb{Q}(\alpha)$. □

References

[1] C. Borcea *K3 surfaces and complex multiplication.* Rev. Roumaine Math. Pures Appl. 31 (1986), no. 6, 499-505.

[2] Y. Zarhin *Hodge groups of K3 surfaces.* J. Reine Angew. Math. 341 (1983), 193-220.