THE LLV DECOMPOSITION OF HYPERKÄHLER COHOMOLOGY

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1. Recall of results on the LLV algebra

For X a compact hyperkähler manifold, the rational second cohomology group $H^2(X,\mathbf{Q})$ is equipped with the Beauville–Bogomolov–Fujiki form q_X. Following [GKLR], we write

$$(V,q) \coloneqq \left(H^2(X,\mathbf{Q}) \oplus \mathbf{Q}^2, q_X \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$$

for the *Mukai completion* (usually this is denoted as $\widetilde{H}(X,\mathbf{Q})$).

Let $h \in \operatorname{End}(H^*(X,\mathbf{Q}))$ be the degree operator such that

$$h|_{H^k(X,\mathbf{Q})} = (k - \dim X)\operatorname{Id},$$

where the degrees are centered at the middle cohomology. The *Looijenga–Lunts–Verbitsky algebra* $\mathfrak{g}$ is the subalgebra of $\operatorname{End}(H^*(X,\mathbf{Q}))$ generated by all Lefschetz operators and dual Lefschetz operators (equivalently, by $\mathfrak{sl}_2$*-triples* (L_a, h, Λ_a) for $a \in H^2(X,\mathbf{Q})$).

Theorem 1.1 (Looijenga–Lunts, Verbitsky)**.**

(1) $\mathfrak{g}$ *is isomorphic to* $\mathfrak{so}(V,q)$*;*
(2) $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$*;*
(3) $\mathfrak{g}_0 = \mathfrak{g}_0' \oplus \mathbf{Q}h$*, and the reduced part* $\mathfrak{g}_0' \coloneqq [\mathfrak{g}_0, \mathfrak{g}_0]$ *is isomorphic to* $\mathfrak{so}(H^2(X,\mathbf{Q}), q_X)$*.*

The cohomology $H^*(X,\mathbf{Q})$ is a $\mathfrak{g}$-module by construction. The main goal is to study the decomposition of $H^*(X,\mathbf{Q})$ into irreducible $\mathfrak{g}$-modules. First we have the following easy result.

Proposition 1.2. $H^*(X,\mathbf{Q})$ *decomposes into* $H^*_{\mathrm{even}}(X,\mathbf{Q}) \oplus H^*_{\mathrm{odd}}(X,\mathbf{Q})$ *as* $\mathfrak{g}$*-modules.*

Another general result is obtained by Verbitsky.

Theorem 1.3 (Verbitsky)**.** *The subalgebra* $SH^2(X,\mathbf{Q}) \subset H^*(X,\mathbf{Q})$ *generated by* $H^2(X,\mathbf{Q})$ *is an irreducible* $\mathfrak{g}$*-submodule. It is isomorphic to* $\operatorname{Sym}^*(H^2(X,\mathbf{Q}))/\langle a^{n+1} \mid q_X(a) = 0\rangle$ *as algebra and* $\mathfrak{g}_0'$*-module.*

Corollary 1.4. *The branching rules for* $\mathfrak{g}_0' \subset \mathfrak{g}$ *show that* $SH^2(X,\mathbf{Q})$ *is isomorphic to* $V_{(n)}$ *as* $\mathfrak{g}$*-module (see below for notations).*

So there is always an irreducible component that is known (and also quite big), which is referred to as the *Verbitsky component*. Here, the method of recovering the $\mathfrak{g}$-module structure by decomposing it further with respect to some smaller subalgebra is very important and will be frequently used later.

2. Representation theory

We introduce the necessary notions for the representation theory of $\mathfrak{g}$. For this section, $\mathfrak{g} \coloneqq \mathfrak{so}(V,q)$ denotes a Lie algebra of type B_r or D_r defined over $\mathbf{Q}$, where $\dim V = 2r+1$ or $\dim V = 2r$. For references, see the Appendices of [GKLR] and the book [FH].

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Type B. Let $\mathfrak{h} \subset \mathfrak{g}_{\mathbf{C}}$ be a Cartan subalgebra. The standard representation V decomposes as

$$V = V(0) \oplus V(\varepsilon_1) \oplus V(-\varepsilon_1) \oplus \cdots \oplus V(\varepsilon_r) \oplus V(-\varepsilon_r),$$

where $0, \pm\varepsilon_1, \ldots, \pm\varepsilon_r \in \mathfrak{h}^\vee$ are the weights of V. An element $h \in \mathfrak{h}$ acts as the scalar $\varepsilon(h)$ on $V(\varepsilon)$. We choose a positive Weyl chamber generated by the fundamental weights

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \text{ for } 1 \leq i \leq r-1, \quad \varpi_r = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r).$$

They correspond to the highest weight of $\bigwedge^i V$ for $1 \leq i \leq r-1$ and the spin module respectively. The set of dominant weights is the following

$$\Lambda^+ = \left\{ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r \;\middle|\; \begin{array}{c} \lambda_1 \geq \cdots \geq \lambda_r \geq 0 \\ \lambda_i \in \frac{1}{2}\mathbf{Z}, \lambda_i - \lambda_j \in \mathbf{Z} \end{array} \right\}.$$

Example. $V_{(1,\ldots,1)} = \bigwedge^k V$, $V_{(k)} = \ker(\operatorname{Sym}^k V \xrightarrow{q} \operatorname{Sym}^{k-2} V)$.

Over $\mathbf{C}$, irreducible $\mathfrak{g}_{\mathbf{C}}$-modules are classified by their highest weight.

Over $\mathbf{Q}$, the Schur–Weyl construction for a $\mathfrak{g}$-module with integral highest weight is still available: let λ be a dominant weight with $\sum \lambda_i = d$, we have

$$V_\lambda \coloneqq \mathbf{S}_\lambda V \cap V^{[d]},$$

where $\mathbf{S}_\lambda$ is the Schur functor, and $V^{[d]}$ is the intersection of all the kernels $\ker(V^d \xrightarrow{q} V^{d-2})$ given by contracting any two components with q. On the other hand, modules with half-integer highest weight are not necessarily defined over $\mathbf{Q}$.

Type D. The standard representation V has weights $\varepsilon_1, \ldots, \pm\varepsilon_r \in \mathfrak{h}^\vee$. The fundamental weights are given by

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \text{ for } 1 \leq i \leq r-2, \quad \varpi_{r-1} = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r), \quad \varpi_r = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r),$$

corresponding to the highest weight of $\bigwedge^i V$ for $1 \leq i \leq r-2$ and the two half-spin modules respectively. The set of dominant weights is the following

$$\Lambda^+ = \left\{ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r \;\middle|\; \begin{array}{c} \lambda_1 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r| \geq 0 \\ \lambda_i \in \frac{1}{2}\mathbf{Z}, \lambda_i - \lambda_j \in \mathbf{Z} \end{array} \right\}.$$

Again, all the representations with integral highest weight are defined over $\mathbf{Q}$ via the Schur–Weyl construction, which is not necessarily the case for those with half-integer highest weight.

The dimension of each V_λ can be obtained using Weyl dimension formula, which we won't state here. We will however need the following corollary of the dimension formula.

Lemma 2.1. *Let λ and $\mu \neq 0$ be dominant integral weights of $\mathfrak{g}$, then* $\dim V_{\lambda+\mu} > \dim V_\lambda$.

Weyl character. We review the results on the Weyl character ring, which more generally hold for any reductive rational Lie algebra $\mathfrak{g}$, although our main interest remains in the cases of type B and D.

Let $\operatorname{Rep}(\mathfrak{g})$ be the category of finite dimensional rational $\mathfrak{g}$-modules. The complexification gives a functor

$$\operatorname{Rep}(\mathfrak{g}) \longrightarrow \operatorname{Rep}(\mathfrak{g}_{\mathbf{C}})$$

to the category of $\mathfrak{g}_{\mathbf{C}}$-modules, which induces an injective morphism

$$K(\mathfrak{g}) \hookrightarrow K(\mathfrak{g}_{\mathbf{C}})$$

at the level of *representation rings*, that is, the Grothendieck rings of the corresponding categories.

The Weyl character of a $\mathfrak{g}_{\mathbf{C}}$-module $V = \bigoplus_\mu V(\mu)$ is given by $\operatorname{ch} V \coloneqq \sum \dim V(\mu) e^\mu$ with value in the group ring $\mathbf{Z}[\Lambda]$, where e^μ is the element corresponding to the weight μ. The character map factors through the representation ring $K(\mathfrak{g}_{\mathbf{C}})$ and has image in $\mathbf{Z}[\Lambda]^{\mathfrak{W}}$, the $\mathfrak{W}$-invariant subring.

Theorem 2.2. *The character map* $\operatorname{ch}\colon K(\mathfrak{g}_{\mathbf{C}}) \to \mathbf{Z}[\Lambda]^{\mathfrak{W}}$ *is a ring isomorphism.*

We describe the Weyl character ring $\mathbf{Z}[\Lambda]^{\mathfrak{W}}$ for $\mathfrak{g}$ of type B_r and D_r.

Proposition 2.3.

(1) When $\mathfrak{g}$ is of type B_r, write $x_i \coloneqq e^{\varepsilon_i}$. Then

$$\mathbf{Z}[\Lambda] = \mathbf{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}].$$

The Weyl group $\mathfrak{W}_{2r+1}$ is isomorphic to $\mathfrak{S}_r \ltimes (\mathbf{Z}/2)^r$, where $\mathfrak{S}_r$ acts as permutations on $x_1, \ldots, x_r$ and the i-th $\mathbf{Z}/2$ acts as $x_i \mapsto x_i^{-1}$.

(2) When $\mathfrak{g}$ is of type D_r, the group ring $\mathbf{Z}[\Lambda]$ is the same as above for B_r, while the Weyl group $\mathfrak{W}_{2r}$ is the index-2 subgroup of $\mathfrak{W}_{2r+1}$ consisting of elements with an even number of non-trivial components in $(\mathbf{Z}/2)^r$.

We have the following result that relates the two.

Proposition 2.4. *Let (V, q) be a rational quadratic space of dimension $\dim V = 2r+1$, and $W \subset V$ a non-degenerate subspace of dimension $\dim W = 2r$. Let $\mathfrak{g} = \mathfrak{so}(V, q)$ and $\mathfrak{m} = \mathfrak{so}(W, q|_W)$. Then the restriction functor $\mathrm{Res}\colon \mathrm{Rep}(\mathfrak{g}) \to \mathrm{Rep}(\mathfrak{m})$ induces an injective morphism for the character rings, and consequently, the (rational) representation rings. We have the following diagram*

$$\begin{tikzcd}
K(\mathfrak{g}) \arrow[r, hook, "\mathrm{Res}"] \arrow[d, hook] & K(\mathfrak{m}) \arrow[d, hook] \\
K(\mathfrak{g}_{\mathbf{C}}) \arrow[r, hook, "\mathrm{Res}"] \arrow[d, "\mathrm{ch}"', "\simeq"] & K(\mathfrak{m}_{\mathbf{C}}) \arrow[d, "\mathrm{ch}"', "\simeq"] \\
\mathbf{Z}[\Lambda]^{\mathfrak{W}_{2r+1}} \arrow[r, hook] & \mathbf{Z}[\Lambda]^{\mathfrak{W}_{2r}}.
\end{tikzcd}$$

In particular, for an arbitrary $\mathfrak{g}$-module, if one can obtain its decomposition as an $\mathfrak{m}$-module via restriction, then its Weyl character is uniquely determined and hence so is its $\mathfrak{g}$-module structure.

Remark 2.5. In the hyperkähler setting, the LLV algebra $\mathfrak{g}$ is of type B_{r+1} or D_{r+1}, and its reduced part $\mathfrak{g}_0'$ is of type B_r or D_r, so the proposition does not apply directly for $\mathfrak{g}_0' \subset \mathfrak{g}$. Instead, in the $\mathrm{K3}^{[n]}$-case, we will take the subalgebra $\mathfrak{m}$ to be $\mathfrak{g}(S)$, the LLV algebra of a K3 surface S.

3. Hodge structures

From this section on, we let $r \coloneqq \lfloor b_2(X)/2 \rfloor$ so $\mathfrak{g}$ is of type B_{r+1} or D_{r+1}, and $\mathfrak{g}_0'$ is of type B_r or D_r. The weights of $\mathfrak{g}$ will be denoted as $\lambda = \lambda_0 \varepsilon_0 + \cdots + \lambda_r \varepsilon_r$, starting from the index 0.

The LLV decomposition is a diffeomorphism invariant, but we can obtain more information using a complex structure. Let $f \in \mathrm{End}(H^*(X, \mathbf{R}))$ be the Weil operator

$$f|_{H^{p,q}(X)} = i(q-p)\,\mathrm{Id}\,.$$

We will use this operator to define Hodge structures on each irreducible component V_λ, and obtain some conditions on the dominant weight λ that can appear.

Proposition 3.1. *We have $f \in (\mathfrak{g}_0')_{\mathbf{R}}$.*

Proof. Denote by I, J, K three complex structures coming from a hyperkähler metric g where I is the complex structure that we are using. We have three Kähler classes $\omega_I = g(I-,-)$, $\omega_J = g(J-,-)$, and $\omega_K = g(K-,-)$, hence three $\mathfrak{sl}_2$-triples (L_I, h, Λ_I), (L_J, h, Λ_J), (L_K, h, Λ_K). These are all operators on $H^*(X, \mathbf{R})$ and lie in $\mathfrak{g}_{\mathbf{R}}$ by construction.

Verbitsky showed that the Weil operator $f = f_I$ for the complex structure I satisfies

$$f_I = -[L_J, \Lambda_K] = -[L_K, \Lambda_J],$$

so $f_I \in (\mathfrak{g}_0)_{\mathbf{R}}$. One may consider Weil operators f_J and f_K for the other two complex structures, and verify that

$$[f_J, f_K] = -2 f_I.$$

So f_I indeed lies in $[(\mathfrak{g}_0)_{\mathbf{R}}, (\mathfrak{g}_0)_{\mathbf{R}}] = (\mathfrak{g}_0')_{\mathbf{R}}$. ☐

Remark 3.2. Recall that the real subalgebra $\mathfrak{g}_g$ generated by the three $\mathfrak{sl}_2$-triples is isomorphic to $\mathfrak{so}(4,1)$: an explicit basis over $\mathbf{R}$ is given by

$$\Lambda_I, \Lambda_J, \Lambda_K, \quad f_I, f_J, f_K, h, \quad L_I, L_J, L_K.$$

In particular, the degree-0 part is generated by h and the three Weil operators.

Under the action of f, the standard representation V decomposes as

$$V = V^{2,0} \oplus V^{1,1} \oplus V^{0,2},$$

where f acts as $-2i$, 0, and $2i$ respectively. Similarly, we have another decomposition under the action of h

$$V = V_{-2} \oplus V_0 \oplus V_2,$$

where h acts as -2, 0, and 2 respectively. Take $\mathfrak{h} \subset \mathfrak{g}_{\mathbf{C}}$ a Cartan subalgebra that contains both h and f, then h and if are among $\pm\varepsilon_i^\vee$. Up to the choice of a Weyl chamber, we may suppose that $h = \varepsilon_0^\vee$ and $if = \varepsilon_1^\vee$. Under this choice, we can also identify $\varepsilon_1, \ldots, \varepsilon_r$ as the weights of $\mathfrak{g}_0'$.

For a $\mathfrak{g}$-module V_λ that appears in $H^*(X, \mathbf{Q})$, we take its weight decomposition with respect to the chosen Cartan subalgebra $\mathfrak{h}$: $(V_\lambda)_{\mathbf{C}} = \bigoplus_\mu V_\lambda(\mu)$, where $V_\lambda(\mu)$ is the component of weight $\mu = \mu_0\varepsilon_0 + \cdots + \mu_r\varepsilon_r$. Then h acts as $2\mu_0$ and if acts as $2\mu_1$ on $V_\lambda(\mu)$

$$\begin{aligned} 2\mu_0 &= p + q - 2n \\ 2\mu_1 &= i \cdot i(q-p) = p - q \end{aligned} \quad \Rightarrow \quad \begin{aligned} p &= \mu_0 + \mu_1 + n \\ q &= \mu_0 - \mu_1 + n \end{aligned} \tag{1}$$

so $V_\lambda(\mu) \subset H^{p,q}(X)$. In other words, $V_\lambda \subset H^*(X, \mathbf{Q})$ is a sub-Hodge structure. More generally, there is a naturally define Hodge structure on any $\mathfrak{g}$-module V_λ, determined by the actions of h and f: we let

$$(V_\lambda)_{\mathbf{C}}^{p,q} \coloneqq \bigoplus_{\mu \text{ satisfying (1)}} V_\lambda(\mu).$$

The Hodge numbers $h^{p,q}$ count the multiplicities of suitable weights. For a given $\mathfrak{g}$-module, these can be easily obtained using `GAP` or `Sage`. On the other hand, the Hodge numbers do not necessarily determine the $\mathfrak{g}$-module structure (such an example will show up in the case of OG6).

Example. The Verbitsky component $SH^2(X, \mathbf{Q})$ contains a non-trivial $H^{2n,0}$-part coming from $\operatorname{Sym}^n H^2(X, \mathbf{Q})$. We have $p = 2n, q = 0$ so $\mu_0 = n, \mu_1 = 0$ which must be the highest weight (n). (In fact this can be used to prove that $SH^2(X, \mathbf{Q}) \simeq V_{(n)}$: we just saw that the highest weight of $SH^2(X, \mathbf{Q})$ dominates (n); on the other hand, we have $\dim SH^2(X, \mathbf{Q}) = \dim V_{(n)}$ due to the description of Verbitsky, so by Lemma 2.1, the highest weight must be exactly (n).)

In particular, since $H^{2n,0}(X)$ is one-dimensional, the component $V_{(n)}$ appears with multiplicity 1 in $H^*(X, \mathbf{Q})$. (It also exhausts all the outermost Hodge numbers $h^{2k,0} = 1$.)

Corollary 3.3.

(1) Each component V_λ of $H^*_{\text{even}}(X, \mathbf{Q})$ has integral highest weight λ;
(2) Each component V_λ of $H^*_{\text{odd}}(X, \mathbf{Q})$ has half-integer highest weight λ;
(3) Each component V_λ other than the Verbitsky component satisfies $\lambda_0 + \lambda_1 \le n-1$ and $\lambda_0 \le n - \frac{3}{2}$.

Proof. For statements (1) and (2), we look at the component $V_\lambda(\lambda)$ and get

$$p + q = 2\lambda_0 + 2n,$$

which allows us to conclude that λ_0 is an integer or a half-integer in the two cases.

For statement (3), since V_λ is not the Verbitsky component, it cannot have a $H^{2n,0}$-part, so by looking at the component $V_\lambda(\lambda)$ we get

$$\lambda_0 + \lambda_1 + n = p \le 2n - 1,$$

which gives the first inequality. By definition, the Verbitsky component exhausts the second cohomology $H^2(X, \mathbf{Q})$ and hence $H^{4n-2}(X, \mathbf{Q})$ by Hodge symmetry, so we also have

$$3 \le p + q = 2\lambda_0 + 2n \le 4n - 3,$$

which gives the second inequality. □

Remark 3.4.

(1) The two inequalities in (3) are tight: for generalized Kummer varieties Kum_n with $n \geq 2$ (whose LLV algebra is of type B_4), we can have the component $V_{(n-\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}$. In fact, the existence of this component is equivalent to the non-vanishing of $b_3(X)$.

(2) When $n = 2$, the statement (3) shows that $\lambda = (2)$ or $\lambda_0 \leq \frac{1}{2}$, so all the possible weights are $(2), (\frac{1}{2}, \cdots, \frac{1}{2}), (0)$. Numerically there are three free variables: the second Betti number $b_2(X)$ and the multiplicities of the latter two components. The half-integer weight component generates $H^*_{\mathrm{odd}}(X, \mathbf{Q})$. Conjecturally, the sum of each weight is bounded by $n = 2$, so the odd cohomologies must vanish entirely when $b_2(X) \geq 8$, which is indeed the case.

(3) When $n = 3$ and V_λ is a component of $H^*_{\mathrm{even}}(X, \mathbf{Q})$ other than $V_{(3)}$, we get $\lambda_0 \leq 1$ so λ is a sequence of ones, and V_λ must be a wedge product $\bigwedge^k V$.[1]

The corollary gives some contraints on the irreducible components that can appear. For O'Grady's 10-dimensional example, this is already enough to determine the full decomposition.

Proposition 3.5. *Let X be a hyperkähler manifold of dimension 10 such that $b_2(X) = 24$, $e(X) = 176904$, and $H^*_{\mathrm{odd}}(X) = 0$. Then we have the following decomposition of $\mathfrak{g}$-modules*

$$H^*(X, \mathbf{Q}) = V_{(5)} \oplus V_{(2,2)}.$$

In particular, O'Grady's example OG10 satisfies these numerical conditions, so we have obtained its LLV decomposition.

Proof. The LLV algebra $\mathfrak{g}$ is of type D_{13}. Write $H^*(X, \mathbf{Q}) = H^*_{\mathrm{even}}(X, \mathbf{Q}) = V_{(5)} \oplus V'$. We have $\dim V' = e(X) - \dim V_{(5)} = 37674$. By using the inequalities in Corollary 3.3 and by considering the dimension bound and Lemma 2.1, the only possible dominant weights that can appear are

$$\{(3), (2,2), (2,1), (2), (1,1,1,1), (1,1,1), (1,1), (1), (0)\}.$$

Each V_λ carries a Hodge structure and therefore has its own Betti numbers. Using Salamon's result on the Betti numbers of a hyperkähler manifold, one may verify that the only possible solution is one copy of $V_{(2,2)}$. □

4. Mumford–Tate algebra

Definition 4.1. Let W be a rational Hodge structure. Let f be the Weil operator

$$f|_{W^{p,q}} = i(q-p)\,\mathrm{Id}\,.$$

The special Mumford–Tate algebra $\mathfrak{m} = \mathfrak{m}(W)$ is the smallest rational subalgebra of $\mathrm{End}(W)$ such that $f \in \mathfrak{m}_{\mathbf{R}}$.

The (full) Mumford–Tate algebra is $\mathfrak{m} \oplus \mathbf{Q}h$ where h is the degree operator $h|_{W^{p,q}} = (p+q)\,\mathrm{Id}$. It coincides with the associated Lie algebra of the Mumford–Tate group of W. (This degree operator differs from the one that we defined earlier, so we need to take a Tate twist $H^*(X, \mathbf{Q})(\dim X)$. But we will not need this notion.)

When W is the cohomology $H^*(X, \mathbf{Q})$ of a hyperkähler manifold X, by Proposition 3.1 we see that $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}_0'$. Conversely, we have the following result.

Proposition 4.2. *For X a very general hyperkähler manifold, the Mumford–Tate algebra $\mathfrak{m}$ is equal to $\mathfrak{g}_0'$.*

Proof. Consider the restriction map

$$\rho\colon\ \mathrm{End}(H^*(X, \mathbf{Q})) \longrightarrow \mathrm{End}(H^2(X, \mathbf{Q})).$$

The Weil operator f_2 on $H^2(X, \mathbf{Q})$ is the restriction of f. Since $\mathfrak{m}$ satisfies $f \in \mathfrak{m}_{\mathbf{R}}$, its restriction $\rho(\mathfrak{m})$ will satisfy $f_2 = \rho(f) \in \rho(\mathfrak{m})_{\mathbf{R}}$. Thus by definition, $\rho(\mathfrak{m})$ contains the special Mumford–Tate algebra $\mathfrak{m}(H^2(X, \mathbf{Q}))$ for the second cohomology. By local Torelli theorem, the latter is equal to $\mathfrak{so}(H^2(X, \mathbf{Q}), q_X) \simeq \mathfrak{g}_0'$ for X very general. So $\rho(\mathfrak{m}) \simeq \mathfrak{g}_0'$, which shows that $\mathfrak{m}$ must coincide with $\mathfrak{g}_0'$. □

Consequently, for a very general X, the decomposition of $H^*(X, \mathbf{Q})$ into $\mathfrak{g}_0'$-modules is the same as decomposition into sub-Hodge structures.

[1] In the article of Sawon on the bound of $b_2(X)$, he wrongly assumed that only $\bigwedge^2 V$ can appear. Note that even if the general conjecture holds, that is, the sum of λ is bounded by 3, we can still have $\bigwedge^3 V$.

Example. For X of K3$^{[2]}$-type, by a dimension count we have $H^*(X,\mathbf{Q}) = V_{(2)}$ as $\mathfrak{g}$-module. Write H for the second cohomology group as a $\mathfrak{g}_0'$-module. Using the description by Verbitsky, we get an isomorphism of $\mathfrak{g}_0'$-modules

$$\begin{aligned} H^*(X,\mathbf{Q}) &= \mathbf{Q} \oplus H \oplus \operatorname{Sym}^2 H \oplus H \oplus \mathbf{Q} \\ &= \mathbf{Q} \oplus H \oplus (H_{(2)} \oplus \mathbf{Q}) \oplus H \oplus \mathbf{Q} \end{aligned},$$

where $H_{(2)}$ is an irreducible $\mathfrak{g}_0'$-module obtained as $\ker(\operatorname{Sym}^2 H \xrightarrow{q_X} \mathbf{Q})$. The 1-dimensional component $\mathbf{Q} \subset H^4(X,\mathbf{Q})$ is generated by the dual of q_X, which is also proportional to $c_2(X)$.

For a Hodge special X, the Mumford–Tate algebra $\mathfrak{m}$ becomes smaller, so $H^*(X,\mathbf{Q})$ may decompose further into smaller components. This is the key idea for determining the LLV decomposition for the other three types of hyperkähler manifolds.

5. K3$^{[n]}$-type

In the K3$^{[n]}$-type case, there is a natural choice of a Hodge special locus: when $X = S^{[n]}$ is actually the Hilbert scheme of a K3 surface S (not necessarily algebraic). We have a decomposition

$$(H^2(X,\mathbf{Q}), q_X) = (H^2(S,\mathbf{Q}), q_S) \oplus \langle -2(n-1)\rangle.$$

So $\mathfrak{g}(S)$ naturally realizes as a subalgebra of $\mathfrak{g} = \mathfrak{g}(X)$, and $\mathfrak{m}(S) = \mathfrak{m}(H^2(S,\mathbf{Q}))$ a subalgebra of $\mathfrak{m} = \mathfrak{m}(H^2(X,\mathbf{Q}))$. We write $W := H^*(S,\mathbf{Q})$, which coincides with the Mukai completion of $H^2(S,\mathbf{Q})$ and is therefore the standard representation for $\mathfrak{g}(S)$. When S is non-algebraic and very general, $\mathfrak{m}(S)$ coincides with $\mathfrak{g}_0'(S) = \mathfrak{so}(H^2(S,\mathbf{Q}), q_S)$ and is of type D_{11}.

$$\begin{tikzcd} \mathfrak{g} & \mathfrak{g}_0' \arrow[l, hook'] \\ \mathfrak{g}(S) \arrow[u, hook] & \mathfrak{g}_0'(S) = \mathfrak{m}(S) \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

The Hodge structure on $H^*(S^{[2]},\mathbf{Q})$ is described by Göttsche–Soergel [GS] (stated for algebraic ones only; the general case is due to de Cataldo–Migliorini).

Theorem 5.1. *Let S be a K3 surface, not necessarily algebraic. We have an isomorphism of Hodge structures*

$$H^*(S^{[n]},\mathbf{Q})(n) \simeq \bigoplus_{\alpha\vdash n} H^*(S^{(a_1)} \times \cdots \times S^{(a_n)},\mathbf{Q})(a_1 + \cdots + a_n).$$

The sum is taken over all partitions α of n, where $\alpha = (a_1,\cdots,a_n)$ satisfies $a_1 \cdot 1 + \cdots + a_n \cdot n = n$. Here $S^{(a)}$ denotes the a-th symmetric power $S^a/\mathfrak{S}_a$ of S, and we have an isomorphism of Hodge structures

$$H^*(S^{(a)},\mathbf{Q}) \simeq \operatorname{Sym}^a H^*(S,\mathbf{Q}).$$

Remark 5.2. We can omit all the Tate twists by considering the grading h on the cohomologies centered at the middle cohomology.

In other words, we have obtained the decomposition of $H^*(X,\mathbf{Q})$ as an $\mathfrak{m}(S)$-module. To deduce the $\mathfrak{g}$-module structure, we first lift this as a $\mathfrak{g}(S)$-module decomposition, and then apply Proposition 2.4.

Theorem 5.3. *We have an isomorphism*

$$H^*(S^{[n]},\mathbf{Q}) \simeq \bigoplus_{\alpha\vdash n} \bigotimes_{i=1}^{n} \operatorname{Sym}^{a_i} H^*(S,\mathbf{Q})$$

of $\mathfrak{g}(S)$-modules. Consequently, the Weyl character of $H^*(S^{[n]},\mathbf{Q})$ as a $\mathfrak{g}(S)$-module is equal to

$$\operatorname{ch} H^*(S^{[n]},\mathbf{Q}) = \sum_{\alpha\vdash n} \prod_{i=1}^{n} \operatorname{ch} \operatorname{Sym}^{a_i} W.$$

In view of Proposition 2.4, this gives the Weyl character of $H^*(X,\mathbf{Q})$ as a $\mathfrak{g}$-module.

Proof. The $\mathfrak{g}(S)$-module structure is a diffeomorphism invariant, so we may assume that S is very general and non-algebraic. Recall that in this case, the special Mumford–Tate algebra $\mathfrak{m}(S)$ coincides with $\mathfrak{g}_0'(S) = \mathfrak{so}(H^2(S,\mathbf{Q}), q_S)$. So the isomorphism of Hodge structures gives an isomorphism of $\mathfrak{g}_0'(S)$-modules.

Since $\mathfrak{g}_0(S) = \mathfrak{g}_0'(S) \oplus \mathbf{Q}h$ and the decomposition respects the grading h, we can lift it to an isomorphism of $\mathfrak{g}_0(S)$-modules. The weight lattice of $\mathfrak{g}_0(S)$ is the same as that of $\mathfrak{g}(S)$, so this is an isomorphism of $\mathfrak{g}(S)$-modules. □

Example. We consider again the $\mathrm{K3}^{[2]}$-type case. The isomorphism is given as

$$H^*(S^{[2]},\mathbf{Q}) \simeq H^*(S^{(2)} \times S^{(0)},\mathbf{Q}) \oplus H^*(S^{(0)} \times S^{(1)},\mathbf{Q}) = \mathrm{Sym}^2 H^*(S,\mathbf{Q}) \oplus H^*(S,\mathbf{Q}).$$

The right hand side decomposes into 3 irreducible $\mathfrak{g}(S)$-modules, and further into 10 irreducible $\mathfrak{g}_0'(S)$-modules.

as $\mathfrak{g}$-module (B_{12}) | as $\mathfrak{g}_0'$-module (B_{11}) (= $\mathfrak{m}(X)$ for X very general)

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & 21 & 1 & \\ 1 & 21 & 232 & 21 & 1 \\ & 1 & 21 & 1 & \\ & & 1 & & \end{array} \xrightarrow{\mathrm{Res}} \begin{array}{c} \boxed{1} \\ \boxed{1\ \ 21\ \ 1} \\ \boxed{1\ \ 21\ \ 231\ \ 21\ \ 1} \\ \boxed{1\ \ 21\ \ 1} \\ \boxed{1} \end{array} \oplus \boxed{1}$$

Hodge special ($X = S^{[2]}$) $\quad \Big\downarrow \mathrm{Res} \quad \Big\downarrow \mathrm{Res}$

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & 20 & 1 & \\ 1 & 20 & 211 & 20 & 1 \\ & 1 & 20 & 1 & \\ & & 1 & & \end{array} \oplus \begin{array}{ccc} & 1 & \\ 1 & 20 & 1 \\ & 1 & \end{array} \oplus 1 \xrightarrow{\mathrm{Res}} \begin{array}{c} \boxed{1} \\ \boxed{1\ \ 20\ \ 1} \\ \boxed{1\ \ 20\ \ 210\ \ 20\ \ 1} \\ \boxed{1\ \ 20\ \ 1} \\ \boxed{1} \end{array} \begin{array}{c} \\ \oplus \\ \oplus \\ \oplus \\ \\ \end{array} \boxed{1} \oplus \begin{array}{c} \boxed{1} \\ \boxed{1\ \ 20\ \ 1} \\ \boxed{1} \end{array} \oplus \boxed{1}$$

as $\mathfrak{g}(S)$-module (D_{12}) | as $\mathfrak{g}_0'(S)$-module (D_{11}) (= $\mathfrak{m}(S)$ for S very general)

We may write the formula for the characters of $H^*(\mathrm{K3}^{[n]},\mathbf{Q})$ in a more succinct fashion by considering all Hilbert powers at the same time. Note that the LLV algebras are a priori not the same in different dimensions. But since we are considering Weyl characters, we only need the complexification $\mathfrak{g}_{\mathbf{C}}$ which is always isomorphic to $\mathfrak{so}(25)$.

Proposition 5.4. *Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{so}(25)$. The generating series of the characters of the $\mathfrak{g}$-modules $H^*(\mathrm{K3}^{[n]})$ for $n \geq 2$ is given by*

$$\sum_{n=0}^{\infty} \mathrm{ch}\, H^*(\mathrm{K3}^{[n]}) q^n = \prod_{n=1}^{\infty} \prod_{i=0}^{11} \frac{1}{(1 - x_i q^n)(1 - x_i^{-1} q^n)}. \tag{2}$$

The identity lives inside the formal power series ring $A[[q]]$ where

$$A \coloneqq \mathbf{Z}[\Lambda]^{\mathfrak{W}} = \mathbf{Z}[x_0^{\pm 1}, \dots, x_{11}^{\pm 1}, (x_0 \cdots x_{11})^{\pm \frac{1}{2}}]^{\mathfrak{W}_{25}}$$

is the Weyl character ring of type B_{12}. Note that when $n = 1$, the cohomology $H^*(\mathrm{K3})$ does not admit a structure of $\mathfrak{g}$-module, so we write formally

$$\mathrm{ch}\, H^*(\mathrm{K3}^{[1]}) \coloneqq \sum_{i=0}^{11} (x_i + x_i^{-1}).$$

Proof. By Theorem 5.3 and Proposition 2.4, it suffices to take $X = S^{[n]}$ and show the identity in the character ring of $\mathfrak{g}(S)$. We write s_a for $\operatorname{ch}\operatorname{Sym}^a W$, where $W = H^*(S, \mathbf{Q})$ is the standard representation of $\mathfrak{g}(S)$. Then

$$\begin{aligned}\sum_{n=0}^{\infty}\operatorname{ch} H^*(S^{[n]})q^n &= \sum_{n=0}^{\infty}\left(\sum_{\alpha\vdash n}\prod_{i=1}^{n}\operatorname{ch}\operatorname{Sym}^{a_i} W\right)q^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{\alpha\vdash n} s_{a_1}s_{a_2}\cdots s_{a_n}\right)q^n \\ &= \sum_{n=0}^{\infty}\sum_{\alpha\vdash n} s_{a_1}q^{a_1}\cdot s_{a_2}q^{2a_2}\cdot\ \cdots\ \cdot s_{a_n}q^{na_n} \\ &= \sum_{\alpha} s_{a_1}q^{a_1}\cdot s_{a_2}q^{2a_2}\cdot\ \cdots\ \cdot s_{a_n}q^{na_n}\end{aligned}$$

where the last sum is over all partitions of integers. Equivalently, we can freely pick the values for each a_i and recover all the partitions in this way. So

$$\sum_{\alpha} s_{a_1}q^{a_1}\cdot s_{a_2}q^{2a_2}\cdot\ \cdots\ \cdot s_{a_n}q^{na_n} = \left(\sum_{a_1=0}^{\infty} s_{a_1}q^{a_1}\right)\cdot\left(\sum_{a_2=0}^{\infty} s_{a_2}q^{2a_2}\right)\cdot\ \cdots = \prod_{n=1}^{\infty}\sum_{a=0}^{\infty} s_a q^{na} = \prod_{n=1}^{\infty} A(q^n)$$

where $A(q) = \sum_{a=0}^{\infty} s_a q^a$. Finally, we have

$$\begin{aligned}A(q) &= \sum_{a=0}^{\infty} s_a q^a \\ &= 1 + \operatorname{ch} W\cdot q + \operatorname{ch}\operatorname{Sym}^2 W\cdot q^2 + \cdots \\ &= 1 + (x_0 + \cdots + x_{11} + x_0^{-1} + \cdots + x_{11}^{-1})q + (x_0^2 + x_0\cdot x_1 + \cdots + x_{11}^{-2})q^2 + \cdots \\ &= \prod_{i=0}^{11}(1 + x_i q + x_i^2 q^2 + \cdots)(1 + x_i^{-1}q + x_i^{-2}q^2 + \cdots) \\ &= \prod_{i=0}^{11}\frac{1}{(1-x_i q)(1-x_i^{-1}q)},\end{aligned}$$

which concludes the proof. ☐

Corollary 5.5. *Let X be a hyperkähler manifold of* $\mathrm{K3}^{[n]}$*-type. Any irreducible component V_λ of the LLV decomposition of $H^*(X, \mathbf{Q})$ with highest weight $\lambda = \lambda_0\varepsilon_0 + \cdots + \lambda_{11}\varepsilon_{11}$ satisfies*

$$\lambda_0 + \cdots + \lambda_{11} \le n.$$

Proof. The weight λ corresponds to the monomial $x_0^{\lambda_0}\cdots x_{11}^{\lambda_{11}}$ in the character ring. When we expand the right hand side of (2) we get

$$\prod_{n=1}^{\infty}\prod_{i=0}^{11}\left(\sum_{j\ge 0}(x_i q^n)^j\right)\left(\sum_{k\ge 0}(x_i^{-1}q^n)^k\right).$$

For each term of this product, its degree in x_i is bounded by its degree in q. So each monomial that appears in the coefficient of q^n has degree $\le n$, which gives the inequality. ☐

Remark 5.6. More generally, for a hyperkähler manifold X of dimension $2n$ with $r = \lfloor b_2(X)/2\rfloor$, it is conjectured that we have the inequality

$$\lambda_0 + \cdots + \lambda_{r-1} + |\lambda_r| \le n$$

for each component V_λ of the LLV decomposition of $H^*(X, \mathbf{Q})$. This holds for all known examples.

Remark 5.7. Once the character of the $\mathfrak{g}$-module structure is known, one can use computer algebra to recover the actual decomposition. One implementation in `Sage` can be found here.

6. Generalized Kummer varieties and OG6

We briefly remark on the remaining two cases. See [GKLR] for details and the references therein.

Generalized Kummer varieties. The LLV algebra $\mathfrak{g}$ is of type B_4.

Similar to the case of $\mathrm{K3}^{[n]}$-type, we consider Hodge special members of the family: we specialize X to an actual generalized Kummer variety associated to a very general complex torus A of dimension 2. The results of Göttsche–Soergel give a complete description of the Hodge structure of $H^*(X)$ in terms of the Hodge structures on $H^*(A)$, which can be seen as a decomposition of $\mathfrak{m}(A)$-modules (of type D_3). We can similarly lift it to a $\mathfrak{g}(A)$-module decomposition (of type D_4) and apply Proposition 2.4 to obtain the character of $H^*(X)$ as a $\mathfrak{g}$-module.

OG6. This last case is more complicated. The LLV algebra $\mathfrak{g}$ is of type D_5.

Using the Hodge numbers of OG6 and the Hodge numbers of the $\mathfrak{g}$-modules, we may obtain two possible decompositions for $H^*(X, \mathbf{Q})$. To determine which case we are in, we specialize X to a Hodge special member with an explicit geometric construction given by Rapagnetta. In this situation, the Mumford–Tate algebra $\mathfrak{m}$ is of type B_2 (that of a very general abelian surface A), and the geometric construction gives a description of the Hodge structure of $H^*(X)$ in terms of $\mathfrak{m} = \mathfrak{m}(A)$-modules (Mongardi–Rapagnetta–Saccà). Then by comparing the restrictions to $\mathfrak{m}$ of the two possible $\mathfrak{g}$-module decompositions, only one agrees with the $\mathfrak{m}$-module decomposition obtained from geometry, so we may conclude.

References

[FH] Fulton, W., Harris, J., *Representation theory*, Graduate Texts in Mathematics **129**, Springer-Verlag, New York, 1991.

[GKLR] Green, M., Kim, Y., Laza, R., Robles, C., The LLV decomposition of hyper-Kähler cohomology, (2019), eprint `arXiv:1906.03432`. .

[GS] Göttsche, L., Soergel, W., Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces, *Math. Ann.* **296** (1993), 235–245.

[V] Verbitsky, M., Action of the Lie algebra of SO(5) on the cohomology of a hyper-Kähler manifold, *Funktsional. Anal. i Prilozhen.* **24** (1990), 70–71.