Plan for the seminar Algebraic Geometry. Summer term 2010

All talks in the seminar are devoted to central results in algebraic geometry but most of them are usually not taught in standard courses (at least not during the first two semesters). Each topic will require more preparation than a usual seminar talk. Some of the talks will take longer than 90 min and usually the speaker will only cover parts of what he has learned while preparing it. Each of the topics is worth the effort.

The talks are only loosely connected to each other, they rather stand alone. The choice of the topics is dictated by my course on K3 surfaces. All topics treated in the seminar will come up again in the course, often in a more special form, namely applied to K3 surfaces.

Prerequisites for the seminar is a solid background in algebraic geometry, e.g. Hartshorne's book. The seminar is recommended for all students attending my class, but everybody else attracted by these topics is very welcome. **Please get in contact** with me (well in advance) if you are interested to give one of the talks. The order of the talks is subject to changes.

1. Riemann–Roch for surfaces

The talk should start with the simple and nice proof of Mumford for the Riemann–Roch theorem for line bundles on surfaces. See Lecture 12 in:

Mumford, David *Lectures on curves on an algebraic surface* Princeton (1966).

The rest of the talk has two parts. Firstly, the formulation of the Hirzebruch–Riemann–Roch theorem in general. Secondly (and this is the main part), its proof for surfaces.

For the first part one may follow Appendix A, 2-4 in Hartshorne. At this point we may work with the much easier Chow groups modulo numerical equivalence. Introduce Chern and Todd classes via the splitting principle. State the Hirzebruch–Riemann–Roch theorem and deduce Noether's formula from it $\chi(\mathcal{O}_X) = (1/12)(c_1^2 + c_2)$.

Then pass to the case of surfaces. The reference here is:

Kurke, Herbert *Vorlesungen über algebraische Flächen*, Teubner Texte zur Mathematik (1982).

Discuss the ad hoc definition of the second Chern class given there and compare it to the general one. Prove the functoriality properties of the ad hoc definition. In this approach the Hirzebruch–Riemann–Roch theorem is proved as a consequence of Noether's formula (p.141-143).

2. The Nakai–Moishezon criterion

This talk is devoted to various characterization of ample line bundles. Clearly, an ample line bundle on a complete scheme has positive degree on each closed curve contained in the scheme. The converse is almost true, which runs under the name Nakai–Moishezon (–Kleiman–Seshadri) criterion. We follow

Hartshorne, Robin *Ample Subvarieties of Algebraic Varieties*, LNM 156 Springer (1979).

Start with the functorial properties of ampleness in Section 1.4. Then pass to Nakai's criterion Theorem 5.1 which says that a line bundle is ample if of positive degree on any closed subscheme. Then pass to Kleiman's result (Theorem 6.1) that non-negativity of the degree on all closed subschemes is equivalent to non-negativity of the degree on all closed curves. Eventually prove Seshadri's version Theorem 7.1 which states that ampleness is equivalent to the degree on each closed curve being a little more positive than 0.

Another recommended reference is

Badescu, Lucian *Algebraic surfaces* Springer Univesitext (2000).

3. GAGA

Serre's GAGA associates to a scheme X of finite type over $\mathbb{C}$ a complex space X^{an} (essentially by forgetting all non-closed points and putting a finer topology on the resulting space). This yields a morphism of ringed spaces $X^{an} \longrightarrow X$. Pulling-back yields a functor from $\mathcal{O}_X$-sheaves on X to sheaves on X^{an}. For complete schemes (or proper morphisms) this commutes with cohomology (resp. higher direct images).

The passage between schemes in algebraic geometry, mostly projective ones over $\mathbb{C}$, and complex manifolds (or rather complex spaces) is often confusing in the beginning, but it is all beautifully explained in Serre's original paper (which is also responsible for the name):

Serre, Jean-Pierre *Géométrie algébrique et géométrie analytique*, Annales de l'institut Fourier, 6 (1956), 1–42.

An account of it should also be contained in the more recent:

Neeman, Ammon *Algebraic and Analytic Geometry*, LMS Lecture Note Series 345 (2007)

The talk should introduce (quickly) the notion of a complex space and outline the proof of the main result in GAGA: $\mathrm{Coh}(X) \simeq \mathrm{Coh}(X^{an})$ for a projective varieties over $\mathbb{C}$.

4. Short introduction to stacks

Moduli spaces of geometric objects (bundles, varieties, etc) often exist, but not necessarily as schemes. In the lecture course last term we have seen examples of representable (as schemes) functors (e.g. Quot and Hilb) but we have also encountered examples where (e.g. the Picard) functor is only representable by an algebraic space. If the objects parametrized by the moduli functor may have additional automorphisms, then we should not seek to represent the functor by a space but by a stack. This takes automatically care of the question whether there is a universal object.

The aim of this talk is to introduce the most basic ideas geared towards the moduli stack of varieties, e.g. curves and K3 surfaces.

There is a short introduction:

Gomez, Tomas *Algebraic Stacks*, arXiv:math/9911199v1.

The classical source is of course (for us only Section 4 would be needed):

Deligne, Pierre; Mumford, David *The irreducibility of the space of curves of given genus*, Publ. Math. IHES 36 (1969), 75–109.

There are two more ambitious ongoing projects. The first is a long-term book project by Behrend, Conrad, Edidin, Fulton, Fantechi, Göttsche, Kresch. Search for it, parts are available already. The other one is an open source text book: ‘The stacks project’ by Johan de Jong et al.

5. Lattice theory

There is a classification theory for indefinite unimodular lattices. This talk should recall the basic theory as explained in Chapter V of:

Serre, Jean-Pierre *A course in arithmetic*, GTM 7 Springer (1973).

The most important result for our purposes is Theorem 5 which we will use to determine the intersection form on the second cohomology of a K3 surface. It is $2(-E_8) \oplus 3U$.

For further applications we will also need a few more results on the embedding of certain (small) lattices into $2(-E_8) \oplus 3U$. The main reference here is:

Nikulin, Viacheslav *Integral symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 111–177.

The talk should provide all the necessary material for the formulation of Theorem 1.14.2 and 1.14.4. In particular a discussion of the discriminant form etc.

6. Frobenius and the formulation of the Weil conjecture

I hope to discuss Deligne’s proof (and/or the one by Piatetski-Shapiro and Shafarevich) of the Weil conjectures for K3 surfaces. In this talk the Weil conjectures should be stated in general. One could follow Appendix C in

Hartshorne, Robin *Algebraic Geometry*, GTM 52 Springer (1977).

The talk should contain a thorough discussion of the Frobenius morphism and the yoga of Zeta functions. We will have to be very brief when it comes to étale cohomology. In particular, only the geometric part of the theory (i.e. over an algebraically closed field) will be of interest to us. The Weil conjectures for curves is not too hard. In particular the Riemann hypothesis for curves uses intersection theory (Riemann–Roch and Hodge index) for surfaces which fits nicely in the rest of the program. (There is a set of very nice notes of talks by Beilinson for this.) (This approach goes back to Mattuck–Tate and Grothendieck.)

7. Bogomolov inequality

Bogomolov’s inequality asserts that for a semi-stable sheaf E (say on a smooth projective surface) X the discriminant $2rc_2(E) - (r-1)^2c_1(E)^2$ is non-negative. The first part of the talk should discuss slope stability and prove the inequality following Section 3.4 in

Huybrechts, Daniel; Lehn, Manfred *The geometry of moduli spaces of sheaves* Vieweg or Cambridge

or Section 4.3 in

Lazarsfeld, Robert *A sampling of vector bundle techniques in the study of linear series*

The second part should contain an account of the relation between 'generic semi-positivity' and 'semi-stability' as in

Miyakoa, Yoichi *The Chern classes and Kodaira dimension of a minimal variety*

The interest for the seminar (and my lecture course) is that Miyaoka's techniques give a purely algebraic proof of the stability of the tangent bundle of a K3 surface