Classical boundedness results and applications

Summer term 2016, Tuesday 2-4pm, 0.011

12 April: Mumford–Castelnuovo regularity and boundedness of semistable sheaves. (Speaker: Ulrike Rieß)
Mumford–Castelnuovo regularity and boundedness of semistable sheaves with given Hilbert polynomial on curves(!). Follow [5, Ch. 1.7]. Special emphasis on Grothendieck's lemma [5, Lem. 1.7.9] essentially saying that the family of quotients of a fixed sheaf with fixed (or just bounded above) slope is bounded. (See [5, Lem. 4.C.5, Thm. 5.2.5] for applications, but there are others.)

26 April: Matsusaka's big theorem [12, 13]. (Speaker: Zhiyuan Li)
The result is that for a fixed numerical polynomial P there exists a k_0 such that for smooth projective polarized varieties (X, L) with Hilbert polynomial $P(X, L) = P$ and $k \geq k_0$ the line bundle L^k is very ample. Follow the account of Lieberman and Mumford in [18] or Raynaud's in [21]. Pay special attention where exactly the smoothness assumption and any assumption on the characteristic being zero come in.

3 May: Kollár–Matsusaka theorem. (Speaker: Wenhao Ou)
The most striking consequence of the KM theorem shows that up to finite ambiguity the Hilbert polynomial $P = P(X, L)$ of a polarized smooth projective variety (X, L) is determined by the two leading coefficients of P. In technical terms, the KM theorem asserts that there exist universal polynomial $Q(d, \xi)(r)$ of degree $< n$ such that for every (X, L) with Hilbert polynomial $P(X, L)(r) = (d/n!)r^n + (\xi/2(n-1)!)r^{n-1} + \ldots$ one has the inequality: $|h^0(X, L^r) - (d/n!)r^n| \leq Q(d, \xi)(r)$. (The notation in [10] is little old fashioned, use instead [9].) There are generalizations of this result to big and nef line bundles [20, 19].

10 May: Refined boundedness of stable vector bundles. (Speaker: Daniel Huybrechts)
The result should be that fixing just $\mathrm{rk}(E)$, $\mathrm{c}_1(E)$ and $\mathrm{c}_2(E)$ of a stable vector bundle E determines all the remaining Chern classes of E up to finite ambiguity (and hence the Hilbert polynomial). This proves the boundedness of this collection. The proof involves the usual restriction theorems [5], which should be discussed at this occasion (possibly without proof), to complete intersection surfaces and a comparison between the Ext-spaces. (There does not seem to be a reference for this.)

24 May: Boundedness for Fanos. (Speaker: Luca Tasin, Enrica Floris)
Following [9, V.2] prove boundedness of smooth Fano varieties in characteristic 0. In particular, sketch the proof of Corollary 2.14.2, which says that for any smooth Fano variety X in characteristic 0, there exists a number d, depending only on the dimension of X, such that any two points of X can be joined by an irreducible rational curve of anticanonical degree at most d. Skip the proof of Corollary 2.14.1 (note that 2.14.1 holds in any characteristic). Discuss unboundedness of Fano 3-folds in positive characteristic [8, Thm 1.4].

31 May: Boundedness of pluricanonical maps. (Speaker: Luca Tasin, Enrica Floris)
The main result is that for any positive integer n there exists an integer r_n such that if X is a smooth projective variety of general type and dimension n, then the map induced by the linear system $|rK_X|$ is birational for any $r > r_n$. See [4].

In particular, this implies that if we fix a polynomial $h \in \mathbb{Q}[t]$, then there exists an integer m such that if X is a canonically polarised variety with canonical singularities and Hilbert polynomial h, then mK_X is very ample (see [3, Thm. 13.6]).

7 June: Finiteness of diffeomorphism types.(Speaker: Andrey Soldatenkov)
This talk should survey the result of Sullivan [23] which shows that the cohomology ring of a compact manifold (of dimension at least five) together with the Pontrjagin classes determine the diffeomorphism type up to finite ambiguity.

21 June: Finiteness of hyperkähler manifolds. (Speaker: Corinne Bedussa, Ulrike Rieß)
There are only finitely many deformation types of compact hyperkähler manifolds with fixed BB form and Fujiki constant, see [6]. See also [2].

28 June: More finiteness results for hyperkähler manifolds.(Speaker: Corinne Bedussa, Ulrike Rieß)
Sawon and Kamenova prove that there exists only finitely many deformation types of Lagrangian fibrations of hyperkähler manifolds with fixed BB form and Fujiki constant, see [7, 22].

12 July: Anti-canonical sections of Fano varieties I. (Speaker: Vlad Lazic)
In a recent paper Birkar [1] proves a version of Matsusaka's big theorem for (singular) Fano varieties, namely that $|-mK_X|$ has a (nice) section for $m \geq m_0$, with m_0 only depending on the dimension.

19 July: Anti-canonical sections of Fano varieties II. (Speaker: Vlad Lazic)

References

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