Moduli Space of Cubic Surfaces

Emanuel Reinecke

Geboren am 18. April 1990 in Braunschweig

23. Juli 2012

Bachelorarbeit Mathematik

Betreuer: Prof. Dr. Daniel Huybrechts

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Modulraum kubischer Flächen

Diese Bachelorarbeit soll einen vollständigen und detaillierten Beweis des folgenden Satzes erarbeiten:

Der Modulraum der kubischen Flächen ist isomorph zum gewichteten projektiven Raum $\mathbb{P}(1, 2, 3, 4, 5)$.

Zu Anfang wollen wir uns einige elementare Grundlagen der Geometrischen Invariantentheorie aneignen, auf denen die Beweise aufbauen. Nach der Behandlung weiterer Voraussetzungen, namentlich der Sylvesterform einer kubischen Fläche und des Begriffes des gewichteten projektiven Raumes, widmen wir uns dem längsten und mühsamsten Teil des Beweises, der Berechnung der Invarianten der kubischen Flächen und ihrer Vollständigkeit. Die Arbeit wird abgeschlossen durch einige Bemerkungen über die aus dem Satz resultierende Korrespondenz von kubischen Flächen und dem gewichteten projektiven Raum. Die theoretischen Grundlagen werden nach Bedarf entwickelt und Beispiele immer in Hinblick auf ihre Anwendung im Spezialfall der kubischen Flächen gegeben.

Contents

1 Introduction **1**

2 Basic results from Geometric Invariant Theory **1**

3 Stable and semistable points in the space of cubic surfaces **7**

4 The Sylvester form of the homogeneous cubic polynomials **13**

5 The weighted projective space **15**

6 The invariants of the cubic forms on $\mathbb{P}^3$ **19**

7 Some special points and hypersurfaces **46**

1 Introduction

This Bachelor's thesis should give a complete and detailed proof of the following theorem known from classical algebraic geometry:

Theorem 1.1. *The moduli space of cubic surfaces $\mathcal{M}$ is isomorphic to the weighted projective space* $\mathbb{P}(1,2,3,4,5)$.

To begin with, we shall recapitulate some basic facts from Geometric Invariant Theory which are built upon in the proofs. Having provided some additional prerequisites, namely the Sylvester form of a cubic surface and the notion of a weighted projective space, we shall proceed to the longest and most tedious part of the proof, the computation of the invariants of the cubic surfaces. The article will conclude with some remarks on the correspondence of the space of cubic surfaces and the weighted projective space resulting from the aforementioned theorem. The theory will be developed as needed and examples will always be given with regard to their application in the special case of cubic surfaces.

Let me start by giving an overview over the notions used in the proof.

We investigate the space of cubic surfaces on $\mathbb{P}^3$ and the natural action of the special linear group $SL(4)$ on it which identifies projectively equivalent orbits. The quotient map from the space of cubic surfaces onto the set of orbits creates difficulties as we would like the map to be continuous and the quotient to be Hausdorff but the orbits under $SL(4)$ are in general not closed, so a good quotient does not always exist. One therefore considers two types of quotients of invariant subvarieties of the space of cubic surfaces, and we shall see shortly that there are sufficiently large subsets, the semi-stable points and the stable points, on which these quotients exist.

The theorem to be proven in this thesis says now that one of these quotients is isomorphic to a generalization of the projective space, called weighted projective space, where $\mathbb{C}^\times$ acts with different weights.

The main part of the proof will involve finding all invariant polynomials on the cubic surfaces, i.e. the polynomials in the coefficients of the cubic polynomials defining the surfaces which are constant on orbits. This has already been done in the early 20th century by the Irish theologian and mathematician G. Salmon. However, his proofs are quite geometrical and not fully exact. The first such complete proof has been given by the Russian mathematician N. Beklemishev. His proof on the other hand is very concise, and it will therefore be the aim of this thesis to treat this matter with more detail, which allows us to correct some minor mistakes.

The author thanks his adviser D. Huybrechts for the statement of the problem and his continued support.

2 Basic results from Geometric Invariant Theory

In order to comprehend the concepts used in the proofs, we need to become acquainted with dual versions of some notions of representation theory which are suited better for the use in algebraic geometry, taken from [Muk03, Ch. 3.3(b), Ch. 4.1]. For the rest of this section, let k denote an algebraically closed field of characteristic 0.

We start by recalling

Definition 2.1. An *affine algebraic group* is an affine scheme $G = \operatorname{Spec} A$, where A is a finitely generated k-algebra, with k-algebra homomorphisms $\mu\colon A \to A \otimes_k A$ (comultiplication), $e\colon A \to k$ (coidentity) and $\iota\colon A \to A$ (coinverse) satisfying

(Ass) The following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[d, "\mu"] & A \otimes_k A \arrow[d, "\mathrm{id}_A \otimes \mu"] \\ A \otimes_k A \arrow[r, "\mu \otimes \mathrm{id}_A"] & A \otimes_k A \otimes_k A \end{tikzcd}$$

(Id) The composition $A \xrightarrow{\mu} A \otimes_k A \xrightarrow{e \otimes \mathrm{id}_A} k \otimes_k A \to A$ is the identity map on A.

(Inv) The composition $A \xrightarrow{\mu} A \otimes_k A \xrightarrow{\iota \otimes \mathrm{id}_A} A \otimes_k A \xrightarrow{m} A$ is the equal to $j \circ e$, where m is the multiplication in the k-algebra A and $j\colon k \to A$ is the canonical inclusion.

One can see easily that these axioms just correspond to the usual associativity, existence of an identity element and of an inverse postulated for an ordinary group.

As examples, consider

Definition 2.2. The algebraic group $\mathbb{G}_\mathrm{m} := \operatorname{Spec}\left(k\left[X, X^{-1}\right]\right)$ together with the comultiplication $\mu\colon k\left[X, X^{-1}\right] \to k\left[X, X^{-1}\right] \otimes k\left[X, X^{-1}\right],\ X \mapsto X \otimes X$, the coidentity $e\colon k\left[X, X^{-1}\right] \to k,\ X \mapsto 1$ and the coinversion $\iota\colon k\left[X, X^{-1}\right] \to k\left[X, X^{-1}\right],\ X \mapsto X^{-1}$ is called the *one-dimensional algebraic torus*.

Definition 2.3. The algebraic group $SL(n,k) := \operatorname{Spec}\left(\left(k\left[X_{ij}\right]\right) / \left(\det(X) - 1\right)\right),\ 1 \leq i, j \leq n$ together with the comultiplication

$$\mu\colon k\left[X_{ij}\right] / \left(\det(X) - 1\right) \to \left(k\left[X_{ij}\right] / \left(\det(X) - 1\right)\right) \otimes_k \left(k\left[X_{ij}\right] / \left(\det(X) - 1\right)\right)$$
$$X_{ij} \mapsto \sum_{l=1}^{n} X_{il} \otimes X_{lj},$$

the coidentity $e\colon k\left[X_{ij}\right] / \left(\det(X) - 1\right) \to k,\ X_{ij} \mapsto \delta_{ij}$ and the coinversion

$$\iota\colon k\left[X_{ij}\right] / \left(\det(X) - 1\right) \to k\left[X_{ij}\right] / \left(\det(X) - 1\right),\ X_{ij} \mapsto \left(\operatorname{adj}(X)\right)_{ij},$$

where $\det(X)$ and $\operatorname{adj}(X)$ denote the determinant and the adjugate matrix of the matrix $X = \left(X_{ij}\right)_{1 \leq i,j \leq n}$ respectively, is called the *special linear group*.

Again, one checks easily that $\mathbb{G}_\mathrm{m}$ and $SL(n,k)$ really are algebraic groups and that their underlying topological spaces are just the multiplicative group $k^\times$ resp. the special linear group known from linear algebra.

Definition 2.4. An *action* of an affine algebraic group $G = \operatorname{Spec} A$ as in Definition 2.1 on an affine variety $X = \operatorname{Spec} R$ is a morphism $a\colon G \times X \to X$ determined by a k-algebra homomorphism $\alpha\colon R \to R \otimes_k A$ (coaction) satisfying

(Ass) The following diagram commutes.

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\alpha"] & R \otimes_k A \arrow[d, "\mathrm{id}_R \otimes \mu"] \\ R \otimes_k A \arrow[r, "\alpha \otimes \mathrm{id}_A"] & R \otimes_k A \otimes_k A \end{tikzcd}$$

(Id) The composition $R \xrightarrow{\alpha} R \otimes_k A \xrightarrow{\mathrm{id}_R \otimes e} R \otimes_k k \cong R$ is the identity map on R.

Once more, this corresponds to the usual axioms for associativity and the identity element of a group action. We use the notation $g.x := a(g, x)$ for $g \in G$ and $x \in X$. The group action is generalized to

Definition 2.5. An *(algebraic) representation* of an affine algebraic group $G = \operatorname{Spec} A$ as in Definition 2.1 is a pair (V, ρ), where V is a k-vector space and $\rho\colon V \to V \otimes_k A$ a k-vector space homomorphism satisfying

(Ass) The following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"] & V \otimes_k A \arrow[d, "\mathrm{id}_V \otimes \mu"] \\ V \otimes_k A \arrow[r, "\rho \otimes \mathrm{id}_A"] & V \otimes_k A \otimes_k A \end{tikzcd}$$

(Id) The composition $V \xrightarrow{\rho} V \otimes_k A \xrightarrow{\mathrm{id}_V \otimes e} V \otimes_k k \cong V$ is the identity map on R.

In particular, every group action of an algebraic group $G = \operatorname{Spec} A$ on an affine scheme $X = \operatorname{Spec} R$ gives rise to the representation (R, α) since every k-algebra is naturally a k-vector space. In [Muk03, Rem. 4.3] it is checked that this definition of a representation is equivalent to the usual one known from representation theory. Beware that under this equivalence a representation in the above sense corresponds to a linear action of G on the dual space V^*.

Let us return to invariant theory.

Definition 2.6. Let (W, ρ) be a representation of an algebraic group $G = \operatorname{Spec} A$. Then a vector $w \in W$ is called *G-invariant* if $\rho(w) = w \otimes 1$. The subspace of G-invariant vectors is denoted by W^G.

If the representation actually comes from a group action of G on some $X = \operatorname{Spec} R$, the vector space R is in addition a ring and R^G a subring of R.

In the case $R = k[X_1, \dots, X_n]$, there already is another notion of invariants from Classical Invariant Theory. Specifically, an $f \in R$ is an invariant in the classical sense, if $f(a(g, p)) = f(p)$ for all closed points (g, p) in the affine scheme $G \times X$. Here a denotes the group action $a\colon G \times X \to X$ corresponding to the coaction homomorphism $\alpha\colon R \to R \otimes_k A$. One sees in the following way that both notions coincide.

Lemma 2.7. *Let G, X and a be as above. Then for all $f \in R = \mathcal{O}(X)$ the equivalence*

$$f \in R^G \Leftrightarrow f(a(g,p)) = f(p) \text{ for all closed points } (g,p) \in G \times X$$

holds.

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha X^\alpha \in R = \mathcal{O}(X)$ which can also be seen as a morphism $X \to \mathbb{A}^1_k$ given by $\varphi\colon k[T] \to R$, $T \mapsto f$.

By definition, f is G-invariant if and only if the diagram

$$\begin{tikzcd} A \otimes_k k[X_1, \dots, X_n] & k[X_1, \dots, X_n] \arrow[l, "\alpha"'] \\ k[X_1, \dots, X_n] \arrow[u, "1 \otimes \mathrm{id}"] & k[T] \arrow[l, "\varphi"] \arrow[u, "\varphi"'] \end{tikzcd}$$

is commutative. But since the $A \mapsto \operatorname{Spec}(A)$ gives an equivalence between the category of commutative rings and the category of affine schemes, the commutativity of this diagram is in turn equivalent to the commutativity of

$$\begin{tikzcd} G \times X \arrow[r, "a"] \arrow[d, "p_2"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & \mathbb{A}^1_k \end{tikzcd},$$

where p_2 is the second projection. This just means $f(a(g,p)) = f(p)$ for all closed points $(g,p) \in G \times X$, which is the classical notion of an invariant. $\square$

Thus, from now on we shall use both definitions of invariants under a group action interchangeably.

Definition 2.8. An algebraic group $G = \operatorname{Spec} A$ is called *linearly reductive*, if for every finite-dimensional representation W of G and every non-trivial G-invariant linear form $l\colon W \to k$, there is a $w \in W^G$ with $l(w) \neq 0$.

There are many different, equivalent definitions for linearly reductive groups, e.g. [Muk03, Def. 4.36, Prop. 4.37, Lem. 4.74].

Proposition 2.9. *Every finite group as well as the groups $\mathbb{G}_{\mathrm{m}}$ and $SL(n)$, $n \in \mathbb{N}^*$ are linearly reductive.*

Proof. [Muk03, Prop. 4.38, Prop. 4.41 and Thm. 4.43]. $\square$

In the literature one may also come across geometrically reductive groups, but, at least in characteristic zero, every linearly reductive group is already geometrically reductive, so all theorems about the latter carry over.

We can now come to the definition of quotients under group actions.

Definition 2.10. Let G be an algebraic group acting on a variety X.

(i) A *categorical quotient* is a G-invariant morphism $\pi\colon X \to Y =: X/\!\!/G$ which satisfies the universal property for G-invariant morphisms, i.e. for any G-invariant morphism $g\colon X \to Z$ there is a unique morphism $\bar{g}\colon Y \to Z$ such that $g = \bar{g} \circ \pi$.

(ii) A *geometric quotient* is a categorical quotient such that $\Psi\colon G\times X \to X\times X$, $(g,x) \mapsto (g.x, x)$ satisfies $\operatorname{im}\Psi = X \times_Y X$.

One can check that categorical and geometric quotient are unique up to isomorphism.

Theorem 2.11. *Let G be a linearly reductive group which acts on an affine variety $X = \operatorname{Spec}(R)$. Then the categorical quotient of X with respect to the action of G exists and is of the form $X/\!\!/G = \operatorname{Spec}\left(R^G\right)$.*

Proof. [Dol03, Thm. 6.1]. □

Remark 2.12. This result actually allows us to compute a wide range of quotients under the action of a linearly reductive group since the universal property defining the categorical quotient is local. Hence in order to check that a map $p\colon X \to Y$ is a categorical quotient, it suffices to do so for all restrictions $p_i\colon p^{-1}(U_i) \to U_i$ where $(U_i)_i$ is an open covering of Y, ideally such that the $p^{-1}(U_i)$ are affine, cf. also [MFK94, Ch. 1, §2, Rem. (5)].

Corollary 2.13. *Let G be a linearly reductive group which acts on a normal affine variety $X = \operatorname{Spec} R$. Then $X/\!\!/G$ is also a normal affine variety.*

Proof. By Theorem 2.11, we know that $X/\!\!/G$ is an affine variety of the form $\operatorname{Spec}\left(R^G\right)$. Recall that an affine variety is normal if and only if its coordinate ring is normal, i.e. an integral domain which is integrally closed in its quotient field. Hence, we have to show that R^G is integrally closed in $\operatorname{Quot}\left(R^G\right)$. First note that $\operatorname{Quot}\left(R^G\right) \subseteq (\operatorname{Quot}(R))^G$. Let now $q \in \operatorname{Quot}\left(R^G\right)$ and $n \in \mathbb{N}^*$, $r_0, \ldots, r_{n-1} \in R^G$ such that $q^n + r_{n-1}q^{n-1} + \ldots + r_1 q + r_0 = 0$. As R is normal by assumption, we have $q \in R \cap (\operatorname{Quot}(R))^G = R^G$. Thus, R^G is normal. □

In particular, the theorem shows that the coordinate ring of the quotient $X/\!\!/G$ of an affine variety $X = \operatorname{Spec} R$ under the action of a linearly reductive group G is $\mathcal{O}(X/\!\!/G) = R^G = (\mathcal{O}(X))^G$. If G is finite (cf. Proposition 2.9), one has even

Lemma 2.14. *Let G be a finite algebraic group which acts on an affine variety $X = \operatorname{Spec} R$. Then the function field of $X/\!\!/G$ is given by $K(X/\!\!/G) = (K(X))^G$.*

Proof. Since

$$\begin{aligned} K(X/\!\!/G) &= \operatorname{Quot}(\mathcal{O}(X/\!\!/G)) = \operatorname{Quot}\left(R^G\right) \quad \text{and} \\ (K(X))^G &= (\operatorname{Quot}(\mathcal{O}(X)))^G = (\operatorname{Quot}(R))^G, \end{aligned}$$

we have to show that $\operatorname{Quot}\left(R^G\right) = (\operatorname{Quot}(R))^G$. Clearly, we have $\operatorname{Quot}\left(R^G\right) \subseteq (\operatorname{Quot}(R))^G$. Let now $\frac{f_1}{f_2} \in (\operatorname{Quot}(R))^G$. Since G is finite, we get the equality $\frac{f_1}{f_2} =$

$\frac{f_1 \cdot \prod_{g\in G\smallsetminus\{e\}} g.f_2}{\prod_{g\in G} g.f_2}$, where e denotes the identity element of G. But now $\prod_{g\in G} g.f_2 \in R^G$ and $\frac{f_1}{f_2} \in (\mathrm{Quot}\,(R))^G$ by assumption, so

$$f_1 \cdot \prod_{g\in G\smallsetminus\{e\}} g.f_2 = \frac{f_1}{f_2} \cdot \prod_{g\in G} g.f_2 \in (\mathrm{Quot}\,(R))^G \cap R = R^G$$

is G-invariant as well, hence $\frac{f_1}{f_2} \in \mathrm{Quot}\left(R^G\right)$. □

It is not yet clear whether there is a similar concrete interpretation of general categorical and geometric quotients. This turns out to be the case for all quotients which satisfy some additional conditions.

Definition 2.15. Let G be an algebraic group acting on a variety X. A G-invariant morphism $p\colon X \to Y$ to another variety Y is called a *good categorical quotient* if it satisfies

(i) For all $U \subseteq Y$ open, the corresponding ring homomorphism $\mathcal{O}\,(U) \to \mathcal{O}\left(p^{-1}\,(U)\right)$ is isomorphic onto the subring $\mathcal{O}\left(p^{-1}\,(U)\right)^G$.

(ii) For all $W \subseteq X$ closed and G-invariant, the image $p\,(W)$ under p is a closed subset of Y.

(iii) For all $W_1, W_2 \subseteq X$ closed and G-invariant, $W_1 \cap W_2 = \varnothing$ implies $p\,(W_1) \cap p\,(W_2) = \varnothing$.

It is called a *good geometric quotient* if $\Psi\colon G \times X \to X \times X$, $(g,x) \mapsto (g.x, x)$ fulfils the additional requirement

(iv) $\mathrm{im}\,\Psi = X \times_Y X$

as above.

Proposition 2.16. *A good categorical quotient is a categorical quotient.*

Proof. [Dol03, Prop. 6.2]. □

Corollary 2.17. *A good geometric quotient is a geometric quotient.*

All categorical quotients treated in this thesis will actually be good categorical quotients.

Lemma 2.18. *Let G be an algebraic group acting on a variety X and $p\colon X \to Y$ be a good categorical quotient. Then p is surjective.*

Proof. By property (i), the corresponding ring homomorphism $\mathcal{O}\,(U) \to \mathcal{O}\left(p^{-1}\,(U)\right)$ is injective for every affine open $U \subseteq Y$, so p is dominant. At the same time, $p\,(X)$ is closed in Y by property (ii), hence $p\,(X) = \overline{p\,(X)} = Y$. □

Corollary 2.19. *Let G be an algebraic group acting on a variety X and $p\colon X \to Y$ be a good categorical quotient. Then for all $V \subseteq X$ open and G-invariant with $p^{-1}(p(V)) = V$, the image $p(V)$ under p is an open subset of Y.*

Proof. By assumption, the set $X \setminus V$ is G-invariant and closed. Hence property (ii) and Lemma 2.18 yield that $p(V) = p(X) \setminus p(X \setminus V) = Y \setminus p(X \setminus V)$ is open. □

Corollary 2.20. *If an algebraic group G acts on an irreducible variety X and $p\colon X \to Y$ is a good categorical quotient, then Y is also an irreducible variety.*

Proof. By the preceding lemma, one has $p(X) = Y$, and images of irreducible spaces under continuous maps are again irreducible. □

Lemma 2.21. *For a good categorical quotient $p\colon X \to X/\!\!/G$, we have:*

(i) For two $x_1, x_2 \in X$ the equality $p(x_1) = p(x_2)$ holds if and only if $\overline{Gx_1} \cap \overline{Gx_2} \neq \varnothing$.

(ii) Each fibre $p^{-1}(y)$ for some $y \in X/\!\!/G$ contains a unique closed orbit.

Proof. [Dol03, Cor. 6.1]. □

This lemma allows us to think of the categorical quotient $X/\!\!/G$ as the set of closed orbits of X or, in other words, the quotient of X by the equivalence relation $x_1 \sim x_2 \Leftrightarrow \overline{Gx_1} \cap \overline{Gx_2} \neq \varnothing$.

3 Stable and semistable points in the space of cubic surfaces

In what follows, we denote by $V := \mathbb{C}[X_0, \ldots, X_3]_3$ the space of homogeneous cubic polynomials of degree 3 on the projective space $\mathbb{P}^3$ and by $|\mathcal{O}(3)| := \mathbb{P}(V)$ the space of the corresponding cubic surfaces, i.e. the space of the zero sets of all $f \in V$. Note that $f_1, f_2 \in V$ have the same zero set if and only if $f_1 = \lambda f_2$ for some $\lambda \in \mathbb{C}^\times$. As there are $\binom{4+3-1}{3} = 20$ different monomials of degree 3 in $X_0, \ldots, X_3$, we have $V \cong \mathbb{A}^{20}$ and $|\mathcal{O}(3)| \cong \mathbb{P}^{19}$. We further fix $k = \mathbb{C}$ and $G = SL(4)$, which acts linearly on V by its natural action on $\mathbb{P}^3$ via $(g.f)(p) = f(g^{-1}p)$ for all $p \in \mathbb{P}^3$.

We shall now start by defining the semi-stable and stable points of V and then show that they have the property mentioned in the introduction, namely the existence of a categorical resp. geometric quotient on them.

Definition 3.1. An element $f \in V$ is said to be

(i) *semi-stable* if $0 \notin \overline{G.f}$ and

(ii) *stable* if $G.f \subseteq V$ is closed and the stabilizer G_f is finite.

The set of semi-stable resp. stable points is denoted by V^{ss} resp. V^s.

Since G acts linearly on V, the stabilizer of 0 is the whole (infinite) group G and consequently one has $V^s \subseteq V^{ss}$.

Remark 3.2. Both subsets are G-invariant open subvarieties.

Proposition 3.3. *There exists a categorical quotient of V^{ss} and a geometric quotient of V^s.*

Proof. [MFK94, Thm. 1.10]. $\square$

Since all $g \in G$, $f \in V$ and $\lambda \in \mathbb{C}^\times$ fulfil $g.(\lambda f) = \lambda (g.f)$, the action of G on V gives rise to an action of G on $|\mathcal{O}(3)| = \mathbb{P}(V)$. The definition of (semi-)stable points is also independent of the multiplication with a non-zero scalar, hence we can say that a cubic surface given by $f = 0$ for some $f \in V$ is (semi-)stable if and only if f is so, which coincides with the general notion of (semi-)stability. The semi-stable resp. stable points of $|\mathcal{O}(3)|$ are again denoted by $|\mathcal{O}(3)|^{ss}$ resp. $|\mathcal{O}(3)|^s$.

In order to determine the semi-stable and stable cubic surfaces, we need to classify some singularities that can occur in a cubic surface.

Definition 3.4. Let X, Y be varieties over $\mathbb{C}$. Then two points $p \in X$ and $q \in Y$ are *analytically isomorphic* if there is a $\mathbb{C}$-algebra isomorphism $\hat{\mathcal{O}}_p \cong \hat{\mathcal{O}}_q$.

Definition 3.5. A cubic surface $V(f) \subseteq \mathbb{P}^3$ has

(i) an *ordinary double point* at $p \in V(f)$ if and only if p is analytically isomorphic to the origin in the affine variety $V\left(x^2 + y^2 + z^2\right) \subset \mathbb{A}^3$.

(ii) an *ordinary cusp* at $p \in V(f)$ if and only if p is analytically isomorphic to the origin in the affine variety $V\left(x^2 + y^2 + z^3\right) \subset \mathbb{A}^3$.

In the more general theory one says that f has a singularity of type A_1 resp. type A_2 in p.

The second type of singularities does not occur in the classification of the surfaces, but will play an important role in its proof.

Theorem 3.6. *(i) The stable cubic surfaces are exactly the ones which are smooth or have only ordinary double points.*

(ii) There is only one point contained in the categorical but not in the geometric quotient. It corresponds to the closed orbit of $X_0^3 - X_1X_2X_3$ (cf. Lemma 2.21).

Some form of this theorem can already be found in [MFK94]. The proof given here follows [Bea09] and relies on the Hilbert-Mumford Numerical Criterion which is being introduced now:

Definition 3.7. A *one-parameter subgroup of G* is a non-trivial algebraic group homomorphism $\lambda\colon \mathbb{G}_\mathrm{m} \to G$.

One important property of these morphisms which will be needed later is stated by

Lemma 3.8. *Let $\lambda\colon \mathbb{G}_\mathrm{m} \to G$ be a one-parameter subgroup of G. Then the matrices of $\lambda(G)$ are simultaneously diagonalizable.*

Proof. As $\lambda\colon \mathbb{G}_{\mathrm{m}} \to G$ is an algebraic group homomorphism, it is a rational representation of $\mathbb{G}_{\mathrm{m}}$. The ring of regular functions on $\mathbb{G}_{\mathrm{m}}$ is the ring of Laurent polynomials

$$A := \mathbb{C}[z, z^{-1}] = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\chi_n,$$

where $\chi_n(z) = z^n$ are characters of $\mathbb{G}_{\mathrm{m}}$ for all $n \in \mathbb{Z}$. Therefore, the character group of $\mathbb{G}_{\mathrm{m}}$ spans A as a $\mathbb{C}$-module and $\mathbb{G}_{\mathrm{m}}$ is by definition a diagonalizable group. It is a standard fact from representation theory of linear algebraic groups that the image of the rational representation λ is then conjugate to a subgroup of the diagonal matrices, see e.g. [Bor91, Prop. 8.4]. □

Since G acts on V, a one-parameter subgroup $\lambda\colon \mathbb{G}_{\mathrm{m}} \to G$ gives rise to an action of $\mathbb{G}_{\mathrm{m}}$ on V which in turn induces a morphism $\mathbb{G}_{\mathrm{m}} \to V$, $t \mapsto \lambda(t).f$ for all $f \in V$. If that morphism extends to a morphism $\mathbb{A}^1 \to X$, the image of the origin is called the *limit of λ at f as $t \to 0$* and denoted by $\lim_{t\to 0} \lambda(t).f$.

The criterion can now be stated as follows:

Proposition 3.9. *An element $f \in V$ is*

(i) not semi-stable *if and only if there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_{\mathrm{m}} \to G$ such that $\lim_{t\to 0} \lambda(t).f = 0$ and*

(ii) not stable *if and only if $f = 0$ or there exists a one-parameter subgroup $\lambda\colon \mathbb{G}^{\mathrm{m}} \to G$ such that $\lim_{t\to 0} \lambda(t).f \notin G.f$ (in particular, the limit exists).*

Proof. [MFK94, Thm. 2.1]. □

The Zariski tangent space is insufficient to describe completely the structure of a surface around a singularity. That is better reflected by a more general construction, called the tangent cone, here again only defined for the special case of a hypersurface.

Definition 3.10. Let $S \subseteq \mathbb{A}^n$ be a hypersurface given by a polynomial $f \in k[x_1, \dots, x_n]$. Let $p \in \mathbb{A}^n$ and

$$f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha (x-p)^\alpha \text{ with } a_\alpha = 0 \text{ for almost all } \alpha \in \mathbb{N}^n$$

be an expansion of f around p, where we use the usual multi-index notation. Set

$$m := \min\{k \in \mathbb{N}\colon a_\alpha \neq 0 \text{ for some } \alpha \in \mathbb{N}^n,\ |\alpha| = k\}.$$

Then $f_p^{\mathrm{in}} = \sum_{|\alpha|=m} a_\alpha (x-p)^\alpha$ is called the *initial form* of f in p and $TC_p(S) = V(f_p^{\mathrm{in}})$ the *tangent cone* of S in p.

Remark 3.11. Of course, there is a more algebraic definition for the tangent cone, but the concrete one given will be completely sufficient for our purposes. In particular, it gives a useful necessary condition for a singularity to be an ordinary double point or cusp: Keeping in mind Definition 3.5, if a cubic surface $f = 0$ has an ordinary double point resp. cusp in $p \in U$ for a $U \subseteq \mathbb{P}^3$ affine open, then f_p^{in} must be a quadric of rank 3 resp. 2.

We are now ready to verify the classification given above.

Proof of Theorem 3.6. **Step 1** *Cubic surfaces with at most singularities of type A_1 or A_2 are semi-stable.*

Let first $S = V(f)$ be a cubic surface corresponding to a $f = \sum_{\alpha \in \mathbb{N}^4,\, |\alpha|=3} a_\alpha X^\alpha \in V$ which is not semi-stable. Then S has a singularity which is not an ordinary double point or cusp. For this it can be assumed that S is irreducible. The Hilbert-Mumford criterion stated in Proposition 3.9 tells us that we can choose a one-parameter subgroup $\lambda\colon \mathbb{G}_\mathrm{m} \to G$ with $\lim_{t\to 0} \lambda(t).f = 0$. By Lemma 3.8, one can, after a possible base change, assume

$$\lambda(t) = \operatorname{diag}(\lambda_0(t), \dots, \lambda_3(t)) \text{ for all } t \in \mathbb{G}_\mathrm{m}$$

for some $\lambda_0(t), \cdots, \lambda_3(t) \in \mathbb{C}[t, t^{-1}]$, as $\lambda(t)$ is a homomorphism of algebraic groups.

From $\lambda(t) \in SL(4)$ for all $t \in \mathbb{G}_\mathrm{m}$ it follows that

$$\lambda_0(t) \cdot \dots \cdot \lambda_3(t) = 1 \text{ for all } t \in \mathbb{G}_\mathrm{m}.$$

If some $\lambda_i(t)$ was not a monomial, it would have a zero at some $t_0 \in \mathbb{G}_\mathrm{m}$, so $\prod_{j\neq i} \lambda_j$ would have a pole in this point, which is impossible since it is a Laurent polynomial. One therefore concludes that $\lambda_i(t) = t^{r_i}$ for $r_i \in \mathbb{Z}$, $i = 0, \dots, 3$, and $\sum_{i=0}^{3} r_i = 0$. Assume further without loss of generality that $r_0 \leq \dots \leq r_3$.

The action of $\lambda(t)$ on f yields

$$\lambda(t).f = \sum_{\alpha \in \mathbb{N}^4,\, |\alpha|=3} a_\alpha \left(t^{r_0} X_0\right)^{\alpha_0} \cdots \left(t^{r_3} X_3\right)^{\alpha_3} = \sum_{\alpha \in \mathbb{N}^4,\, |\alpha|=3} t^{r\cdot\alpha} a_\alpha X^\alpha.$$

As $\lim_{t\to 0} \lambda(t) f = 0$, we get $r \cdot \alpha > 0$ for all $\alpha \in \mathbb{N}^4$ with $a_\alpha \neq 0$.

Since f is irreducible by assumption, not every monomial X^α can be divided by X_3, in other words, there exists an $\alpha \in \mathbb{N}^4$ such that $a_\alpha \neq 0$ and $\alpha_3 = 0$. But then $r \cdot \alpha > 0$ and the linear ordering of the r_i imply $r_2 > 0$. This also means that no monomial can be of the form $X_0^2 X_i$ or $X_0 X_1 X_i$ with $i = 0, \dots, 3$, because

$$2r_0 + r_i \leq r_0 + r_1 + r_i \leq r_0 + r_1 + r_3 = -r_2 < 0,$$

which is impossible.

Hence on the open affine subset $\{X_0 \neq 0\}$, the polynomial f is given in local coordinates

$$x_1 = \frac{X_1}{X_0},\ x_2 = \frac{X_2}{X_0},\ x_3 = \frac{X_3}{X_0}$$

by $f\left(x_1, x_2, x_3\right) = f_2\left(x_1, x_2, x_3\right) + f_3\left(x_1, x_2, x_3\right)$ with $f_2 = a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2$ and f_3 homogeneous of degree 3. We are going to show that f has a singularity not of type A_1 or A_2 in $p := (0,0,0) \in \{X_0 \neq 0\}$.

If $f_2 = 0$, then f_p^{in} is not a quadric and f has no ordinary double point or cusp in p by Remark 3.11.

If $f_2 \neq 0$, then $f_p^{\mathrm{in}} = f_2$. Again by Remark 3.11, we know that for p to be an ordinary double point or cusp, f_2 must have rank 3 or 2. The former is impossible since f_2 depends only on x_2 and x_3. The latter is possible only if $a_{22} \neq 0$ or $a_{23} \neq 0$, which in turn implies $r_0+2r_2 = (r_0, r_1, r_2, r_3) \cdot (1, 0, 2, 0) > 0$ or $r_0+r_2+r_3 = (r_0, r_1, r_2, r_3) \cdot (1, 0, 1, 1) > 0$. Since $r_2 \leq r_3$, in either case one has $r_1 = -(r_0 + r_2 + r_3) < 0$. In particular, $r \cdot (0, 3, 0, 0) < 0$ and x_1^3 cannot be contained in f_3, so f does not have an ordinary cusp in p.

It can thus be concluded that a cubic surface which is smooth or has only ordinary double points and cusps must be semi-stable.

In fact, one can even show:

Step 2 *Cubic surface with at most singularities of type A_1 are already stable.*

To prove this, let $S = V(f)$ now be a non-stable cubic surface and $\lambda \colon \mathbb{G}_\mathrm{m} \to G$ a one-parameter subgroup for which the limit $\lim_{t \to 0} \lambda(t).f$ exists. One again can conclude

$$\lambda(t).f = \sum_{\alpha \in \mathbb{N}^4,\, |\alpha| = 3} t^{r \cdot \alpha} a_\alpha X^\alpha$$

for some $r_0 \leq r_1 \leq r_2 \leq r_3$, only this time the existence of the limit gives merely the condition $r \cdot \alpha \geq 0$ from which follows only $r_2 \geq 0$.

If $r_2 > 0$, one concludes as before that no monomial of f can be of the form $X_0^2 X_i$ or $X_0 X_1 X_i$, $i = 0 \ldots, 3$, and therefore f_p^{in}, $p := (0, 0, 0) \in \{X_0 \neq 0\}$, cannot be a quadric of rank 3, meaning that f does not have an ordinary double point in p.

If $r_2 = 0$, one has $r_0+r_1+r_3 = r_0+r_1+r_2+r_3 = 0$. Hence, if $r_0 < r_1$, then f still contains neither any monomial of the form $X_0^2 X_i$, since $2r_0 + r_i < r_0 + r_i + r_3 = 0$, nor of the form $X_0 X_1^2$ or $X_0 X_1 X_2$, i.e. the only additionally possible monomial would be $X_0 X_1 X_3$. On the other hand, $X_0 X_2^2$ cannot occur because of $r_0 + 2r_2 = r_0 < 0$. In consequence, f_p^{in} is given either by a cubic form or a quadric of the form $f_2 = a_{13} x_1 x_3 + a_{23} x_2 x_3 + a_{33} x_3^2$ which corresponds to the matrix

$$\begin{pmatrix} 0 & 0 & \frac{a_{13}}{2} \\ 0 & 0 & \frac{a_{23}}{2} \\ \frac{a_{13}}{2} & \frac{a_{23}}{2} & a_{33} \end{pmatrix}$$

and has thus rank ≤ 2. In either case, f does not have an ordinary double point in p. The remaining case $r_2 = 0$, $r_0 = r_1$ leads, together with $\sum_i r_i = 0$, to $r = (-n, -n, 0, 2n)$ for some $n \in \mathbb{N}$. One checks that in this case, S has at least one cusp.

We have seen by now that a cubic surface with only ordinary double points or ordinary cusps is semi-stable and one with only ordinary double points is stable. It remains to show that these conditions are already sufficient, i.e. that cubic surfaces with a singularity of another type than A_1, A_2 cannot be semi-stable and that the ones with ordinary cusps cannot be stable.

Step 3 *Semi-stable cubic surfaces have at most singularities of type A_1 or A_2.*

Let $S = V(f)$, with $f = \sum_{\alpha \in \mathbb{N}^4,\, |\alpha| = 3} a_\alpha X^\alpha \in V$ as before, have a singularity not of type A_1. After a possible coordinate change, this singularity lies in the point $p := [1 : 0 : 0 : 0]$. In the standard local affine coordinates of the neighbourhood $\{X_0 \neq 0\}$ of p the function $f(x_1, x_2, x_3)$ must not contain a linear or constant term, since this would mean that the differential at p would not vanish and hence f would not have a singularity in p. Thus, we have

$$f = X_0 g(X_1, X_2, X_3) + h(X_1, X_2, X_3)$$

with g homogeneous of degree 2, h homogeneous of degree 3 and the rank of g less than 3 because f has no ordinary double point in p.
If the quadric g has rank 0 or 1, we can, after a further coordinate change, assume $g = 0$ or $g = X_3^2$. In either case, the one-parameter subgroup $\lambda\colon \mathbb{G}_{\mathrm{m}} \to G$ corresponding to $r = (-5, 1, 1, 3)$, when acting on f, has the limit

$$\lim_{t\to 0} \lambda(t).f = \lim_{t\to 0} \sum_{\alpha\in\mathbb{N}^4,\, |\alpha|=3} t^{r\cdot\alpha} a_\alpha X^\alpha = 0,$$

as $r \cdot \alpha > 0$ for $\alpha = (1, 0, 0, 2)$ and every α with non-vanishing coefficient in h. So f is not semi-stable.
If the quadric g has rank 2, we can assume $g = X_2^2 + X_3^2 = (X_2 + \mathrm{i}X_3)(X_2 - \mathrm{i}X_3)$ and after another coordinate change $g = X_2X_3$.
If f further does not have an ordinary cusp point at p, f, and thus h, cannot contain the monomial X_1^3. But then $(-5, -1, 3, 3) \cdot \alpha > 0$ for $\alpha = (1, 0, 1, 1)$ and every α with non-vanishing coefficient in h, and by the same reasoning as above, f is not semi-stable.
It can already be concluded that cubic surfaces with singularities not of type A_1 or A_2 are not semi-stable.

Step 4 *Stable cubic surfaces have at most singularities of type A_1.*

If, at last, f has an ordinary cusp, it is semi-stable, but not stable: It has already been shown that it is semi-stable. As f has an ordinary cusp, the coefficient c of the monomial X_1^3 in f must not vanish. To see that f is not stable, it suffices to show that the cubic polynomial $f_0 = X_0X_2X_3 + cX_1^3$ is in the closure of the orbit of f. For if f was stable, its orbit would be closed and would contain f_0 which would therefore be stable as well. But the stabilizer of f_0 is infinite, since it contains the diagonal matrices $\operatorname{diag}(\xi_0, 1, \xi_2, \xi_3) \in SL(4, \mathbb{C})$ with $\xi_0\xi_2\xi_3 = 1$.

So let $\lambda\colon \mathbb{G}_{\mathrm{m}} \to G$ be the one-parameter subgroup corresponding to $r = (-2, 0, 1, 1)$. Recalling that f is of the form $f = X_0X_2X_3 + h(X_1, X_2, X_3)$, one sees that the only α for which $r \cdot \alpha = 0$ and the coefficient $a_\alpha \neq 0$ are $(1, 0, 1, 1)$ and $(0, 3, 0, 0)$, so

$$\lim_{t\to 0} \lambda(t).f = X_0X_2X_3 + cX_1^3$$

is in the closure of the orbit of f under the euclidean topology and hence also under the Zariski topology.

To conclude the proof of the second part of the theorem, notice that with $X_0X_2X_3 + cX_1^3$ also the surface S_0 corresponding to $X_0X_2X_3 - X_1^3$ is contained in the closure of the orbit of each semi-stable, but not stable cubic surface. Since a quadric X_iX_j can be written as $X_i^2 + X_j^2$ after a change of coordinates as seen above, the singular points $[1 : 0 : 0 : 0]$, $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$ of S_0 in the open affine subsets $\{X_0 \neq 0\}$, $\{X_2 \neq 0\}$ and $\{X_3 \neq 0\}$ are all ordinary cusps. In $\{X_1 \neq 0\}$, the surface S_0 has no singularities because the differential of $x_0x_2x_3 - 1$ vanishes only in points in which the polynomial itself does not vanish, namely where at least two of the coordinates vanish.

The surface S_0 is therefore semi-stable, as it contains only ordinary cusps. It is in the orbit closure of each semi-stable, not stable surface, so there is only one point contained in the categorical but not in the geometric quotient by Lemma 2.21(i). By Lemma 2.21(ii), there is a unique closed orbit which maps to this point. Since the closure of this orbit is the orbit itself, S_0 must already be contained in that orbit which concludes the proof. $\square$

Remark 3.12. In particular, from Step 3 of the proof follows that a non-stable cubic polynomial $f \in V$ is, after a possible linear change of coordinates, given by a polynomial of the form

(a) $h(X_1, X_2, X_3)$,

(b) $X_0X_3^2 + h(X_1, X_2, X_3)$ or

(c) $X_0X_2X_3 + h(X_1, X_2, X_3)$,

where $h(X_1, X_2, X_3)$ is a homogeneous polynomial of degree 3.

Since we have now familiarized ourselves with the semi-stable points V^{ss} and $|\mathcal{O}(3)|^{ss}$, we can finally see

Definition 3.13. The categorical quotient $\mathcal{M} := |\mathcal{O}(3)|^{ss}/\!\!/G$ is called the *moduli space of cubic surfaces.*

In order to compute the the moduli space of cubic surfaces, we need an analogue to Theorem 2.11 in the projective case.

Lemma 3.14. *There is an isomorphism* $|\mathcal{O}(3)|^{ss}/\!\!/G \cong \operatorname{Proj}\left(\mathcal{O}(V)^G\right)$.

Proof. [Dol03, Prop. 8.1] with $L = \mathcal{O}_{\mathbb{P}^{19}}(1)$. $\square$

Thus, it suffices to compute $\mathcal{O}(V)^G$, the ring of invariants of the cubic forms on $\mathbb{P}^3$.

4 The Sylvester form of the homogeneous cubic polynomials

For the computation of $\mathcal{O}(V)^G$, it is not advisable to consider cubic polynomials $f \in V$ in their most general form because we would have to look at a 20-dimensional space whose structure is quite difficult to grasp. It will be shown that it is sufficient to restrict oneself to an open subset of that space with a somewhat easier structure, the set of the cubics with a Sylvester form:

A general $f \in V$ can be written as the linear combination of the cubes of five linear forms, each four of which are linearly independent and whose sum vanishes. In other words, one embeds (non-canonically) $\mathbb{P}^3$ into $\mathbb{P}^4$ as the hyperplane $\sum_{i=0}^4 X_i = 0$ and can then write f as the restriction of a homogeneous cubic polynomial on $\mathbb{P}^4$ to that hyperplane.

Theorem 4.1. *A general homogeneous polynomial $f \in V$ can be written as*

$$f = \sum_{i=0}^{4} \lambda_i X_i^3, \quad \sum_{i=0}^{4} X_i = 0,$$

where the coefficients $\lambda_0, \dots, \lambda_4 \in \mathbb{C}$ are unique up to permutation and a common scalar factor.

Proof. A modern proof can be found in [Dol12, Cor. 9.4.2]. □

Definition 4.2. A cubic surface $V(f)$ given by the equations

$$f = \sum_{i=0}^{4} \lambda_i X_i^3 = 0, \quad \sum_{i=0}^{4} X_i = 0$$

is said to have a *non-degenerate Sylvester form*, if $\lambda_i \neq 0$ for all $i = 0, \ldots, 5$. Otherwise, it has a *degenerate Sylvester form.*

If a surface is given by a Sylvester form, its corresponding Hessian surface is also of a simple form. In what follows, let $\mathrm{He}(f) = \left(\frac{\partial^2 f}{\partial X_i \partial X_j}\right)_{i,j=0,\ldots,n}$ denote the Hessian matrix of a homogeneous polynomial f on $\mathbb{P}^n$ and H_f its determinant.

Lemma 4.3. *Let $V(f)$ be a surface given by a Sylvester equation*

$$f = \sum_{i=0}^{4} \lambda_i X_i^3 = 0, \quad \sum_{i=0}^{4} X_i = 0$$

and $N = |\{i \colon \lambda_i = 0\}|$ be the number of vanishing coefficients. If $N \leq 4$, the Hessian surface $V(H_f)$ is a surface of degree $4 - N$. If in addition $V(f)$ is non-degenerate, the Hessian is given by

$$\sum_{i=0}^{4} \prod_{j \neq i} \lambda_j X_j = 0, \quad \sum_{i=0}^{4} X_i = 0. \tag{1}$$

Proof. A proof which uses techniques from Classical Algebraic Geometry can be found in [Dol12, Sect. 9.4.2]. □

In the following, we need

Definition 4.4. Let $p_{ijk} \in V\left(\sum_{i=0}^4 X_i\right) \subseteq \mathbb{P}^4$ denote the points given by $X_i = X_j = X_k = 0$ (in the hyperplane the vanishing of three coordinates already defines a unique point), $l_{ij} \subset V\left(\sum_{i=0}^4 X_i\right) \subset \mathbb{P}^4$ the lines given by $X_i = X_j = 0$ and $E_i \subset V\left(\sum_{i=0}^4 X_i\right) \subset \mathbb{P}^4$ the hyperplanes given by $X_i = 0$.

Proposition 4.5. *Let $V(f)$ be a cubic surface given by a non-degenerate Sylvester equation. Then every singular point of $V(f)$ is a singular point of the Hessian $V(H_f)$ and the p_{ijk} are the only points which are singular points of $V(H_f)$ and not of $V(f)$.*

Proof. For this proof, let us not embed $\mathbb{P}^3$ into $\mathbb{P}^4$, i.e. drop the condition $\sum_{i=0}^4 X_i = 0$ and instead regard $X_4 = -(X_0 + \ldots + X_3)$ as a function of $X_0, \ldots, X_3$, which of course amounts to the same thing. This will simplify significantly the differentiating needed for the computations of the singular points.

Let $S := V\left(\prod_{i=0}^{4} X_i\right)$. We show first that on the open set $\mathbb{P}^3 \setminus S$, the singular points of $V(f)$ coincide with the singular points of $V(H_f)$ and then that on S, $V(f)$ has no singular points and $V(H_f)$ only the singular points p_{ijk}.

On $\mathbb{P}^3 \setminus S$, the Hessian is given by the equation $\sum_{i=0}^{4} \frac{1}{\lambda_i X_i} = 0$ because the λ_i do not vanish by assumption and we can divide Equation (1) from Lemma 4.3 by $\prod_{i=0}^{4} \lambda_i X_i$. A point $p = [p_0 : p_1 : p_2 : p_3] \in \mathbb{P}^3 \setminus S$ is a singular point of $V(H_f)$ if and only for all $i = 0, \ldots, 3$

$$\frac{\partial H_f}{\partial X_i}(p) = 0 \Leftrightarrow \frac{1}{\lambda_i p_i^2} - \frac{1}{\lambda_4 p_4^2} = 0 \Leftrightarrow \lambda_i p_i^2 - \lambda_4 p_4^2 = 0 \Leftrightarrow \frac{\partial f}{\partial X_i}(p) = 0,$$

where $p_4 = -(p_0 + p_1 + p_2 + p_3)$. Hence, p is a singular point of $V(H_f)$ if and only if it is a singular point of $V(f)$.

On S, the surface $V(f)$ has no singularities. For if there was a singular point $p = [p_0 : p_1 : p_2 : p_3] \in S$, again $p_4 := -(p_0 + p_1 + p_2 + p_3)$, one would have $3\lambda_i p_i^2 - 3\lambda_4 p_4^2 = \frac{\partial f}{\partial X_i}(p) = 0$ for all $i = 0, \ldots, 3$ from which follows immediately $\lambda_i p_i^2 = \lambda_j p_j^2$ for all $i, j \in \{0, \ldots, 4\}$. Then $p_i = 0$ for an $i \in \{0, \ldots, 4\}$ implies $p_j = 0$ for all $j = 0, \ldots, 3$ because the λ_j do not vanish by assumption, but this is impossible.

It remains to identify the singular points of $V(H_f)$ which lie in in S. Let $p = [p_0 : p_1 : p_2 : p_3] \in S$, $p_4 := -(p_0 + p_1 + p_2 + p_3)$ and $i \in \{0, \ldots, 4\}$ such that $p_i = 0$. The point p is in the zero locus of H_f if and only if $0 = H_f(p) = \sum_{j=0}^{4} \prod_{l \neq j} \lambda_l p_l = \prod_{l \neq i} \lambda_l p_l$, which is in turn equivalent to the existence of an $l \in \{0, \ldots 4\}$, $l \neq i$ such that $p_l = 0$ (because $\lambda_j \neq 0$, $j = 0, \ldots, 4$ by assumption). Hence, the Hessian in S is given by $V(H_f) \cap S = \bigcup_{i \neq j} l_{ij}$. One sees easily from the explicit form of the Hessian in Equation (1) that the only potentially singular points of $V(H_f)$ in S are the vertices p_{ijk} given by $X_i = X_j = X_k = 0$. On the other hand, these actually are singular points of the Hessian since differentiating Equation (1) with respect to the X_i leaves a sum of terms which contain the product of three different coordinates and therefore vanish when evaluated at the p_{ijk}. □

Definition 4.6. The pentahedron spanned by the planes E_i is called the *Sylvester pentahedron.*

5 The weighted projective space

To clarify the statement of the main theorem we need to become acquainted with the notion of the weighted projective space. We shall follow [Dol82] where the reader may also find more advanced results which cannot be discussed here.

Definition 5.1. Let $Q := (q_0, \ldots, q_r)$ be a finite tuple of non-zero natural numbers and $S(Q) := \mathbb{C}[T_0, \ldots, T_r]$ the algebra graded by $\deg T_i = q_i$.
Then $\mathbb{P}(Q) := \mathrm{Proj}(S(Q))$ is called the *weighted projective space of type Q.*

Note that we get the usual projective space as the special case $\mathbb{P}^r = \mathbb{P}(1, \ldots, 1)$. We shall use the notation of the previous definition for the entire section.

In the following, we shall often use the following

Definition 5.2. Let S be a graded ring, $a \in \mathbb{N}^*$ a natural number. Then one denotes by $S^{(a)} := \bigoplus_{n=0}^{\infty} S_{na}$ the subring of S obtained as the direct sum of all homogeneous parts of degree divisible by a and graded by taking the S_{na} to be the homogeneous elements of degree n.

One checks easily that $S^{(a)}$ really defines a graded ring. One of its relations to the original ring S is stated as

Lemma 5.3. *Let S be a graded ring, $a \in \mathbb{N}^*$. Then there exists an isomorphism of schemes from* $\operatorname{Proj}(S)$ *onto* $\operatorname{Proj}\left(S^{(a)}\right)$.

Proof. [Gro61, Prop. 2.4.7]. □

Using this, we get

Lemma 5.4. *Let $Q = (q_0, \ldots, q_r)$, $Q' = (aq_0, \ldots, aq_r)$ for some $a \in \mathbb{N}^*$. Then there is an isomorphism $\mathbb{P}(Q) \cong \mathbb{P}(Q')$.*

Proof. From $S(Q')_m = S(Q)_{am}$ for all $m \in \mathbb{N}$, one concludes $S(Q') = S(Q)^{(a)}$ and, by Lemma 5.3,

$$\mathbb{P}(Q) = \operatorname{Proj}(S(Q)) \cong \operatorname{Proj}\left(S(Q)^{(a)}\right) = \operatorname{Proj}\left(S\left(Q'\right)\right) = \mathbb{P}\left(Q'\right). \qquad \square$$

By the preceding lemma, we can always assume without loss of generality that

$$\gcd(q_0, \ldots, q_r) = 1$$

In fact, it can, by the following proposition, even be assumed that each r numbers are coprime.

Proposition 5.5. *Let $Q = (q_0, \ldots, q_r) \in (\mathbb{N}^*)^{r+1}$. Then a tuple $Q' = (q'_0, \ldots, q'_r) \in (\mathbb{N}^*)^{r+1}$ exists such that there is an isomorphism $\mathbb{P}(Q) \cong \mathbb{P}(Q')$ and for $i = 0, \ldots, r$, one has* $\gcd\left(q'_0, \ldots, q'_{i-1}, q'_{i+1}, \ldots, q'_r\right) = 1$.

Proof. Define for all $i \in \{0, \ldots, r\}$

$$\begin{aligned} t_i &= \gcd(q_0, \ldots, q_{i-1}, q_{i+1}, \ldots, q_r) \\ a_i &= \operatorname{lcm}(t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_r) \\ a &= \operatorname{lcm}(t_0, \ldots, t_r). \end{aligned}$$

From $t_j \mid q_i \ \forall i \neq j$ one concludes $a_i \mid q_i$. In particular,

$$\gcd(a_i, t_i) \mid \gcd(q_i, t_i) = \gcd(q_0, \ldots, q_r) = 1,$$

which yields $\gcd(a_i, t_i) = 1$. Furthermore,

$$\begin{aligned} a_i t_i &= \gcd(a_i, t_i) \cdot \operatorname{lcm}(a_i, t_i) \\ &= 1 \cdot \operatorname{lcm}(\operatorname{lcm}(t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_r), t_i) \\ &= \operatorname{lcm}(t_0, \ldots, t_r) = a. \end{aligned}$$

Now let $Q' := \left(\frac{q_0}{a_0}, \ldots, \frac{q_r}{a_r}\right)$ and let us prove that it has got the desired properties.

The first part of the statement will be another application of Lemma 5.3. To compute $S(Q)^{(a)}$, note that a monomial $T_1^{\alpha_1} \cdot \ldots \cdot T_r^{\alpha_r} \in S(Q)$ has degree an for some $n \in \mathbb{N}$ if and only if $a \mid \alpha_0 q_0 + \cdots + \alpha_r q_r$, which is in turn equivalent to

$$t_i \mid \alpha_0 q_0 + \cdots + \alpha_r q_r \text{ for all } i = 1, \ldots, r. \tag{2}$$

On the other hand, one has for all i that $\gcd(q_i, t_i) = \gcd(q_1, \ldots, q_r) = 1$ and $t_i \mid q_j$, $j \neq i$. Consequently, (2) holds if and only if $t_i \mid \alpha_i$ for all i, so $S(Q)^{(a)} = \mathbb{C}\left[T_0^{t_0}, \ldots, T_r^{t_r}\right]$. Since the $T_i^{t_i}$ are of degree $q_i t_i = a\frac{q_i}{a_i} = aq_i'$ in $S(Q)$, they are of degree q_i' in $S(Q)^{(a)}$ (cf. Definition 5.2). Therefore, $S(Q') \cong S(Q)^{(a)}$ and, by Lemma 5.3,

$$\mathbb{P}(Q) = \operatorname{Proj}(S(Q)) \cong \operatorname{Proj}\left(S(Q)^{(a)}\right) \cong \operatorname{Proj}\left(S\left(Q'\right)\right) = \mathbb{P}\left(Q'\right).$$

It remains to show the second part. Let $i \in \{0, \ldots, r\}$. As $t_i \mid a_j \; \forall j \neq i$, there are $a_j' \in \mathbb{N}^*$, $j \neq i$ such that $a_j = a_j' t_i$. Hence,

$$\begin{aligned}\gcd\left(q_0', \ldots, q_{i-1}', q_{i+1}', \ldots, q_r'\right) &= \gcd\left(\frac{q_0}{a_0}, \ldots, \frac{q_{i-1}}{a_{i-1}}, \frac{q_{i+1}}{a_{i+1}}, \ldots, \frac{q_r}{a_r}\right) \\ &= \frac{1}{t_i} \gcd\left(\frac{q_0}{a_0'}, \ldots, \frac{q_{i-1}}{a_{i-1}'}, \frac{q_{i+1}}{a_{i+1}'}, \ldots, \frac{q_r}{a_r'}\right)\end{aligned}$$

divides $\frac{1}{t_i}\gcd(q_0, \ldots, q_{i-1}, q_{i+1}, \ldots, q_r) = 1$, from what follows already

$$\gcd\left(q_0', \ldots, q_{i-1}', q_{i+1}', \ldots, q_r'\right) = 1. \qquad \square$$

From now on, we shall therefore assume without loss of generality that

$$\gcd(q_0, \ldots, q_{i-1}, q_{i+1}, \ldots, q_r) = 1 \text{ for all } i = 0, \ldots, r.$$

Interpretations of the weighted projective spaces

Since the previous discussion of the weighted projective space has been quite abstract, let us introduce two interpretations designed to make this notion more concrete.

Analogously to the definition of $\mathbb{P}^r$ as the quotient of $\mathbb{A}^{r+1} \setminus \{(0)\}$ under the linear action of the one-dimensional torus $\mathbb{G}_\mathrm{m}$, one may consider the action of $\mathbb{G}_\mathrm{m}$ on $\mathbb{A}^{r+1} = \operatorname{Spec}(\mathbb{C}[Y_0, \ldots, Y_r])$ with $\deg Y_i = 1$ for all $i = 0, \ldots, r$, defined by the coaction homomorphism

$$\alpha \colon \mathbb{C}[Y_0, \ldots, Y_r] \to \mathbb{C} \otimes_\mathbb{C} \mathbb{C}\left[X, X^{-1}\right], \; Y_i \mapsto Y_i \otimes X^{q_i} \text{ for } i = 0, \ldots r.$$

Defining the $\mathbb{G}_\mathrm{m}$-invariant set $U := \mathbb{A}^{r+1} \setminus \{(0)\}$, we have an open affine cover $U = \bigcup_{i=0,\ldots,r} D(Y_i)$. By Theorem 2.11 and Proposition 2.9, for all $i = 0, \ldots, r$, we get a categorical quotient of $D(Y_i)$ of the form

$$D(Y_i) \cong \operatorname{Spec}\left(\mathbb{C}[Y_0, \ldots, Y_n]_{Y_i}\right) \twoheadrightarrow \operatorname{Spec}\left(\left(\mathbb{C}[Y_0, \ldots, Y_n]_{Y_i}\right)^{\mathbb{G}_\mathrm{m}}\right).$$

But $f \in \mathbb{C}[Y_0, \ldots, Y_n]_{Y_i}$ is invariant with respect to the action of $\mathbb{G}_\mathrm{m}$ if and only if the powers of X which arise when applying α are the same for numerator and denominator of f. This just means $\operatorname{Spec}\left(\left(\mathbb{C}[Y_0, \ldots, Y_n]_{Y_i}\right)^{\mathbb{G}_\mathrm{m}}\right) \cong \operatorname{Spec}\left(\left(\mathbb{C}[T_0, \ldots, T_n]_{T_i}\right)_0\right)$, where $\deg T_j = q_j$ for all j.

On the other hand,

$$\operatorname{Spec}\left(\left(\mathbb{C}[T_0, \ldots, T_n]_{T_i}\right)_0\right) = \operatorname{Spec}\left(\mathbb{C}[T_0, \ldots, T_n]_{(T_i)}\right) = D_+(T_i).$$

This gives us for all $i = 0, \ldots, r$ the categorical quotient $D(Y_i) \twoheadrightarrow D_+(T_i)$. Since the quotient maps glue and $(D_+(T_i))_i$ is an open cover of $\operatorname{Proj}(S(Q)) = \mathbb{P}(Q)$, we get the morphism $p\colon U \to \mathbb{P}(Q)$ which is again a categorical quotient by Remark 2.12.

Let us examine the action of $\mathbb{G}_\mathrm{m}$ on the closed points of $\mathbb{A}^{r+1}$:
Let $(t, a) \in \mathbb{G}_\mathrm{m} \times \mathbb{A}^{r+1}$ be a closed point, given by the maximal ideal

$$\mathfrak{m} = (Y_0 - a_0, \ldots, Y_r - a_r, X - t).$$

Then $(Y_0 - t^{q_0}a_0, \ldots, Y_r - t^{q_n}a_r) \subseteq \alpha^{-1}(\mathfrak{m})$ because for all $i = 0, \ldots, r$

$$\begin{aligned}\alpha(Y_i - t^{q_i}a_i) &= Y_iX^{q_i} - t^{q_i}a_i = X^{q_i}(Y_i - a_i) + a_i(X^{q_i} - t^{q_i}) \\ &= X^{q_i}(Y_i - a_i) + a_i(X - t)\sum_{l=1}^{q_i} X^{q_i - l}t^{l-1} \in \mathfrak{m}.\end{aligned}$$

As $(Y_0 - t^{q_0}a_0, \ldots, Y_n - t^{q_n}a_n)$ is already a maximal ideal in $\mathbb{C}[Y_0, \ldots, Y_n]$, we must have $\alpha^{-1}(\mathfrak{m}) = (Y_0 - t^{q_0}a_0, \ldots, Y_n - t^{q_n}a_n)$. In other words, $(t, (a_0, \ldots, a_r))$ is mapped to $(a_0t^{q_0}, \ldots, a_rt^{q_r})$, and $\mathbb{P}(Q)$ is the complex analytic quotient space $\left(\mathbb{C}^{r+1} \smallsetminus \{0\}\right)/\mathbb{C}^\times$ for this action.

This first interpretation allows us in particular to give coordinates to the weighted projective space, similar to those of the usual one. Since the construction is very similar to that of the ordinary projective space, one might try to construct the weighted projective space as a quotient of the projective space by another, possibly smaller group. This is the idea of the second interpretation of the weighted projective space.

For fixed $Q = (q_0, \ldots, q_r) \in (\mathbb{N}^*)^{r+1}$ and $0 \le i \le r$, consider

$$\mu_{q_i} := \operatorname{Spec}\left((\mathbb{C}[X_i]) / ((X_i^{q_i} - 1))\right)$$

as a subgroup of $\mathbb{G}_\mathrm{m}$, the group of q_i-th roots of unity. Then one checks easily that

$$\begin{aligned}\beta\colon \mathbb{C}[Y_0, \ldots, Y_r] &\to (\mathbb{C}[Y_0, \ldots, Y_r] \otimes_\mathbb{C} \mathbb{C}[X_0, \ldots, X_r]) / ((X_0^{q_0} - 1, \ldots, X_r^{q_r} - 1)) \\ Y_i &\mapsto Y_i \otimes \bar{X}_i\end{aligned}$$

defines an action of $\mu_Q := \mu_{q_o} \times \ldots \times \mu_{q_r}$ on $\mathbb{P}^r$.

Clearly, a power Y_i^m is an invariant under this action, i.e. $\beta(Y_i^m) = Y_i^m \otimes 1$, if and only if $q_i \mid m$. Since under β every Y_i is mapped to a tensor product which contains only X_i, a polynomial $f \in \mathbb{C}[Y_0, \ldots, Y_r]$ is invariant if and only if it contains only products of $Y_0^{q_0}, \ldots, Y_r^{q_r}$, so $\mathbb{C}[Y_0, \ldots, Y_r]^{\mu_Q} = \mathbb{C}[Y_0^{q_0}, \ldots, Y_r^{q_r}]$. Hence, if $S(Q) = \mathbb{C}[T_0, \ldots, T_r]$ is graded as above by $\deg T_i = q_i$, then $\varphi\colon S(Q) \to \mathbb{C}[Y_0, \dots, Y_r]^{\mu_Q}$, $T_i \mapsto Y_i^{q_i}$ is a graded ring isomorphism. Similarly to the first interpretation, we choose an open affine cover $(D_+(Y_i))_{i=0,\dots,r}$ of $\mathbb{P}^r = \operatorname{Proj}(\mathbb{C}[Y_0, \dots, Y_r])$ and an open affine cover $(D_+(T_i))_{i=0,\dots,r}$ of $\mathbb{P}(Q) = \operatorname{Proj}(\mathbb{C}[T_0, \dots, T_r])$. By Theorem 2.11 and Proposition 2.9, the categorical quotient of $D_+(Y_i)$ by μ_Q is then

$$D_+(Y_i) \cong \operatorname{Spec}\left(\mathbb{C}\left[\frac{Y_0}{Y_i}, \dots, \frac{Y_r}{Y_i}\right]\right) \twoheadrightarrow \operatorname{Spec}\left(\mathbb{C}\left[\frac{Y_0}{Y_i}, \dots, \frac{Y_r}{Y_i}\right]^{\mu_Q}\right).$$

But $\operatorname{Spec}\left(\mathbb{C}\left[\frac{Y_0^{q_0}}{Y_i^{q_i}}, \dots, \frac{Y_r^{q_r}}{Y_i^{q_i}}\right]\right) \cong \operatorname{Spec}\left(\left(\mathbb{C}\left[\frac{T_0}{T_i}, \dots, \frac{T_r}{T_i}\right]\right)_0\right) \cong D_+(T_i)$, so we have again a categorical quotient $D_+(Y_i) \twoheadrightarrow D_+(T_i)$ for all i and the quotient maps glue. By Remark 2.12, we know that we have a categorical quotient $\mathbb{P}(Q) = \mathbb{P}^r /\!\!/ \mu_Q$.

While the usual projective space is smooth, the weighted projective space may have singularities which correspond to points with non-trivial stabilizers for the group action from the second interpretation.

Proposition 5.6. *An arbitrary point $a = [a_0 : \ldots : a_r] \in \mathbb{P}(Q)$ is singular if and only if* $\gcd(i\colon a_i \neq 0) > 1$.

Proof. [DD85, Prop. 7]. □

In the special case $\mathbb{P}(1,2,3,4,5)$, this gives

$$\mathbb{P}(1,2,3,4,5)_{\text{sing}} = \{[0:0:1:0:0], [0:0:0:0:1], [0:z_1:0:z_2:0] : z_1 z_2 \neq 0\}.$$

6 The invariants of the cubic forms on $\mathbb{P}^3$

The aim of this section is to prove the following theorem which goes back to [Sal65]:

Theorem 6.1. *The ring of invariants of the cubic forms on $\mathbb{P}^3$ is*

$$\mathcal{O}(V)^G = \mathbb{C}\left[\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}, \hat{I}_{40}, \hat{I}_{100}\right].$$

Here, the $\hat{I}_n$ are invariants of degree n which will be defined in the following. The invariants $\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}$ and $\hat{I}_{40}$ are algebraically independent and there is a polynomial P such that $\hat{I}_{100}^2 = P\left(\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}, \hat{I}_{40}\right)$.

From this result we can easily follow the theorem to be proved in this thesis.

Theorem 6.2. *The moduli space $\mathcal{M}$ of cubic surfaces $S \subset \mathbb{P}^3$ is isomorphic to the weighted projective space $\mathbb{P}(1,2,3,4,5)$.*

$$\mathcal{M} \cong \mathbb{P}(1,2,3,4,5)$$

Proof. Let $A := \mathcal{O}(V)^G$. In the computation to come, it will be of advantage to know the form of $A^{(8)}$. A monomial $\hat{I}_8^{\alpha_8}\hat{I}_{16}^{\alpha_{16}}\hat{I}_{24}^{\alpha_{24}}\hat{I}_{32}^{\alpha_{32}}\hat{I}_{40}^{\alpha_{40}}\hat{I}_{100}^{\alpha_{100}} \in A$ has the degree

$$\begin{aligned}\deg \hat{I}_8^{\alpha_8}\hat{I}_{16}^{\alpha_{16}}\hat{I}_{24}^{\alpha_{24}}\hat{I}_{32}^{\alpha_{32}}\hat{I}_{40}^{\alpha_{40}}\hat{I}_{100}^{\alpha_{100}} &= \sum_{i=1}^{5} \alpha_{8i} \deg \hat{I}_{8i} + \alpha_{100} \deg \hat{I}_{100} \\ &= \sum_{i=1}^{5} \alpha_{8i} \cdot 8i + \alpha_{100} \cdot 100,\end{aligned}$$

and is therefore in $A^{(8)}$ if and only if $8 \mid \alpha_{100} \cdot 100$ which is in turn equivalent to α_{100} being even. Hence, by Theorem 6.1 we have

$$\begin{aligned}A^{(8)} &= \mathbb{C}\left[\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}, \hat{I}_{40}, \hat{I}_{100}^2\right] \\ &= \mathbb{C}\left[\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}, \hat{I}_{40}, P\left(\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}, \hat{I}_{40}\right)\right] \\ &= \mathbb{C}\left[\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}, \hat{I}_{40}\right]\end{aligned}$$

and using Lemma 3.14, Lemma 5.3, the definition of the weighted projective space and Lemma 5.4, we obtain

$$\begin{aligned}\mathcal{M} = |\mathcal{O}(3)|^{ss}/\!\!/SL(4) &\cong \operatorname{Proj}(A) \cong \operatorname{Proj}\left(A^{(8)}\right) \cong \operatorname{Proj}\left(\mathbb{C}\left[\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}, \hat{I}_{40}\right]\right) \\ &\cong \mathbb{P}(8, 16, 24, 32, 40) \cong \mathbb{P}(1, 2, 3, 4, 5),\end{aligned}$$

which concludes the proof of the theorem. □

Before we see the longsome proof of Theorem 6.1 which goes back to [Bek82], it is, for the sake of clarity, best to outline the main ideas of the proof, and bring forward some necessary definitions and explicit calculations. Especially the last part will be fairly lengthy but hopefully treating it at this point helps presenting the actual proof of the theorem in a more coherent manner.

Since it is quite difficult to compute the invariants of $\mathcal{O}(V)$ with respect to the action of the rather big group G, the idea is now to simplify that computation by finding a much smaller subgroup H of G, at best finite, which acts on a much smaller subspace S of V such that $\mathcal{O}(V)^G = \mathcal{O}(S)^H$. Unfortunately this will not be possible, but equality will hold if we consider only a subset of $\mathcal{O}(S)^H$ which satisfies some additional condition. More concretely, the proof consists of the following steps:

Step 1 Find a linear subspace S of V and a finite subgroup H of G which acts on S and show that there is a birational map $S/\!\!/H \to V/\!\!/G$. In particular, every $F \in \mathcal{O}(S)^H$ can be continued to a rational function $\hat{F}$ on all of V (see the actual proof for details).

Step 2 Find another linear subspace S' of V such that

$$\mathcal{O}(V)^G = \{F \in \mathcal{O}(S)^H : \hat{F} \text{ is regular on } S'\}.$$

Step 3 Compute $\{F \in \mathcal{O}(S)^H : \hat{F} \text{ is regular on } S'\}$.

Recall that by Theorem 2.11 and Proposition 2.9, the categorical quotients $S /\!\!/ H$ and $V /\!\!/ G$ of the affine varieties S resp. V really exist. Note that $V /\!\!/ G$ is not the moduli space $\mathcal{M}$, which is computed by means of Lemma 3.14.

In order to find the subspace and the subgroup from Step 1, the ideal solution would be to have one representative in each orbit such that all representatives together form a sufficiently good subspace. This cannot be achieved, but there is a generalization which can.

Definition 6.3. Let $\pi \colon V \to V /\!\!/ G$ be the categorical quotient of V under the action of the linearly reductive group G. A linear (affine) subspace $W \subseteq V$ is said to be a *section* if $\dim W = \dim V /\!\!/ G$ and $\pi|_W \colon W \to V /\!\!/ G$ is dominant.

As in Section 4, we embed $\mathbb{P}^3$ into $\mathbb{P}^4$ as the hyperplane $V\left(\sum_{i=0}^4 X_i\right)$ and leave this embedding fixed.

The correct definitions for the linear subspaces of V turn then out to be

$$S := \{\lambda_0 X_0^3 + \lambda_1 X_1^3 + \lambda_2 X_2^3 + \lambda_3 X_3^3 + \lambda_4 X_4^3 \colon \lambda_i \in \mathbb{C}\} \text{ and}$$
$$S' := \{3\alpha_0 X_0^2 X_1 + \frac{1}{4}\alpha_1 X_1^3 + \alpha_2 X_2^3 + \alpha_3 X_3^3 + \alpha_4 X_4^3 \colon \alpha_i \in \mathbb{C}\}.$$

Before we check that S and S' are sections, let us work out the relation between the two subspaces.

Lemma 6.4. *The orbit of each element $f = 3\alpha_0 X_0^2 X_1 + \frac{1}{4}\alpha_1 X_1^3 + \alpha_2 X_2^3 + \alpha_3 X_3^3 + \alpha_4 X_4^3$ in the open subset $W := S' \smallsetminus V(\alpha_0 \alpha_1)$ intersects S. The coefficients λ_i of one representative in the intersection are*

$$\lambda_0 = z^{-3/8}\left(1 - \frac{z}{4}\right)^{3/4} \frac{8\alpha_0 z}{\left(z^{1/2}+2\right)^3}$$
$$\lambda_1 = z^{-3/8}\left(1 - \frac{z}{4}\right)^{3/4} \frac{-8\alpha_0 z}{\left(z^{1/2}-2\right)^3}$$
$$\lambda_i = z^{-3/8}\left(1 - \frac{z}{4}\right)^{3/4} \alpha_i, \quad i = 2, 3, 4,$$

where $z = \frac{\alpha_1}{\alpha_0}$.

Note that there is an ambiguity in taking the eighth root which corresponds to the fact, that the orbit $G.f$ intersects S several times and therefore brings about several representatives in S.

Proof. Given a cubic form $f \in W$, we have to find a linear transformation $g \in G = SL(4)$ such that $g.f \in S$.

At first, we shall only require g to be in $GL(4)$ and subsequently normalize it accordingly. In accordance with the fixed embedding of $\mathbb{P}^3$ into $\mathbb{P}^4$, it suffices to find a

$g \in GL(5)$, which leaves the hyperplane $V\left(\sum_{i=0}^{4} X_i\right)$ invariant and whose restriction to that hyperplane fulfils $g.f \in S$.

Making the ansatz

$$\begin{aligned} g(X_0) &= \alpha X_0 + \beta X_1 \\ g(X_1) &= \gamma X_0 + \delta X_1 \\ g(X_i) &= X_i,\ i = 2, 3, 4, \end{aligned}$$

the requirement

$$\begin{aligned} g.f &= 3\alpha_0 (\alpha X_0 + \beta X_1)^2 (\gamma X_0 + \delta X_1) + \frac{1}{4}\alpha_1 (\gamma X_0 + \delta X_1)^3 + \alpha_2 X_2^3 + \alpha_3 X_3^3 + \alpha_4 X_4^3 \\ &= \left(3\alpha_0\alpha^2 + \frac{\alpha_1}{4}\gamma^2\right)\gamma X_0^3 + \left(3\alpha_0\left(\alpha^2\delta + 2\alpha\beta\gamma\right) + \frac{3\alpha_1}{4}\gamma^2\delta\right) X_0^2 X_1 \\ &+ \left(3\alpha_0\left(\beta^2\gamma + 2\alpha\beta\delta\right) + \frac{3\alpha_1}{4}\gamma\delta^2\right) X_0 X_1^2 + \left(3\alpha_0\beta^2 + \frac{\alpha_1}{4}\delta^2\right)\delta X_1^3 \\ &+ \alpha_2 X_2^3 + \alpha_3 X_3^3 + \alpha_4 X_4^3 \in S \end{aligned} \tag{3}$$

yields the conditions

$$0 = 3\alpha_0\left(\alpha^2\delta + 2\alpha\beta\gamma\right) + \frac{3\alpha_1}{4}\gamma^2\delta \quad \text{and} \tag{4}$$

$$0 = 3\alpha_0\left(\beta^2\gamma + 2\alpha\beta\delta\right) + \frac{3\alpha_1}{4}\gamma\delta^2. \tag{5}$$

Furthermore, we must have

$$\sum_{i=0}^{4} X_i = 0 = \sum_{i=0}^{4} g(X_i) = (\alpha X_0 + \beta X_1) + (\gamma X_0 + \delta X_1) + X_2 + X_3 + X_4,$$

and equating coefficients, we obtain $\alpha + \gamma = \beta + \delta = 1$. Plugging in these relations into (4) and (5), we get

$$\begin{aligned} 0 &= 3\alpha_0\left(\alpha^2(1-\beta) + 2\alpha\beta(1-\alpha)\right) + \frac{3\alpha_1}{4}(1-\alpha)^2(1-\beta) \\ \Leftrightarrow -\alpha_0\left(\alpha^2(1-\beta) + 2\alpha\beta(1-\alpha)\right) &= \frac{\alpha_1}{4}(1-\alpha)^2(1-\beta) \end{aligned} \tag{6}$$

$$\begin{aligned} 0 &= 3\alpha_0\left(\beta^2(1-\alpha) + 2\alpha\beta(1-\beta)\right) + \frac{3\alpha_1}{4}(1-\alpha)(1-\beta)^2 \\ \Leftrightarrow -\alpha_0\left(\beta^2(1-\alpha) + 2\alpha\beta(1-\beta)\right) &= \frac{\alpha_1}{4}(1-\alpha)(1-\beta)^2. \end{aligned} \tag{7}$$

If we had $\beta = 1$, equation (7) would become

$$\alpha_0(1-\alpha) = 0 \Leftrightarrow \alpha = 1,$$

because $\alpha_0 \neq 0$. But this would mean

$$\det g = \alpha\delta - \beta\gamma = \alpha(1-\beta) - \beta(1-\alpha) = 0 - 0 = 0,$$

contradicting $g \in GL(4)$. Thus, $\beta \neq 1$.

Analogously, equation (6) in combination with $\det g \neq 0$ yields $\alpha \neq 1$. Hence, the right side of (7) (and thus also the left side) does not vanish and we can divide (6) by (7) to obtain

$$\begin{aligned}
&\frac{\alpha^2(1-\beta)+2\alpha\beta(1-\alpha)}{\beta^2(1-\alpha)+2\alpha\beta(1-\beta)} = \frac{1-\alpha}{1-\beta} \\
\Rightarrow\ &\alpha^2(1-\beta)^2 + 2\alpha\beta(1-\alpha)(1-\beta) = \beta^2(1-\alpha)^2 + 2\alpha\beta(1-\alpha)(1-\beta) \\
\Rightarrow\ &\alpha^2(1-\beta)^2 = \beta^2(1-\alpha)^2 .
\end{aligned}$$

Since $0 \neq \det g = \alpha\delta - \beta\gamma$ implies $\alpha(1-\beta) \neq \beta(1-\alpha)$ we can follow immediately

$$\alpha(1-\beta) = -\beta(1-\alpha). \tag{8}$$

Plugging in (8) into (6), we get

$$\begin{aligned}
0 &= \alpha_0\left(\alpha^2(1-\beta) - 2\alpha^2(1-\beta)\right) + \frac{\alpha_1}{4}(1-\alpha)^2(1-\beta) \\
&= -\alpha_0\alpha^2(1-\beta) + \frac{\alpha_1}{4}(1-\alpha)^2(1-\beta) \\
\Rightarrow\ \alpha_0\alpha^2 &= \frac{\alpha_1}{4}(1-\alpha)^2 \\
\Rightarrow\quad \alpha^2 &= \frac{z}{4}(1-\alpha)^2 ,
\end{aligned} \tag{9}$$

with $z := \frac{\alpha_1}{\alpha_0}$ as in the formulation of the theorem.

Similarly, we get be plugging in (8) into (7)

$$\begin{aligned}
0 &= \alpha_0\left(\beta^2(1-\alpha) - 2\beta^2(1-\alpha)\right) + \frac{\alpha_1}{4}(1-\alpha)(1-\beta)^2 = 0 \\
&= -\alpha_0\beta^2(1-\alpha) + \frac{\alpha_1}{4}(1-\alpha)(1-\beta)^2 \\
\Rightarrow\ \beta^2 &= \frac{z}{4}(1-\beta)^2 .
\end{aligned}$$

Taking square roots, one follows

$$\alpha = \pm\frac{z^{1/2}}{2}(1-\alpha) \Rightarrow \left(1 \pm \frac{2}{z^{1/2}}\right)\alpha = 1 \Rightarrow \alpha = \frac{z^{1/2}}{z^{1/2} \pm 2}$$

and analogously

$$\beta = \frac{z^{1/2}}{z^{1/2} \pm 2}.$$

Since $0 \neq \det g = \alpha(1-\beta) - \beta(1-\alpha)$, we have to choose different signs in the formulae for α and β. The particular choice is not important because there is also an ambiguity in the choice of the square root which corresponds to choosing different signs in the formulae.

Hence, we can just set

$$\alpha = \frac{z^{1/2}}{z^{1/2}+2} \quad \text{and} \quad \beta = \frac{z^{1/2}}{z^{1/2}-2}$$

and conclude

$$\gamma = 1 - \alpha = 1 - \frac{z^{1/2}}{z^{1/2}+2} = \frac{2}{z^{1/2}+2} \quad \text{and} \quad \delta = 1 - \beta = 1 - \frac{z^{1/2}}{z^{1/2}-2} = \frac{-2}{z^{1/2}-2}$$

to get the desired linear transformation $g \in GL(4)$.

For g to be in $SL(4)$, it needs to be normalized. Therefore, first compute

$$\begin{aligned}
\det g &= \alpha\delta - \beta\gamma \\
&= \frac{z^{1/2}}{z^{1/2}+2} \cdot \frac{-2}{z^{1/2}-2} - \frac{z^{1/2}}{z^{1/2}-2} \cdot \frac{2}{z^{1/2}+2} \\
&= \frac{4z^{1/2}}{4-z} = z^{1/2}\left(1 - \frac{z}{4}\right)^{-1}.
\end{aligned}$$

As g acts on a four-dimensional vector space, the desired transformation in $SL(4)$, obtained by multiplying g with $(\det g)^{-1/4}$ and by abuse of notation again called g, is given by

$$\begin{aligned}
g(X_0) &= z^{-1/8}\left(1 - \frac{z}{4}\right)^{1/4}\left(\frac{z^{1/2}}{z^{1/2}+2}X_0 + \frac{z^{1/2}}{z^{1/2}-2}X_1\right) \\
g(X_1) &= z^{-1/8}\left(1 - \frac{z}{4}\right)^{1/4}\left(\frac{2}{z^{1/2}+2}X_0 + \frac{-2}{z^{1/2}-2}X_1\right) \\
g(X_i) &= z^{-1/8}\left(1 - \frac{z}{4}\right)^{1/4} X_i,\ i = 2,3,4.
\end{aligned}$$

As the coordinates λ_i of $g.f \in S$, we obtain by equating coefficients in equation (3) (of course for the new, normalized $\alpha, \beta, \gamma, \delta$)

$$\begin{aligned}
\lambda_0 &= z^{-3/8}\left(1 - \frac{z}{4}\right)^{3/4} \frac{2}{z^{1/2}+2}\left(3\alpha_0 \frac{z}{\left(z^{1/2}+2\right)^2} + \frac{\alpha_0 z}{4}\frac{4}{\left(z^{1/2}+2\right)^2}\right) \\
&= z^{-3/8}\left(1 - \frac{z}{4}\right)^{3/4} \frac{8\alpha_0 z}{\left(z^{1/2}+2\right)^3} \\
\lambda_1 &= z^{-3/8}\left(1 - \frac{z}{4}\right)^{3/4} \frac{-2}{z^{1/2}-2}\left(3\alpha_0 \frac{z}{\left(z^{1/2}-2\right)^2} + \frac{\alpha_0 z}{4}\frac{4}{\left(z^{1/2}-2\right)^2}\right) \\
&= z^{-3/8}\left(1 - \frac{z}{4}\right)^{3/4} \frac{-8\alpha_0 z}{\left(z^{1/2}-2\right)^3} \\
\lambda_i &= z^{-3/8}\left(1 - \frac{z}{4}\right)^{3/4} \alpha_i, \quad i = 2,3,4.
\end{aligned}$$

□

A very similar calculation shows

Lemma 6.5. *The orbit of each element in an open subset of S intersects S'.*

Proof. Proceed like in the proof of the previous lemma. □

One can now easily follow

Lemma 6.6. *The subspaces S and S' are both sections.*

Proof. Let us first compute $\dim V/\!\!/G$. Since $V \cong \mathbb{A}^{20}_{\mathbb{C}}$ is an irreducible variety, so is the good categorical quotient $V/\!\!/G$, cf. Corollary 2.20. Together with Lemma 2.18, this implies that $\pi\colon V \to V/\!\!/G$ is a surjective morphism of irreducible varieties, hence there is a non-empty open subset $U \subseteq V/\!\!/G$ such that for all $\bar{f} \in U$, $\dim V/\!\!/G = \dim V - \dim \pi^{-1}\left(\bar{f}\right)$.

Since the stable points V^s form a dense open subset of V, a general $f \in V$ has a finite stabilizer under the action of G. Again by the irreducibility of V, one can even pick an $f \in V$ such that f lies in $V^s \cap \pi^{-1}(U)$. But as the quotient map on the stable points is that of a geometric quotient, one has $\pi^{-1}(\pi(f)) = G.f$. Then $\dim G.f = \dim G - \dim G_f = \dim G = \dim SL(4, \mathbb{C}) = 15$ because f has finite stabilizer (the first equality holds due to a basic property of G-varieties, cf. [TY05, Prop. 21.4.3]). Therefore,

$$\dim V/\!\!/G = \dim V - \dim \pi^{-1}(\pi(f)) = \dim V - \dim G.f = 20 - 15 = 5.$$

Since every cubic form in S resp. S' is determined by the coefficients λ_i resp. α_i, which can be chosen independently, one has $S \cong S' \cong \mathbb{A}^5$ and thus $\dim S = \dim S' = 5 = \dim V/\!\!/G$.

It remains to show that $\pi|_S\colon S \to V/\!\!/G$ and $\pi|_{S'}\colon S' \to V/\!\!/G$ are dominant.

Theorem 4.1 yields that a general $f \in V$ can be written in the form $f = \sum_{i=0}^4 \lambda_i l_i^3$ where the l_i are linear forms such that $\sum_{i=0}^4 l_i = 0$ and each four l_i are linearly independent. Given such an f, consider the linear transformation $\tilde{g}$ given by the matrix whose i-th row is exactly l_i, i.e. for all $(p_0, \ldots, p_3) \in \mathbb{A}^4$ one has $\tilde{g}((p_0, \ldots, p_3))_i = l_i((p_0, \ldots, p_3))$, $i = 0, \ldots, 3$. Since the l_i, $i = 0, \ldots, 3$ are linearly independent, one has $\tilde{g} \in GL(4)$, hence $g = \frac{1}{\det \tilde{g}}\tilde{g} \in G = SL(4)$ is well-defined. But then $g.f = f \circ g^{-1} = \sum_{i=0}^4 \lambda_i \det \tilde{g}^3 X_i^3$ in the standard embedding $\sum_{i=0}^4 X_i = 0$ of $\mathbb{P}^3$ into $\mathbb{P}^4$ described above and thus $g.f \in S$. In other words, the orbit of a general $f \in V$ intersects S. Since $\pi\colon V \to V/\!\!/G$ is constant on G-orbits and open on V^s by Corollary 2.19, the fibre of a general $\bar{f} \in V/\!\!/G$ has non-trivial intersection with S, hence $\pi|_S\colon S \to V/\!\!/G$ is dominant.

By Lemma 6.5, there is an open subset $W \subseteq S$ such that the orbit of every $f \in W$ intersects S', i.e. $\pi(W) \subseteq \pi(S')$. A basic fact about the closure operator known from point-set topology then leads to

$$\overline{\pi(S')} \supseteq \overline{\pi(W)} \supseteq \pi\left(\overline{W}\right) = \pi(S),$$

which gives rise to the chain of inclusions $V/\!\!/G \supseteq \overline{\pi(S')} \supseteq \overline{\pi(S)} = V/\!\!/G$. Consequently, $\overline{\pi(S')} = V/\!\!/G$ and $\pi|_{S'}\colon S' \to V/\!\!/G$ is dominant. □

There is still one other longer computation left which will be very useful in the proof.

For the rest of this section, let σ_i, $i = 1, \ldots, 5$ denote the elementary symmetric polynomials and v the Vandermonde determinant in the λ_j and τ_1, τ_2, τ_3 denote the elementary symmetric polynomials and w the Vandermonde determinant in $\alpha_2, \alpha_3, \alpha_4$. If $f \in S' \setminus V(\alpha_0\alpha_1)$, then Lemma 6.4 specifies the coordinates λ_i of one element $\tilde{f}$ in $G.f \cap S$. In the proof we shall need the explicit form of the elementary symmetric polynomials and the Vandermonde determinant in the coordinates λ_i of $\tilde{f}$ as functions of the coordinates α_i of f.

Lemma 6.7. *Let $f = 3\alpha_0 X_0^2 X_1 + \frac{1}{4}\alpha_1 X_1^3 + \alpha_2 X_2^3 + \alpha_3 X_3^3 + \alpha_4 X_4^3 \in S' \smallsetminus V(\alpha_0\alpha_1)$. Then the elementary symmetric polynomials σ_i, $i = 1, \dots, 5$ and the Vandermonde determinant v in the coordinates λ_i of the representative of $G.f$ in S chosen in Lemma 6.4 as functions of the coordinates α_i of f are given by*

$$\begin{aligned}
\sigma_1 &= z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\left(\tau_1 + 2\alpha_0 z\frac{1+\frac{3z}{4}}{\left(1-\frac{z}{4}\right)^3}\right),\\
\sigma_2 &= z^{-3/4}\left(1-\frac{z}{4}\right)^{3/2}\left(\tau_2 + 2\alpha_0\tau_1 z\frac{1+\frac{3z}{4}}{\left(1-\frac{z}{4}\right)^3} + \frac{\alpha_0^2 z^2}{\left(1-\frac{z}{4}\right)^3}\right),\\
\sigma_3 &= z^{-9/8}\left(1-\frac{z}{4}\right)^{9/4}\left(\tau_3 + 2\alpha_0\tau_2 z\frac{1+\frac{3z}{4}}{\left(1-\frac{z}{4}\right)^3} + \frac{\alpha_0^2\tau_1 z^2}{\left(1-\frac{z}{4}\right)^3}\right),\\
\sigma_4 &= z^{-1/2}\left(2\alpha_0\tau_3\left(1+\frac{3z}{4}\right) + \alpha_0^2\tau_2 z\right),\\
\sigma_5 &= \alpha_0^2\tau_3 z^{1/8}\left(1-\frac{z}{4}\right)^{3/4},\\
v &= \alpha_0\left(3+\frac{z}{4}\right)z^{-9/4}\left(1-\frac{z}{4}\right)^{-9/2}\left[\prod_{j=2}^{4} z^2\alpha_0^2 - 2\alpha_0\alpha_j z\left(1+\frac{3z}{4}\right) + \alpha_j^2\left(1-\frac{z}{4}\right)^3\right] w,
\end{aligned}$$

where $z = \frac{\alpha_1}{\alpha_0}$ as before.

Proof. With the explicit form of the λ_i from Lemma 6.4, one can first compute

$$\begin{aligned}
\lambda_0 \cdot \lambda_1 &= z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\frac{8\alpha_0 z}{\left(z^{1/2}+2\right)^3}\cdot z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\frac{-8\alpha_0 z}{\left(z^{1/2}-2\right)^3}\\
&= -z^{-3/4}\left(1-\frac{z}{4}\right)^{3/2}\frac{64\alpha_0^2 z^2}{(z-4)^3}\\
&= z^{5/4}\left(1-\frac{z}{4}\right)^{-3/2}\alpha_0^2\\
\lambda_0 + \lambda_1 &= z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\frac{8\alpha_0 z}{\left(z^{1/2}+2\right)^3} - z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\frac{8\alpha_0 z}{\left(z^{1/2}-2\right)^3}\\
&= z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4} 8\alpha_0 z\frac{\left(z^{1/2}-2\right)^3 - \left(z^{1/2}+2\right)^3}{(z-4)^3}\\
&= -z^{5/8}\left(1-\frac{z}{4}\right)^{-9/4}\frac{\alpha_0}{8}\left(z^{3/2} - 6z + 12z^{1/2} - 8 - z^{3/2} - 6z - 12z^{1/2} - 8\right)\\
&= -z^{5/8}\left(1-\frac{z}{4}\right)^{-9/4}\frac{\alpha_0}{8}(-12z-16)\\
&= 2\alpha_0 z^{5/8}\left(1-\frac{z}{4}\right)^{-9/4}\left(1+\frac{3z}{4}\right)\\
\lambda_1 - \lambda_0 &= z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\frac{-8\alpha_0 z}{\left(z^{1/2}-2\right)^3} - z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\frac{8\alpha_0 z}{\left(z^{1/2}+2\right)^3}
\end{aligned}$$

$$
\begin{aligned}
&= -z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4} 8\alpha_0 z \frac{\left(z^{1/2}+2\right)^3+\left(z^{1/2}-2\right)^3}{(z-4)^3} \\
&= z^{5/8}\left(1-\frac{z}{4}\right)^{-9/4} \frac{\alpha_0}{8}\left(z^{3/2}+6z+12z^{1/2}+8+z^{3/2}-6z+12z^{1/2}-8\right) \\
&= z^{5/8}\left(1-\frac{z}{4}\right)^{-9/4} \frac{\alpha_0}{8}\left(2z^{3/2}+24z^{1/2}\right) \\
&= z^{9/8}\left(1-\frac{z}{4}\right)^{-9/4} \alpha_0\left(3+\frac{z}{4}\right).
\end{aligned}
$$

One can now express the values of the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_5$ and the Vandermonde determinant v in the λ_i at $g.f$ in the coordinates α_i of f.

$$
\begin{aligned}
\sigma_1 &= \sum_{0\le i\le 4} \lambda_i \\
&= 2\alpha_0 z^{5/8}\left(1-\frac{z}{4}\right)^{-9/4}\left(1+\frac{3z}{4}\right)+\sum_{i=2}^{4} z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_i \\
&= z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\left(\tau_1+2\alpha_0 z\frac{1+\frac{3z}{4}}{\left(1-\frac{z}{4}\right)^3}\right), \\
\sigma_2 &= \sum_{0\le i<j\le 4} \lambda_i\lambda_j \\
&= \sum_{2\le i<j\le 4} \lambda_i\lambda_j + \left[(\lambda_0+\lambda_1)\sum_{2\le j\le 4}\lambda_j\right]+\lambda_0\lambda_1 \\
&= \sum_{2\le i<j\le 4} z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_i \cdot z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_j \\
&\quad + \left(2\alpha_0 z^{5/8}\left(1-\frac{z}{4}\right)^{-9/4}\left(1+\frac{3z}{4}\right)\right)\sum_{2\le j\le 4} z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_j \\
&\quad + z^{5/4}\left(1-\frac{z}{4}\right)^{-3/2}\alpha_0^2 \\
&= z^{-3/4}\left(1-\frac{z}{4}\right)^{3/2}\left(\tau_2+2\alpha_0\tau_1 z\frac{1+\frac{3z}{4}}{\left(1-\frac{z}{4}\right)^3}+\frac{\alpha_0^2 z^2}{\left(1-\frac{z}{4}\right)^3}\right), \\
\sigma_3 &= \sum_{0\le i<j<k\le 4} \lambda_i\lambda_j\lambda_k \\
&= \sum_{2\le i<j<k\le 4} \lambda_i\lambda_j\lambda_k + \left[(\lambda_0+\lambda_1)\sum_{2\le j<k\le 4}\lambda_j\lambda_k\right]+\left[\lambda_0\lambda_1\sum_{2\le k\le 4}\lambda_k\right] \\
&= \sum_{2\le i<j<k\le 4} z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_i\cdot z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_j\cdot z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_k \\
&\quad + \left(2\alpha_0 z^{5/8}\left(1-\frac{z}{4}\right)^{-9/4}\left(1+\frac{3z}{4}\right)\right)\sum_{2\le j<k\le 4} z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_j\cdot z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_k
\end{aligned}
$$

$$
\begin{aligned}
&\quad + z^{5/4}\left(1-\frac{z}{4}\right)^{-3/2}\alpha_0^2\sum_{2\le k\le 4} z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_k\\
&= z^{-9/8}\left(1-\frac{z}{4}\right)^{9/4}\left(\tau_3+2\alpha_0\tau_2 z\frac{1+\frac{3z}{4}}{\left(1-\frac{z}{4}\right)^3}+\frac{\alpha_0^2\tau_1 z^2}{\left(1-\frac{z}{4}\right)^3}\right),\\
\sigma_4 &= \sum_{0\le i\le 4}\prod_{j\ne i}\lambda_j\\
&= \left[(\lambda_0+\lambda_1)\sum_{2\le i<j<k\le 4}\lambda_i\lambda_j\lambda_k\right]+\left[\lambda_0\lambda_1\sum_{2\le j<k\le 4}\lambda_j\lambda_k\right]\\
&= 2\alpha_0 z^{5/8}\left(1-\frac{z}{4}\right)^{-9/4}\left(1+\frac{3z}{4}\right)\sum_{2\le i<j<k\le 4}\left[z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_i\cdot\right.\\
&\quad \left.\cdot z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_j\cdot z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_k\right]+\\
&\quad + z^{5/4}\left(1-\frac{z}{4}\right)^{-3/2}\alpha_0^2\sum_{2\le j<k\le 4} z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_j\cdot z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_k\\
&= z^{-1/2}\left(2\alpha_0\tau_3\left(1+\frac{3z}{4}\right)+\alpha_0^2\tau_2 z\right),\\
\sigma_5 &= \prod_{i=0}^{4}\lambda_i\\
&= z^{5/4}\left(1-\frac{z}{4}\right)^{-3/2}\alpha_0^2\cdot\prod_{i=2}^{4} z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_i\\
&= \alpha_0^2\tau_3 z^{1/8}\left(1-\frac{z}{4}\right)^{3/4},\\
v &= \prod_{0\le i<j\le 4}(\lambda_j-\lambda_i)\\
&= (\lambda_1-\lambda_0)\cdot\prod_{j=2}^{4}(\lambda_j-\lambda_0)(\lambda_j-\lambda_1)\cdot\prod_{2\le i<j\le 4}(\lambda_j-\lambda_i)\\
&= (\lambda_1-\lambda_0)\cdot\prod_{j=2}^{4}\left(\lambda_j^2-(\lambda_0+\lambda_1)\lambda_j+\lambda_0\lambda_1\right)\cdot\prod_{2\le i<j\le 4}(\lambda_j-\lambda_i)\\
&= z^{9/8}\left(1-\frac{z}{4}\right)^{-9/4}\alpha_0\left(3+\frac{z}{4}\right)\cdot\\
&\quad \cdot\prod_{j=2}^{4}\left[z^{-3/4}\left(1-\frac{z}{4}\right)^{3/2}\alpha_j^2-2\alpha_0 z^{5/8}\left(1-\frac{z}{4}\right)^{-9/4}\left(1+\frac{3z}{4}\right)\cdot\right.\\
&\quad \left.\cdot z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}\alpha_j+z^{5/4}\left(1-\frac{z}{4}\right)^{-3/2}\alpha_0^2\right]\cdot\\
&\quad \cdot\prod_{2\le i<j\le 4}\left(z^{-3/8}\left(1-\frac{z}{4}\right)^{3/4}(\alpha_j-\alpha_i)\right)
\end{aligned}
$$

$$= \alpha_0 \left(3 + \frac{z}{4}\right) \left[\prod_{j=2}^{4} z^{-3/4} \left(1 - \frac{z}{4}\right)^{-3/2} \left(\left(1 - \frac{z}{4}\right)^3 \alpha_j^2 - 2\alpha_0 z \left(1 + \frac{3z}{4}\right) \alpha_j + z^2 \alpha_0^2 \right) \right] w$$

$$= \alpha_0 \left(3 + \frac{z}{4}\right) z^{-9/4} \left(1 - \frac{z}{4}\right)^{-9/2} \left[\prod_{j=2}^{4} z^2 \alpha_0^2 - 2\alpha_0 \alpha_j z \left(1 + \frac{3z}{4}\right) + \alpha_j^2 \left(1 - \frac{z}{4}\right)^3 \right] w. \quad \square$$

Note that there are three errors in the original paper [Bek82], namely a 2 instead of a 3 in the power of the first denominator in the formula for σ_3 and an additional factor (-1) and $\left(3 + \frac{z}{8}\right)$ instead of $\left(3 + \frac{z}{4}\right)$ in the formula for v.

By means of σ_5, one can also express the set of cubic forms which admit a non-degenerate Sylvester equation (cf. Definition 4.2) with linear forms $l_i = X_i$ more elegantly as $S_0 := S \setminus V(\sigma_5)$.

The importance of S_0 lies in the possibility to apply Proposition 4.5. In what follows, we shall need

Lemma 6.8. *Let $n \in \mathbb{N}^*$. Then for all homogeneous polynomials f on $\mathbb{P}^n$ and all $g = (g_{ij}) \in SL(n+1)$ one has $H_{f \circ g} = H_f \circ g$.*

Proof. One first computes $\mathrm{He}(f \circ g)$.

By the chain rule, one has for all $p \in \mathbb{P}^n$

$$D(f \circ g)(p) = (Df)(gp) \cdot Dg(p) = (Df)(gp) \cdot g$$

and hence for all $j \in \{0, \ldots, n\}$

$$\frac{\partial (f \circ g)}{\partial X_j}(p) = \sum_{k=0}^{n} \frac{\partial f}{\partial X_k}(gp)\, g_{kj}.$$

Carrying out the same calculation for all $\frac{\partial f}{\partial X_k}$, one obtains for all $i, j \in \{0, \ldots, n\}$

$$\begin{aligned} \frac{\partial^2 (f \circ g)}{\partial X_i \partial X_j}(p) &= \frac{\partial}{\partial X_i} \left(\sum_{k=0}^{n} \frac{\partial f}{\partial X_k}(gp)\, g_{kj} \right) \\ &= \sum_{k=0}^{n} \frac{\partial}{\partial X_i} \left(\frac{\partial f}{\partial X_k}(gp)\, g_{kj} \right) \\ &= \sum_{l,k=0}^{n} \frac{\partial^2 f}{\partial X_l \partial X_k}(gp)\, g_{li} g_{kj} \\ &= \left(g^T \cdot ((\mathrm{He}(f))(gp)) \cdot g \right)_{ij}, \end{aligned}$$

whence $\mathrm{He}(f \circ g) = g^T \cdot (\mathrm{He}(f) \circ g) \cdot g$ and, as $g \in SL(n+1)$,

$$\begin{aligned} H_{f \circ g} &= \det(\mathrm{He}(f \circ g)) = \det\left(g^T \cdot (\mathrm{He}(f) \circ g) \cdot g\right) \\ &= (\det(g))^2 \cdot \det(\mathrm{He}(f) \circ g) = 1 \cdot \det(\mathrm{He}(f) \circ g) = H_f \circ g. \end{aligned} \qquad \square$$

Recall that the action of $G = SL(4)$ on $V = \mathbb{C}[X_0, \ldots, X_3]_3$ is given by $(g.f)(p) = f(g^{-1}p)\ \forall p \in \mathbb{P}^3$. In order to identify the group H acting on S, we shall need

Lemma 6.9. *Let $f \in S_0$ and $g \in G$ such that $g.f \in S$. Then g is an automorphism of the Sylvester pentahedron (cf. Definition 4.4).*

Proof. Lemma 6.8 implies that the Hessian surface of $g.f$ is given as the zero locus of the polynomial $H_{f \circ g^{-1}} = H_f \circ g^{-1}$ and therefore has the same degree as the Hessian surface of f. Since by Lemma 4.3, the degree of a Hessian of a degenerate Sylvester form is strictly less than the degree of a Hessian of a non-degenerate Sylvester form, the Sylvester form of $g.f$ (which exists as $g.f \in S$) must again be non-degenerate, i.e. $g.f \in S_0$.

Let furthermore $P := \{p_{ijk} \colon i, j, k \in \{0, \ldots, 4\}$ pairwise distinct$\}$. Then g maps the ten points $p_{ijk} \in P$ (cf. Definition 4.4) again into P: Indeed, since for all $p \in V\left(\sum_{i=0}^{4} X_i\right)$ one has $(g.f)(gp) = f\left(g^{-1}gp\right) = f(p)$ and likewise $H_{g.f}(gp) = H_f\left(g^{-1}gp\right) = H_f(p)$ and $f \in S_0$, the $g(p_{ijk})$ are singular points of $H_{g.f}$, but not of $g.f$ by Proposition 4.5. But $g.f \in S_0$ as well, so another application of Proposition 4.5 gives that $g(p_{ijk}) \in P$ for all i, j, k pairwise distinct. Since g is injective, it even induces a bijection $P \to P$.

One can now determine the image of the E_i (cf. Definition 4.4) under g. Let $i \in \{0, \ldots, 4\}$. There are $\binom{4}{2} = 6$ possibilities to choose two out of the four indices $\{0, \ldots, 4\} \smallsetminus \{i\}$, so let $\{j_l, k_l\}$, $l = 1, \ldots, 6$ be all subsets of $\{0, \ldots, 4\} \smallsetminus \{i\}$ with two elements. Then $p_{ij_lk_l} \in E_i$ for all $l \in \{1, \ldots, 6\}$. We already know that $g(p_{ij_lk_l}) \in P$ for all $l = 1, \ldots, 6$, let therefore $a_l, b_l, c_l \in \{0, \ldots, 4\}$, $l = 1, \ldots, 6$ such that $g(p_{ij_lk_l}) = p_{a_lb_lc_l}$. As g is injective, the sets $\{a_l, b_l, c_l\}$ are pairwise distinct.

One of the five indices $0, \ldots, 4$ must occur in at least four of these sets because if every index were to occur in at most three, one could only choose $\frac{5 \cdot 3}{3} = 5$ points from P. Let without loss of generality $a_1 = a_2 = a_3 = a_4 = a \in \{0, \ldots, 4\}$, in other words, $g(p_{ij_lk_l})$ lie in E_a for $l = 1, \ldots, 4$. One can now show that $g(E_i) = E_a$. First note that since $g \in SL(4)$, it maps linear subspaces to linear subspaces of the same dimension, hence it suffices to show that $g(E_i) \subseteq E_a$. As $g(p_{ij_lk_l}) \in E_a$, $l = 1, \ldots, 4$, it is enough to prove that the span $\overline{p_{ij_1k_1}, p_{ij_2k_2}, p_{ij_3k_3}, p_{ij_4k_4}}$ is equal to E_i. That is the case if the span is two-dimensional because it is a subspace of E_i.

In order to show this, choose $l_1, l_2 \in \{1, \ldots, 4\}$, $l_1 \neq l_2$ such that $\{j_{l_1}, k_{l_1}\} \cap \{j_{l_2} \cap k_{l_2}\} \neq \varnothing$, i.e. $p_{ij_{l_1}k_{l_1}}$ and $p_{ij_{l_2}k_{l_2}}$ have a common non-vanishing coordinate. This is of course always possible because every point $p_{ij_lk_l}$ is given by choosing two out of four indices $j_l, k_l \in \{0, \ldots, 4\} \smallsetminus \{i\}$. Without loss of generality let $j_{l_1} = j_{l_2} = j$. We get

$$\begin{aligned}
\dim(\overline{p_{ijk_{l_1}}, p_{ijk_{l_2}}}) &= \dim(p_{ijk_{l_1}}) + \dim(p_{ijk_{l_2}}) - \dim(p_{ijk_{l_1}} \cap p_{ijk_{l_2}}) \\
&= \dim(p_{ijk_{l_1}}) + \dim(p_{ijk_{l_2}}) - \dim(\varnothing) = 0 + 0 - (-1) = 1.
\end{aligned}$$

Furthermore, there is an $l_3 \in \{1, \ldots, 4\}$ such that $\{j_{l_3}, k_{l_3}\} \cap \{j\} = \varnothing$ because if j occurred as an index in $p_{ij_1k_1}, p_{ij_2k_2}, p_{ij_3k_3}, p_{ij_4k_4}$, two of the points would be equal. Since it follows from $\{j_{l_3}, k_{l_3}\} \cap \{j\} = \varnothing$ that the j-th coordinate of $p_{ijk_{l_1}}$, $p_{ijk_{l_2}}$ vanishes, but the j-th coordinate of $p_{ij_{l_3}k_{l_3}}$ does not, $p_{ij_{l_3}k_{l_3}}$ does not lie in the span $\overline{p_{ijk_{l_1}}, p_{ijk_{l_2}}}$ and hence

$$\begin{aligned}2 &= \dim(E_i) \geq \dim(\overline{p_{ij_1k_1}, p_{ij_2k_2}, p_{ij_3k_3}, p_{ij_4k_4}}) \geq \dim(\overline{p_{ijk_{l_1}}, p_{ijk_{l_2}}, p_{ij_{l_3}k_{l_3}}}) \\ &= \dim(\overline{p_{ijk_{l_1}}, p_{ijk_{l_2}}}) + \dim(p_{ij_{l_3}k_{l_3}}) - \dim(\overline{p_{ijk_{l_1}}, p_{ijk_{l_2}}} \cap p_{ij_{l_3}k_{l_3}}) \\ &= 1 + 0 - \dim(\varnothing) = 2,\end{aligned}$$

from which follows $\overline{p_{ij_1k_1}, p_{ij_2k_2}, p_{ij_3k_3}, p_{ij_4k_4}} = E_i$ and therefore $g(E_i) = E_a$.

Since i has been chosen arbitrarily and no two hyperplanes of the Sylvester pentahedron get mapped to the same hyperplane by the injectivity of g, one obtains the assertion. □

Before we come to the main proof, let us show

Lemma 6.10. *Let $F_i(z)$ be rational functions in z and $\tilde{F}_t(z) = \sum_{i=0}^{N} t^i F_i(z)$. If $\tilde{F}$ is regular as $z \to 0$ for all $t \neq 0$, F_i is regular as $z \to 0$ for all i.*

In the following proof, regular means always regular as $z \to 0$, i.e. the limit $\lim_{z\to 0} F_i(z)$ of the scalar functions $F_i(z)$ exists in $\mathbb{C}$.

Proof. By partial fraction decomposition, every rational function in z can, in a neighbourhood of 0, be expressed in a Laurent series with only finitely many negative powers of z. The part with the non-negative powers forms of course a regular function, so for all $i = 0, \ldots, N$ there are regular functions R_i, an $n_i \in \mathbb{N}$ and $c_{ij} \in \mathbb{C}$, $j = 1, \ldots, n_i$ such that $F_i(z) = R_i + \sum_{j=1}^{n_i} \frac{c_{ij}}{z^j}$. By adding zeros, we can assume without loss of generality that $n_0 = \ldots = n_N = n$. As

$$\tilde{F}_t(z) = \sum_{i=0}^{N} t^i F_i(z) = \sum_{i=0}^{N} t^i \left(R_i + \sum_{j=1}^{n} \frac{c_{ij}}{z^j} \right) = \left(\sum_{i=0}^{N} t^i R_i \right) + \sum_{j=1}^{n} \left(\sum_{i=0}^{N} t^i c_{ij} \right) \frac{1}{z^j}$$

is regular for all $t \neq 0$, we must have $\sum_{i=0}^{N} t^i c_{ij} = 0$ for all $t \neq 0$, $j = 1, \ldots, n$. Plugging in $N+1$ pairwise different values $t_0, \ldots, t_N$ for t, we get

$$\begin{pmatrix} 1 & t_0 & \ldots & t_0^N \\ \vdots & \vdots & & \vdots \\ 1 & t_N & \ldots & t_N^N \end{pmatrix} \cdot \begin{pmatrix} c_{0j} \\ \vdots \\ c_{Nj} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

But Vandermonde matrix on the left-hand side has determinant $\prod_{0 \leq k < l \leq N} (t_l - t_k) \neq 0$ because the t_k were chosen to be pairwise different. Hence, it is injective and we must have $c_{0j} = \ldots = c_{Nj} = 0$ for all $j = 1, \ldots, n$. It follows that $F_i = R_i$ is regular for all $i = 0, \ldots, N$. □

We are now finally ready to tackle the main proof of this chapter.

Proof of Theorem 6.1. Recall that the proof consists of the following steps:

Step 1 Find a finite subgroup H of G which acts on S and show that there is a birational map $S/\!\!/H \to V/\!\!/G$. In particular, every $F \in \mathcal{O}(S)^H$ can be continued to a rational function $\hat{F}$ on all of V (see the actual proof for details).

Step 2 Show that $\mathcal{O}(V)^G = \{F \in \mathcal{O}(S)^H : \hat{F} \text{ is regular on } S'\}$.

Step 3 Compute $\{F \in \mathcal{O}(S)^H : \hat{F} \text{ is regular on } S'\}$.

Let us now concern with the details.

Step 1 *Find a finite subgroup H of G which acts on S and show that there is a birational map $S/\!\!/H \to V/\!\!/G$.*

It seems natural to take as H the stabilizer subgroup of S in G, i.e. all $h \in G$ such that for all $f \in S$ one has $h.f \in S$. Since S_0 is dense in S, we can equivalently require that $h.f \in S$ for all $f \in S_0$. By the proof of Proposition 6.9, one gets already $h.f \in S_0$ for all $f \in S_0$.

Proposition 6.9 also yields that every $h \in H$ is an automorphism of the Sylvester pentahedron. Let us therefore determine the group of the automorphisms of the Sylvester pentahedron $\bigcup_{i=0}^4 E_i$ in $\mathbb{P}^4$ which are in G. Since a (linear) automorphism $h = (h_{ij})_{i,j=0,\ldots,4}$ of the Sylvester pentahedron maps every plane $E_i = V(X_i)$ into a plane $E_j = V(X_j)$ with $i, j \in \{0, \ldots, 4\}$, it has to be a permutation of the coordinates together with a rescaling of these. More concretely, there are $\lambda_0, \ldots, \lambda_4 \in \mathbb{C}^\times$ and a permutation $\sigma \in \mathfrak{S}_5$, $\sigma\colon \{0, \ldots, 4\} \to \{0, \ldots, 4\}$ such that

$$h_{ij} = \begin{cases} \lambda_i & \text{if } j = \sigma(i), \\ 0 & \text{if } j \neq \sigma(i). \end{cases}$$

On the other hand, h is in $G = SL(4)$, hence an automorphism of $\mathbb{P}^3$. As $\mathbb{P}^3$ is embedded into $\mathbb{P}^4$ as the hypersurface $V\left(\sum_{i=0}^4 X_i\right)$, h has to leave this hypersurface invariant. In particular, let for all $i, j \in \{0, \ldots, 4\}$, $i \neq j$ be $q_{ij} = [q_0 : \ldots : q_4]$ with $q_l = \delta_{li} - \delta_{lj}$. Then $q_{ij} \in V\left(\sum_{i=0}^4 X_i\right)$, hence also $g(q_{ij}) \in V\left(\sum_{i=0}^4 X_i\right)$. But $g(q_{ij})$ has λ_i as the $\sigma(i)$-th and $-\lambda_j$ as the $\sigma(j)$-th coordinate and zeros in the other coordinates, so

$$0 = \sum_{l=0}^4 (g(q_{ij}))_l = \lambda_i - \lambda_j \Leftrightarrow \lambda_i = \lambda_j.$$

Thus, $\lambda_0 = \ldots = \lambda_4 =: \lambda$ and h has to be of the form $h = \lambda P_\sigma$, where P_σ is the permutation matrix corresponding to σ. In fact, the map λP_σ leaves $V\left(\sum_{i=0}^4 X_i\right)$ invariant and $\lambda P_\sigma \in GL(4)$.

Recall that G acts on a four-dimensional vector space, so for λP_σ to be in G, we still need

$$\lambda^4 \cdot \operatorname{sgn}(\sigma) = \lambda^4 \cdot \det(P_\sigma) = \det(\lambda P_\sigma) = 1.$$

Let ζ_8 denote a primitive eighth root of unity.
If $\sigma \in \mathfrak{A}_5$, one must have $\lambda^4 = 1$ or, equivalently, $\lambda = \zeta_8^{2n}$, $n = 0, \ldots, 3$.
If $\sigma \in \mathfrak{S}_5 \setminus \mathfrak{A}_5$, one must have $\lambda^4 = -1$ or, equivalently, $\lambda = \zeta_8^{2n+1}$, $n = 0, \ldots, 3$.
Since every odd permutation is the product of an even permutation with the transposition $\langle 0, 1\rangle$ and $\left(\zeta_8 \cdot P_{\langle 0,1\rangle}\right)^2 = \zeta_8^2 \cdot \mathbb{1}$, the group of the automorphisms of the Sylvester pentahedron which lie in G is generated by $\mathfrak{A}_5$ and $\zeta_8 \cdot P_{\langle 0,1\rangle}$.

Obviously all these automorphisms leave the section S invariant, i.e. they map an $f \in S$ again into S, and are therefore in H. Since it has already been established in Proposition 6.9 that every $h \in H$ is an automorphism of the Sylvester pentahedron, we can identify H with the automorphisms of the Sylvester pentahedron which lie in G, so

$$H = \left\langle \mathfrak{A}_5, \zeta_8 \cdot P_{\langle 0,1\rangle} \right\rangle .$$

In particular, H is a finite group of order $|H| = |\mathfrak{A}_5| \cdot |\left\langle \zeta_8 P_{\langle 0,1\rangle} \right\rangle| = \frac{5!}{2} \cdot 8 = 60 \cdot 8 = 480$. For any $f \in S_0$, the orbit $H.f$ is exactly the intersection $G.f \cap S$ because $g.f \in S$ for any $g \in G$ already implies that g is an automorphism of the Sylvester pentahedron and therefore in H.

Next, one sees that every $f \in S_0$ is stable. By Remark 3.12, there are, after a linear change of coordinates, the following possibilities for the determinant of the Hessian matrix of a non-stable $f \in V$. In all cases, $h\left(X_1, X_2, X_3\right)$ denotes a homogeneous polynomial of degree 3.

(a) $f\left(X_0, X_1, X_2, X_3\right) = h\left(X_1, X_2, X_3\right)$

$$\mathrm{He}\,(f) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & * & \\ 0 & & & \end{pmatrix}$$

$H_f = \det\left(\mathrm{He}\,(f)\right) = 0$

(b) $f = X_0 X_3^2 + h\left(X_1, X_2, X_3\right)$

$$\mathrm{He}\,(f) = \begin{pmatrix} 0 & 0 & 0 & 2X_3 \\ 0 & & & \\ 0 & & * & \\ 2X_3 & & & \end{pmatrix}$$

$H_f = \det\left(\mathrm{He}\,(f)\right) = 4X_3^2 \cdot \alpha\left(X_0, X_1, X_2, X_3\right)$ for some quadratic polynomial $\alpha\left(X_0, X_1, X_2, X_3\right)$ since by the Leibniz rule every term needs to contain a coefficient from the first row and from the first column as a factor.

(c) $f = X_0 X_2 X_3 + h\left(X_1, X_2, X_3\right)$

$$\mathrm{He}\,(f) = \begin{pmatrix} 0 & 0 & X_3 & X_2 \\ 0 & & & \\ X_3 & & * & \\ X_2 & & & \end{pmatrix}$$

$H_f = \det\left(\mathrm{He}\,(f)\right) = X_2^2 \cdot \beta_1\left(X_0, X_1, X_2, X_3\right) + X_2 X_3 \cdot \beta_2\left(X_0, X_1, X_2, X_3\right) + X_3^2 \cdot \beta_3\left(X_0, X_1, X_2, X_3\right)$ for some quadratic polynomials $\beta_i\left(X_0, X_1, X_2, X_3\right),\ i = 1, 2, 3$ since again by the Leibniz rule every term needs to contain a coefficient from the first row and from the first column as a factor.

In the first case, the Hessian surface of f, which is given by the polynomial 0, has no singularities at all. Thus, $f \notin S_0$ by Proposition 4.5. In the second and third case, every term in the derivation of H_f with respect to every X_i contains a factor X_2 or X_3, hence the Hessian surface of f has the line l_{23} (cf. Definition 4.4) of singular points. Again by Proposition 4.5, $f \notin S_0$.

In consequence, every $f \in S_0$ is stable. In particular, $\pi^{-1} \circ \pi (f) = G.f$ holds for every $f \in S_0$. Using this and the form of H, we can show the assertion of Step 1.

For all $f \in S_0$, the fibre of $\pi|_S (f)$ under $(\pi|_S)^{-1}$ is the orbit $H.f$. Indeed,

$$(\pi|_S)^{-1} (\pi|_S (f)) = (\pi|_S)^{-1} (\pi (f)) = \pi^{-1} (\pi (f)) \cap S = G.f \cap S = H.f$$

because it is known from Lemma 6.9 that $g.f \in S$ already implies $g \in H$. Hence, the quotient map $\pi|_S /\!\!/ H \colon S /\!\!/ H \to V /\!\!/ G$, which is well-defined since H is a subgroup of G, is one-to-one on the set $S_0 /\!\!/ H$. As $S_0 \subset S$ is open, so is $S_0 /\!\!/ H \subset S /\!\!/ H$ by Corollary 2.19. Since S is a section, $\pi|_S /\!\!/ H$ is also dominant and therefore birational.

As a birational map between two varieties induces an isomorphism of the corresponding function fields, we have $K (S /\!\!/ H) \cong K (V /\!\!/ G)$. On the other hand, Lemma 2.11 and its proof show that $K (S)^H \cong K (S /\!\!/ H)$ and $K (V /\!\!/ G) \subseteq K (V)^G$. Hence, we have the inclusion

$$\mathcal{O} (S)^H \subseteq K (S)^H = K (S /\!\!/ H) \cong K (V /\!\!/ G) \subseteq K (V)^G .$$

Denote in the following the image of an $F \in \mathcal{O} (S)^H$ under this inclusion by $\hat{F}$. Note that in particular $\hat{F} (gs) = F (s)$ for all $s \in S$, $g \in G$.

Before we proceed to Step 2, let us look at some examples of invariants in $\mathcal{O} (S)^H$ which are continued to invariants in $\mathcal{O} (V)^G$. Define

$$\begin{gathered} I_8 = \sigma_4^2 - 4\sigma_3\sigma_5,\ I_{16} = \sigma_5^3\sigma_1,\ I_{24} = \sigma_5^4\sigma_4,\ I_{32} = \sigma_5^6\sigma_2, \\ I_{40} = \sigma_5^8,\ I_{48} = \sigma_5^9\sigma_3 \text{ and } I_{100} = \sigma_5^{18}v, \end{gathered}$$

where the σ_i are the elementary symmetric functions and v is the Vandermonde determinant in the parameters λ_i of the section S.

Since $\deg \sigma_i = i$ and $\deg v = 10$, we have $\deg I_n = n$, $n = 8, 16, 24, 32, 40, 48, 100$. In order to show that the I_n are invariant under $H = \left\langle \mathfrak{A}_5, \zeta_8 \cdot P_{\langle 0,1 \rangle} \right\rangle$, we have to show that they are invariant under even permutations of the λ_i and under swapping λ_0 and λ_1 followed by multiplying every λ_i with ζ_8.

In the cases $n = 8, 16, 24, 32, 40, 48$, the I_n are polynomials in the elementary symmetric polynomials, thus invariant under any permutation of the coordinates. They are also invariant under the multiplication of the coordinates with an eighth root of unity because their degree is divisible by 8.

In the case $n = 100$, σ_5 and v are invariant under even permutations. Swapping λ_0 and λ_1 leaves σ_5^{18} invariant and creates an additional factor -1 in v. That factor is cancelled out by a second -1 which arises at the multiplication of the λ_i with ζ_8 since $\deg I_{100} = 100$ and $\zeta_8^{100} = -1$.

Therefore, $I_n \in \mathcal{O} (S)^H$, $n = 8, 16, 24, 32, 40, 48, 100$, and they can be continued to $\hat{I}_n \in K (V)^G$, which are the invariants appearing in the formulation of the theorem. Note also that $I_{48} = \sigma_5^9\sigma_3 = \frac{1}{4}\left(\sigma_5^8\sigma_4^2 - \sigma_5^8\sigma_4^2 + \sigma_5^8 \cdot 4\sigma_3\sigma_5\right) = \frac{1}{4}\left(I_{24}^2 - I_{40}I_8\right)$, which accords with the assertions of the theorem. Furthermore, $S_0 = S \setminus V(\sigma_5) = S \setminus V\left(\sigma_5^8\right) = S \setminus V(I_{40})$.

Step 2 *Show that $\mathcal{O}(V)^G = \{F \in \mathcal{O}(S)^H : \hat{F}$ is regular on $S'\}$.*

This step is divided into three parts:

Step 2a The set of points whose orbits do not intersect S_0 is given by the hypersurface $V(\hat{I}_{40})$.

Step 2b The section S' intersects the orbits of an open set of points of $V(\hat{I}_{40}) \subset V$.

Step 2c If $F \in \mathcal{O}(S)^H$, then $\hat{F} \in \mathcal{O}(V)^G$ if and only if $\hat{F}$ is regular on S'.

Step 2a *The set of points whose orbits do not intersect S_0 is given by the hypersurface $V(\hat{I}_{40})$.*

Since $S \cong \mathbb{A}^5$, the subvariety $S_0 = S \setminus V(I_{40})$ is affine with coordinate ring $\mathcal{O}(S_0) = \mathbb{C}[\lambda_0, \ldots, \lambda_4]_{I_{40}}$ (cf. [Har77, Lemma I.4.2]), which is normal as the localization of a normal ring. Hence, S_0 and V are normal and irreducible affine varieties, and so are their quotients $S_0/\!\!/H$ and $V/\!\!/G$ by Corollary 2.13 and Corollary 2.20. As it is also known from Step 1 that $\pi|_{S_0}/\!\!/H\colon S_0/\!\!/H \to V/\!\!/G$ is birational and that $\pi|_{S_0}/\!\!/H$ fulfils $(\pi|_{S_0}/\!\!/H)^{-1}\circ(\pi|_{S_0}/\!\!/H)\left(\bar{f}\right) = \{\bar{f}\}$ for all $\bar{f} \in S_0/\!\!/H$ and is therefore quasi-finite, it follows from a corollary of Zariski's main theorem on birational transformations that $\pi|_{S_0}/\!\!/H$ is an open embedding, see e.g. [Now96, Cor. 1].

In particular, $\pi|_{S_0}/\!\!/H\,(S_0/\!\!/H) \subseteq V/\!\!/G$ is open affine. An application of point-set topology yields again

$$\overline{\pi|_{S_0}/\!\!/H\,(S_0/\!\!/H)} = \overline{\pi|_{S_0}(S_0)} = \overline{\pi|_S(S_0)} \supseteq \pi|_S\left(\overline{S_0}\right) = \pi|_S(S)$$

and by the dominance of $\pi|_S$, $\overline{\pi|_{S_0}/\!\!/H\,(S_0/\!\!/H)} \supseteq \overline{\pi|_S(S)} = V/\!\!/G$, from which follows that $\pi|_{S_0}/\!\!/H\,(S_0/\!\!/H)$ is dense in $V/\!\!/G$. It is known that the complement of a dense open affine subset in any variety is of pure codimension one, cf. e.g. [RV04, Cor. 2.4]. In particular, this holds for the complement of $\pi|_{S_0}/\!\!/H\,(S_0/\!\!/H)$ in $V/\!\!/G$, which is exactly the image under π of the set of points whose orbits do not intersect S_0 (because all points in S_0 are stable). It has to be shown that this complement is the zero locus of the G-invariant function $\hat{I}_{40} \in K(V/\!\!/G) \subseteq K(V)^G$ and that already $\hat{I}_{40} \in \mathcal{O}(V/\!\!/G) \cong \mathcal{O}(V)^G$.

Let $X \subseteq (V/\!\!/G) \setminus (\pi|_{S_0}/\!\!/H)(S_0/\!\!/H)$ be an irreducible component of the complement. Since X is of pure codimension one, it is a hypersurface given by a non-constant irreducible polynomial $f \in \mathcal{O}(V/\!\!/G) \cong \mathcal{O}(V)^G$ (of course, irreducible means here irreducible as a polynomial in $\mathcal{O}(V)^G$), cf. [Har77, Prop. I.1.13]. Since the inclusion $\mathcal{O}(V/\!\!/G) \subset K(V/\!\!/G) \cong K(S/\!\!/H)$ is given by the restriction of a function on V to S and the restriction of a regular function is again a regular function, we have an injective map $\varphi\colon \mathcal{O}(V/\!\!/G) \to \mathcal{O}(S/\!\!/H)$, $\alpha \mapsto \alpha|_S$. Then $V(\varphi(f)) \subseteq (S/\!\!/H) \setminus (S_0/\!\!/H) = V(I_{40})$ in $S/\!\!/H$ (use again $\mathcal{O}(S/\!\!/H) \cong \mathcal{O}(S)^H$ to regard I_{40} as a regular function on $S/\!\!/H$). But $V(\varphi(f))$ is of codimension one in $S/\!\!/H$. Hence, every other irreducible closed subset of $S/\!\!/H$ containing it is either equal or the whole space $S/\!\!/H$.

Let us show that I_{40} is irreducible in $\mathcal{O}(S/\!\!/H) \cong \mathcal{O}(S)^H$. It suffices to show that any regular function on $S/\!\!/H$ which does not vanish on $S_0/\!\!/H$ has degree greater or equal to 40. Let $r \in \mathcal{O}(S)^H$ with $V(r) \subseteq V(\sigma_5)$. Then we have $\sqrt{(\sigma_5)} \subseteq \sqrt{(r)}$ by the Nullstellensatz, i.e. there exists an $n \in \mathbb{N}$ and an $r_1 \in \mathcal{O}(S)$ such that $\sigma_5^n = r_1 \cdot r$. Recall that σ_5 is the fifth elementary symmetric function in the coordinates $\lambda_0, \ldots, \lambda_4$ of S, so there are $n_0, \ldots, n_4 \in \{0, \ldots, n\}$ such that $\lambda_0^{n_0} \cdot \ldots \cdot \lambda_4^{n_4} = r$. But r has to be invariant under the action of $H = \langle \mathfrak{A}_5, \zeta_8 \cdot P_{\langle 0,1 \rangle} \rangle$, so $n_0 = \ldots = n_4$ and $8 \mid \deg r$. The first condition implies $r = \sigma_5^m$ for some $m \leq n$ and then the second condition implies that $\sigma_5^8 \mid r$, or $\deg r \geq 40$. Hence, I_{40} is irreducible in $\mathcal{O}(S)^H$ and $V(\varphi(f)) = V(I_{40})$ since $V(I_{40}) \neq S/\!\!/H$.

Let us show next that we even have $\varphi(f)^m = I_{40}^n$ up to units for some $m, n \in \mathbb{N}$. By yet another application of the Nullstellensatz, there are $\nu_1, \nu_2 \in \mathbb{N}$ and $\alpha_1, \alpha_2 \in \mathcal{O}(S)$ such that $\varphi(f)^{\nu_1} = \alpha_1 \cdot I_{40}$ and $I_{40}^{\nu_2} = \alpha_2 \cdot \varphi(f)$. From $\varphi(f), I_{40} \in (\mathcal{O}(S))^H$ we can follow already $\alpha_1, \alpha_2 \in (\mathcal{O}(S))^H$. On the other hand, we have $(I_{40})^{\nu_1\nu_2} = (\alpha_2 \cdot \varphi(f))^{\nu_1} = \alpha_1 \alpha_2^{\nu_1} \cdot I_{40}$. As in the previous paragraph, $\alpha_1 \in \mathcal{O}(S)^H$ and $\alpha_1 \mid I_{40}^{\nu_1\nu_2} = \sigma_5^{8\nu_1\nu_2}$ implies that α_1 is some power of $\sigma_5^8 = I_{40}$ up to units. Let now $n \in \mathbb{N}$ and $\mu \in \mathbb{C}^\times$ such that $\alpha_1 = \mu \cdot I_{40}^{n-1}$ and $m = \nu_1$, then $\varphi(f)^m = \alpha_1 \cdot I_{40} = \mu \cdot I_{40}^n$.

Under the inclusion $\mathcal{O}(S)^H \subseteq K(V)^G$ this equality becomes $f^m = \widehat{\varphi(f)^m} = \widehat{\mu \cdot I_{40}^n} = \mu \cdot \hat{I}_{40}^n$ since the map φ is injective. Therefore, we have $V(f) = V(\hat{I}_{40})$ and $\hat{I}_{40} \in \mathcal{O}(V)^G$ because $f \in \mathcal{O}(V)^G$. As $V(f)$ was an arbitrary irreducible component of the complement of $(\pi|_{S_0}/\!\!/H)(S_0/\!\!/H)$ in $V/\!\!/G$, Step 2a is shown.

Step 2b *The section S' intersects the orbits of an open set of points of $V(\hat{I}_{40}) \subset V$.*

Recall that

$$S' := \{3\alpha_0 X_0^2 X_1 + \frac{1}{4}\alpha_1 X_1^3 + \alpha_2 X_2^3 + \alpha_3 X_3^3 + \alpha_4 X_4^3 \colon \alpha_i \in \mathbb{C}\}$$

and define the hypersurface $N := V(\alpha_1) \subset S'$. Our aim is to show that $\overline{G.N} = V(\hat{I}_{40})$. Since the orbits of varieties under algebraic group actions are constructible sets, this of course implies that $G.N$ contains a dense open subset of $V(\hat{I}_{40}) = \overline{G.N}$ and therefore proves the assertion of Step 2b.

Let us show first that $G.N \subseteq V(\hat{I}_{40})$. By Step 2a, it suffices to prove that that the orbits of points of N do not intersect S_0. Let $f = 3\alpha_0 X_0^2 X_1 + \alpha_2 X_2^3 + \alpha_3 X_3^3 + \alpha_4 X_4^3 \in N$. We want to compute the Hessian surface $V(H_f)$ and then use Lemma 4.3 and Proposition 4.5. As in Lemma 4.3, computing the Hessian requires techniques from Classical Algebraic Geometry which are too extensive to be introduced in this thesis. Nonetheless, the calculation will be given here since the author knows of no presence of it in the literature. The required knowledge is treated in [Dol12, Ch. 1].

The cubic f defines a cubic surface in $\mathbb{P}^3$, again embedded into $\mathbb{P}^4$ as the hyperplane $V\left(\sum_{i=0}^4 X_i\right)$. Its corresponding Hessian surface $V(H_f)$ is the locus of points $a = [a_0 : a_1 : a_2 : a_3 : a_4] \in V\left(\sum_{i=0}^4 X_i\right)$ such that the polar quadric

$$P_a(V(f)) = \sum_{i=0}^{4} a_i \frac{\partial f}{\partial X_i} = 6a_0\alpha_0 X_0 X_1 + 3a_1\alpha_0 X_0^2 + \sum_{i=2}^{4} 3a_i\alpha_i X_i^2 = 0$$

is singular. In the hypersurface $V\left(\sum_{i=0}^{4} X_i\right)$ this means that the polar $a_1\alpha_0 X_0^2 + 2a_0\alpha_0 X_0 X_1 + \sum_{i=2}^{4} a_i\alpha_i X_i^2 = 0$ is tangent to $V\left(\sum_{i=0}^{4} X_i\right)$, or equivalently that the point $[1:1:1:1:1]$ lies in the dual quadric $\frac{2}{a_0\alpha_0}U_0U_1 - \frac{a_1}{a_0^2\alpha_0}U_1^2 + \sum_{i=2}^{4}\frac{1}{a_i\alpha_i}U_i^2 = 0$, where the terms with $\alpha_i = 0$ are left out. Multiplication with the common denominator now gives the equation $H_f = 0$ which defines the Hessian surface on all of $\mathbb{P}^3$.

Thus, if $\alpha_i = 0$ for some $i \in \{0,2,3,4\}$, the Hessian surface has degree strictly less than 4. By Lemma 6.8, for every $g \in G$ the Hessian of $g.f$ is the zero locus of $H_{f\circ g^{-1}} = H_f \circ g^{-1}$ and therefore also of degree strictly less than 4. In particular, $g.f \notin S_0$ for all $g \in G$ by Lemma 4.3.

If $\alpha_i \neq 0$ for $i = 0,2,3,4$, the polynomial H_f defining the Hessian surface of f is (up to a non-zero scalar factor) given by

$$H_f = (2X_0 - X_1)\prod_{j=2}^{4}\alpha_j X_j + X_0\sum_{i=2}^{4}\prod_{j\neq 1,i}\alpha_j X_j.$$

Let now $p = [p_0 : p_1 : p_2 : p_3 : p_4] \in V\left(\sum_{i=0}^{4} X_i\right)$. The surface of f resp. its Hessian surface is given by $f = \sum_{i=0}^{4} X_i = 0$ resp. $H_f = \sum_{i=0}^{4} X_i = 0$ and therefore singular at p if and only if

$$\operatorname{rank}\begin{pmatrix} \frac{\partial f}{\partial X_0}(p) & \frac{\partial f}{\partial X_1}(p) & \frac{\partial f}{\partial X_2}(p) & \frac{\partial f}{\partial X_3}(p) & \frac{\partial f}{\partial X_4}(p) \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = 1 \quad \text{resp.}$$
$$\operatorname{rank}\begin{pmatrix} \frac{\partial H_f}{\partial X_0}(p) & \frac{\partial H_f}{\partial X_1}(p) & \frac{\partial H_f}{\partial X_2}(p) & \frac{\partial H_f}{\partial X_3}(p) & \frac{\partial H_f}{\partial X_4}(p) \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = 1,$$

which is in turn equivalent to $\frac{\partial f}{\partial X_0}(p) = \frac{\partial f}{\partial X_1}(p) = \frac{\partial f}{\partial X_2}(p) = \frac{\partial f}{\partial X_3}(p) = \frac{\partial f}{\partial X_4}(p) = c_1$ resp. $\frac{\partial H_f}{\partial X_0}(p) = \frac{\partial H_f}{\partial X_1}(p) = \frac{\partial H_f}{\partial X_2}(p) = \frac{\partial H_f}{\partial X_3}(p) = \frac{\partial H_f}{\partial X_4}(p) = c_2$ for some $c_1, c_2 \in \mathbb{C}$.

If $p \in V\left(\prod_{i=0}^{4} X_i\right)$, p cannot be a singular point of $V(f)$ since $p_i = 0$ for some $i \in \{0,\ldots,4\}$ implies $c_1 = 0$ and therefore $p_j = 0$ for $j = 0,2,3,4$, but this is impossible as $p \in V\left(\sum_{i=0}^{4} X_i\right)$.

Suppose that $p \in V\left(\prod_{i=0}^{4} X_i\right)$ is a singular point of the Hessian surface.
If $p_0 = 0$, then $-\prod_{j=2}^{4}\alpha_j p_j = \frac{\partial H_f}{\partial X_1}(p) = \frac{\partial H_f}{\partial X_0}(p) = 2\prod_{j=2}^{4}\alpha_j p_j$, hence $\prod_{j=2}^{4}\alpha_j p_j = 0$ and there is a $j_1 \in \{2,3,4\}$ such that $p_{j_1} = 0$. Furthermore, $0 = -\prod_{j=2}^{4}\alpha_j p_j = \frac{\partial H_f}{\partial X_1}(p) = \frac{\partial H_f}{\partial X_{j_1}}(p) = -p_1\prod_{j\neq 0,1,j_1}\alpha_j p_j$, so there is a $j_2 \in \{1,2,3,4\}$, $j_2 \neq j_1$ such that $p_{j_2} = 0$. Thus, $p = p_{0j_1j_2}$ for some $j_1, j_2 \in \{1,2,3,4\}$, $j_1 \neq j_2$.
If $p_0 \neq 0$ and $p_1 = 0$, then $-\prod_{j=2}^{4}\alpha_j p_j = \frac{\partial H_f}{\partial X_1}(p) = \frac{\partial H_f}{\partial X_0}(p) = 2\prod_{j=2}^{4}\alpha_j p_j + 2\sum_{i=2}^{4}\prod_{j\neq 1,i}\alpha_j p_j$, whence it follows that $\sum_{i=2}^{4}\prod_{j\neq 1,i}\alpha_j p_j = -\frac{3}{2}\prod_{j=2}^{4}\alpha_j p_j$. Since p is in the zero locus of H_f, we must have $0 = H_f(p) = (2p_0 - p_1)\prod_{j=2}^{4}\alpha_j p_j + p_0\sum_{i=2}^{4}\prod_{j\neq 1,i}\alpha_j p_j = \frac{p_0}{2}\prod_{j=2}^{4}\alpha_j p_j$. As $p_0 \neq 0$, there must be a $j_1 \in \{2,3,4\}$ such

that $p_{j_1} = 0$. As before, this implies that $c_2 = \frac{\partial H_f}{\partial X_1}(p) = 0$. Thus, $0 = \frac{\partial H_f}{\partial X_0}(p) = 2\prod_{j\neq 1,j_1} \alpha_j p_j$, so there is a $j_2 \in \{2,3,4\}$, $j_2 \neq j_1$ such that $p_{j_2} = 0$. But now $0 = \frac{\partial H_f}{\partial X_{j_1}}(p) = p_0 \alpha_{j_1} \prod_{j\neq 1,j_1,j_2} \alpha_j p_j$, which is impossible because at most three coordinates of $p \in V\left(\sum_{i=0}^4 X_i\right)$ can vanish. Hence, we must have either $p_0 = 0$ or $p_0, p_1 \neq 0$.
If $p_0, p_1 \neq 0$ and $p_{j_1} = 0$ for some $j_1 \in \{2,3,4\}$, it follows again that $c_2 = \frac{\partial H_f}{\partial X_1}(p) = -\prod_{j=2}^4 \alpha_j p_j = 0$ and further $0 = \frac{\partial H_f}{\partial X_0}(p) = 2\prod_{j\neq 1,j_1} \alpha_j p_j$. Since $p_0 \neq 0$, there must be a $j_2 \in \{2,3,4\}$, $j_2 \neq j_1$ such that $p_{j_2} = 0$. Therefore, $0 = \frac{\partial H_f}{\partial X_{j_1}}(p) = \alpha_{j_1} p_0 \prod_{j\neq 1,j_1,j_2} \alpha_j p_j$, which implies $p = p_{234}$ because $p_0 \neq 0$.
On the other hand it is checked easily that p_{0ij}, $i,j \in \{1,2,3,4\}$ and p_{234} really are singular points of $V(H_f)$, so we can conclude that these are the only singular points of the Hessian surface in $V\left(\prod_{i=0}^4 X_i\right)$.

Now we consider the case $p \in V\left(\sum_{i=0}^4 X_i\right) \setminus V\left(\prod_{i=0}^4 X_i\right)$.
As we have seen above, in order for the Jacobian matrix of the surface of f to have rank 1 at p, we must have $\frac{\partial f}{\partial X_i}(p) = c_1$ with $c_1 \in \mathbb{C}$ for all $i \in \{0,\ldots,4\}$. From $\frac{\partial f}{\partial X_0}(p) = 6\alpha_0 p_0 p_1$, $\frac{\partial f}{\partial X_1}(p) = 3\alpha_0 p_0^2$ and $\frac{\partial f}{\partial X_i}(p) = 3\alpha_i p_i^2$, $i = 2,3,4$ it follows that this is the case if and only if $p = \left[\frac{\varepsilon_0}{\sqrt{\alpha_0}} : \frac{\varepsilon_0}{2\sqrt{\alpha_0}} : \frac{\varepsilon_2}{\sqrt{\alpha_2}} : \frac{\varepsilon_3}{\sqrt{\alpha_3}} : \frac{\varepsilon_4}{\sqrt{\alpha_4}}\right]$, where $\varepsilon_i \in \{-1,1\}$ for $i = 0,2,3,4$ (recall $p_i \neq 0$).
On the open set $V\left(\sum_{i=0}^4 X_i\right) \setminus V\left(\prod_{i=0}^4 X_i\right)$ (open as a subset of $\mathbb{P}^3 \cong V\left(\sum_{i=0}^4 X_i\right)$), the Hessian surface of f is given by $\frac{2}{\alpha_0 X_0} - \frac{X_1}{\alpha_0 X_0^2} + \sum_{i=2}^4 \frac{1}{\alpha_i X_i} = 0$. In order for the Jacobian matrix of H_f to have rank 1, we must have $\frac{\partial H_f}{\partial X_i}(p) = c_2$ with $c_2 \in \mathbb{C}$ for all $i \in \{0,\ldots,4\}$.
From $\frac{\partial H_f}{\partial X_0}(p) = -\frac{2}{\alpha_0 p_0^2} + \frac{2p_1}{\alpha_0 p_0^3}$, $\frac{\partial H_f}{\partial X_1}(p) = -\frac{1}{\alpha_0 p_0^2}$ and $\frac{\partial H_f}{\partial X_i}(p) = -\frac{1}{\alpha_i p i^2}$, $i = 2,3,4$ it follows again that this holds if and only if $p = \left[\frac{\varepsilon_0}{\sqrt{\alpha_0}} : \frac{\varepsilon_0}{2\sqrt{\alpha_0}} : \frac{\varepsilon_2}{\sqrt{\alpha_2}} : \frac{\varepsilon_3}{\sqrt{\alpha_3}} : \frac{\varepsilon_4}{\sqrt{\alpha_4}}\right]$, where $\varepsilon_i \in \{-1,1\}$ for $i = 0,2,3,4$.
Furthermore,

$$\begin{aligned} f\left(\left[\frac{\varepsilon_0}{\sqrt{\alpha_0}} : \frac{\varepsilon_0}{2\sqrt{\alpha_0}} : \frac{\varepsilon_2}{\sqrt{\alpha_2}} : \frac{\varepsilon_3}{\sqrt{\alpha_3}} : \frac{\varepsilon_4}{\sqrt{\alpha_4}}\right]\right) &= \frac{3}{2} \cdot \frac{\varepsilon_0}{\sqrt{\alpha_0}} + \frac{\varepsilon_2}{\sqrt{\alpha_2}} + \frac{\varepsilon_3}{\sqrt{\alpha_3}} + \frac{\varepsilon_4}{\sqrt{\alpha_4}} \\ &= H_f\left(\left[\frac{\varepsilon_0}{\sqrt{\alpha_0}} : \frac{\varepsilon_0}{2\sqrt{\alpha_0}} : \frac{\varepsilon_2}{\sqrt{\alpha_2}} : \frac{\varepsilon_3}{\sqrt{\alpha_3}} : \frac{\varepsilon_4}{\sqrt{\alpha_4}}\right]\right), \end{aligned}$$

so $p = \left[\frac{\varepsilon_0}{\sqrt{\alpha_0}} : \frac{\varepsilon_0}{2\sqrt{\alpha_0}} : \frac{\varepsilon_2}{\sqrt{\alpha_2}} : \frac{\varepsilon_3}{\sqrt{\alpha_3}} : \frac{\varepsilon_4}{\sqrt{\alpha_4}}\right]$ lies in the zero locus of f if and only if it lies in the zero locus of H_f. We can conclude that on $V\left(\sum_{i=0}^4 X_i\right) \setminus V\left(\prod_{i=0}^4 X_i\right)$, the singular points of $V(f)$ coincide with the singular points of $V(H_f)$

Our preceding considerations have shown that the only points which are singular points of $V(H_f)$ and not of $V(f)$ in $V\left(\sum_{i=0}^4 X_i\right)$ are the seven points p_{0ij}, $i,j \in \{1,2,3,4\}$, $i \neq j$, and p_{234}. But it has been shown in the proof of Lemma 6.9 that under a linear transformation $g \in G$, singular points of H_f which are no singular points of f must be mapped to singular points of $H_{g.f}$ which are no singular points of $g.f$. Since

g is injective, Proposition 4.5 implies that there cannot be a $g \in G$ such that $g.f \in S_0$. Hence,

$$G.N \subset V(\hat{I}_{40}).$$

In order to prove that $\overline{G.N} = V(\hat{I}_{40})$, it suffices to show that $\dim\left(\overline{G.N}\right) = 19 = \dim(V(\hat{I}_{40}))$ (the second equation holds since $V(\hat{I}_{40})$ is an irreducible hypersurface of $V \cong \mathbb{A}^{20}$, cf. Step 2a). We can use the same method as in the proof of Lemma 6.6. The map $\psi\colon G \times N \to \overline{G.N}$, $(g, f) \mapsto g.f$ is a dominant morphism of varieties, hence we have for a general $f_0 \in \overline{G.N}$

$$\dim G + \dim N = \dim (G \times N) = \dim\left(\overline{G.N}\right) + \dim\left(\psi^{-1}(f_0)\right).$$

Since the constructible set $G.N$ contains an open dense subset of $\overline{G.N}$ and we have for all $g \in G$ and all $f_0 \in G.N$ that $\dim\left(\psi^{-1}(f_0)\right) = \dim\left(\psi^{-1}(g.f_0)\right)$, we can assume without loss of generality that $f_0 \in N$.

But then

$$\begin{aligned}\psi^{-1}(f_0) &= \{(g, f) \in G \times N\colon g.f = f_0\} = \left\{(g, f) \in G \times N\colon g^{-1}.f_0 = f\right\} \\ &\cong \{g \in G\colon g.f_0 \in N\}.\end{aligned}$$

The set $\operatorname{Tran}(f_0, N) := \{g \in G\colon g.f_0 \in N\}$ is called the *transporter of* f_0 *and* N. Since the seven singular points p_{0ij}, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, and p_{234} span the hyperplane $V\left(\sum_{i=0}^{4} X_i\right)$, every $g \in G$ is already determined by the action on these points. If g maps some $f_0 \in N$ again into N, it induces a permutation on these points, hence $\operatorname{Tran}(f_0, N)$ is finite for all $f_0 \in N$. We get

$$\dim\left(\overline{G.N}\right) = \dim(G) + \dim(N) - \dim(\operatorname{Tran}(f_0, N)) = 15 + 4 - 0 = 19,$$

which concludes the proof of Step 2b.

Step 2c *If* $F \in \mathcal{O}(S)^H$, *then* $\hat{F} \in \mathcal{O}(V)^G$ *if and only if* $\hat{F}$ *is regular on* S'.

Let $F \in \mathcal{O}(S)^H$. By the canonical inclusion $\mathcal{O}(S)^H \subset K(V)^G$ discussed above, there are $R_1, R_2 \in \mathcal{O}(V)$ such that $\hat{F} = \frac{R_1}{R_2}$ is G-invariant and extends F on the larger domain V. In particular, $\frac{R_1(gs)}{R_2(gs)} = \hat{F}(gs) = \hat{F}(s) = F(s)$ for all $s \in S$, $g \in G$. Hence, R_2 can only vanish on points whose orbit do not intersect S. By Step 2a, this means $V(R_2) \subseteq V(\hat{I}_{40})$ and by the Nullstellensatz there exists an $\alpha \in \mathcal{O}(V)$ and an $n \in \mathbb{N}$ such that $\alpha \cdot R_2 = \hat{I}_{40}^n$. We can therefore, after a possible expansion of the fraction, assume without loss of generality that $\hat{F} = \frac{R_1}{\hat{I}_{40}^n}$. Since $\hat{I}_{40}^n, \hat{F} \in K(V)^G$, we must also have $R_1 \in \mathcal{O}(V) \cap K(V)^G = \mathcal{O}(V)^G$.

If $\hat{F}$ is regular on V, this obviously implies that $\hat{F}$ is regular on the subset S'.

Conversely, if $\hat{F}$ is regular on S', by its G-invariance and Step 2b, $\hat{F}$ must be regular on an open subset of $V(\hat{I}_{40})$. But $\hat{F} = \frac{R_1}{\hat{I}_{40}^n}$, so $n = 0$ and $\hat{F} = R_1 \in \mathcal{O}(V)^G$ and we have shown that

$$\boxed{\mathcal{O}(V)^G = \{F \in \mathcal{O}(S)^H : \hat{F} \text{ is regular on } S'\}.}$$

Step 3 *Compute* $\{F \in \mathcal{O}(S)^H : \hat{F}$ *is regular on* $S'\}$.
This step is again divided into three parts:

Step 3a Show $\mathcal{O}(S)^H = A_0 \oplus A_1$, where A_0 are the symmetric polynomials on S whose monomials have degree $8k$, $k \in \mathbb{N}$ and A_1 are the polynomials of the form vs, with v the Vandermonde determinant in the coefficients λ_i and s a symmetric polynomial whose monomials have degree $8k-6$, $k \in \mathbb{N}$.

Step 3b Show $\bigoplus_{k\in\mathbb{N}} \mathcal{O}(V)^G_{8k} = \mathbb{C}[\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}, \hat{I}_{40}]$.

Step 3c Show $\bigoplus_{k\in\mathbb{N}} \mathcal{O}(V)^G_{8k+4} = \left(\bigoplus_{k\in\mathbb{N}} \mathcal{O}(V)^G_{8k}\right)[\hat{I}_{100}]$.

Step 3a *Show $\mathcal{O}(S)^H = A_0 \oplus A_1$, where A_0 are the symmetric polynomials on S whose monomials have degree $8k$, $k \in \mathbb{N}$ and A_1 are the polynomials of the form vs, with v the Vandermonde determinant in the coefficients λ_i and s a symmetric polynomial whose monomials have degree $8k-6$, $k \in \mathbb{N}$.*

Recall that $H = \langle \mathfrak{A}_5, \zeta_8 \cdot P_{\langle 0,1\rangle} \rangle$, where we identify an element $\sigma \in \mathfrak{A}_5$ with the corresponding permutation matrix $P_\sigma \in GL(S)$. We therefore have to find the polynomials $P \in \mathcal{O}(S) = \mathbb{C}[\lambda_0, \ldots, \lambda_4]$ which are invariant under the action of $\mathfrak{A}_5$ and of $\zeta_8 \cdot P_{\langle 0,1\rangle}$. The invariant ring $A := \mathbb{C}[\lambda_0, \ldots, \lambda_4]^{\mathfrak{A}_5}$ is just the polynomial algebra generated by the symmetric and the alternating polynomials. By the fundamental theorem of alternating functions, we have $A = \mathbb{C}[\lambda_0, \ldots, \lambda_4]^{\mathfrak{A}_5} = \mathbb{C}[\sigma_1, \ldots, \sigma_5, v]$ (cf. [Rom05]), where the symmetric polynomials are those in which v occurs only in even powers and the alternating polynomials those in which v occurs only in odd powers.
A symmetric polynomial is invariant under $P_{\langle 0,1\rangle}$, hence it is invariant under $\zeta_8 \cdot P_{\langle 0,1\rangle}$ if and only if its monomials have degree $8k$, $k \in \mathbb{N}$.
The action of $P_{\langle 0,1\rangle}$ on an alternating polynomial produces a factor -1. Hence, an alternating polynomial is invariant under $\zeta_8 \cdot P_{\langle 0,1\rangle}$ if and only if its monomials have degree $8k+4$, $k \in \mathbb{N}$. Since the Vandermonde determinant occurs in every alternating polynomial only in odd powers, we may equivalently say that an alternating polynomial is invariant under $\zeta_8 \cdot P_{\langle 0,1\rangle}$ if and only if it is of the form vs, where s is a symmetric polynomial whose monomials have degree $8k-6$, $k \in \mathbb{N}$.

If we denote the subspaces of the symmetric resp. alternating polynomials described above by A_0 resp. A_1, we obtain $\mathcal{O}(S)^H = A_0 \oplus A_1$.

Step 3b *Show $\bigoplus_{k\in\mathbb{N}} \mathcal{O}(V)^G_{8k} = \mathbb{C}[\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}, \hat{I}_{40}]$.*

It can be checked easily that the $\hat{I}_n$, $n = 8, 16, 24, 32, 40$ are regular on S'.
For every cubic polynomial $f \in S' \setminus V(\alpha_0\alpha_1)$, we have found in Lemma 6.7 a representative $\tilde{f} \in S \cap G.f$ and expressed the values of the elementary symmetric functions σ_i in $\tilde{f}$ in terms of the coordinates of f. Since the $\hat{I}_n$ are G-invariant, we can compute their values on $S' \setminus V(\alpha_0\alpha_1)$ simply by plugging in the formulae for the σ_i from Lemma 6.7 into the formulae for the I_n from the end of Step 1. The arising functions in the coordinates $z, \alpha_0, \tau_1, \tau_2, \tau_3$ are polynomials in $\alpha_0, \tau_1, \tau_2, \tau_3$, so one only has to check that they do not have poles for the critical values $z = 0$ and $z = 4$, which is elementary calculus.
Since $V(\alpha_0) = V(\lambda_0) \subset S \cap S'$, the $\hat{I}_n$ are regular on $V(\alpha_0)$ as well.
Since it has been shown in Step 2b that $V(\alpha_1) \subset V(\hat{I}_{40})$ and $I_{40} = \sigma_5^8$, checking the regularity of the $\hat{I}_n$ on $V(\alpha_1)$ becomes also trivial.

Hence, the $\hat{I}_n$ are regular on S' which proves $\mathbb{C}[\hat{I}_8, \hat{I}_{16}, \hat{I}_{24}, \hat{I}_{32}, \hat{I}_{40}] \subseteq \bigoplus_{k\in\mathbb{N}} \mathcal{O}(V)^G_{8k}$ by Step 2.

Verifying the other inclusion causes more trouble. We know that on S, every invariant in $\bigoplus_{k\in\mathbb{N}} \mathcal{O}(V)^G_{8k}$ can be written in the form $P(\sigma_1, \dots, \sigma_5)$ where P is a polynomial. We can now use again that the function is G-invariant and that we have found formulae expressing for all $f \in S' \smallsetminus V(\alpha_0\alpha_1)$ the values of the elementary symmetric functions σ_i in some representative of the orbit of f in S in the coordinates of f. It follows from the regularity condition for $\hat{P}$ that $P(\sigma_1(z), \dots, \sigma_5(z))$ must be regular as a function on S'. In particular, $P(\sigma_1(z), \dots, \sigma_5(z))$ must be regular as $z \to 0$ for any $\alpha_0 \neq 0, \tau_1, \tau_2, \tau_3$ where the τ_i denote the elementary symmetric polynomials in $\alpha_2, \alpha_3, \alpha_4$ as above. Here and in the following, regular will always mean regular as $z \to 0$, i.e. the limit $\lim_{z\to 0} P(\sigma_1(z), \dots, \sigma_5(z))$ of the scalar functions $P(\sigma_1(z), \dots, \sigma_5(z))$ exists in $\mathbb{C}$.

In order to show the assertion of Step 3b, it therefore suffices to prove

Proposition 6.11. *Let $P(y_1, \dots, y_5)$ be a polynomial such that $P(\sigma_1(z), \dots, \sigma_5(z))$ is regular for all $\alpha_0 \neq 0$ and τ_1, τ_2, τ_3. Then P is a polynomial in*

$$J_8 = y_4^2 - 4y_3y_5,\ J_{16} = y_5^3 y_1,\ J_{24} = y_5^4 y_4,$$
$$J_{32} = y_5^6 y_2,\ J_{40} = y_5^8 \text{ and } J_{48} = y_5^9 y_3.$$

Note that $I_n = J_n(\sigma_1, \dots, \sigma_5)$ and that again $J_{48} = \frac{1}{4}\left(J_{24}^2 - J_8 J_{40}\right)$.

The complicated formulae for the $\sigma_i(z)$ from Lemma 6.7 still make it very difficult for us to prove the proposition. We help ourselves with the following workaround:

Recall from Definition 3.10 that the initial form of a polynomial is its homogeneous part of least degree. We define σ_i' to be the initial forms of the σ_i in α_0 for all $i = 1, \dots, 5$. From Lemma 6.7 we get

$$\sigma_1' = \tau_1 \cdot z^{-3/8}\left(1 - \frac{z}{4}\right)^{3/4}, \sigma_2' = \tau_2 \cdot z^{-3/4}\left(1 - \frac{z}{4}\right)^{3/2}$$
$$\sigma_3' = \tau_3 \cdot z^{-9/8}\left(1 - \frac{z}{4}\right)^{9/4}, \sigma_4' = 2\alpha_0\tau_3 \cdot z^{-1/2}\left(1 + \frac{3z}{4}\right)$$
$$\sigma_5' = \alpha_0^2 \tau_3 z^{1/8}\left(1 - \frac{z}{4}\right)^{3/4}.$$

We prove now the proposition with the modified condition that $P(\sigma_1'(z), \dots, \sigma_5'(z))$ instead of $P(\sigma_1(z), \dots, \sigma_5(z))$ is regular for all $\alpha_0 \neq 0$ and τ_1, τ_2, τ_3. This will allow us to prove the original proposition by induction.

Let us first show the proposition with the modified condition for a monomial

$$P = y_1^{m_1} \cdot \ldots \cdot y_5^{m_5}.$$

For a polynomial $F(y_1, \dots, y_5)$, $\nu(F)$ will denote in the following the order of the zero of $F(\sigma_1'(z), \dots, \sigma_5'(z))$ in $z = 0$ (if $F(\sigma_1'(z), \dots, \sigma_5'(z))$ has a pole in $z = 0$, $\nu(F)$ will be negative with absolute value the pole order of $F(\sigma_1'(z), \dots, \sigma_5'(z))$). For example, we have

$$\nu(y_1) = -\frac{3}{8},\ \nu(y_2) = -\frac{3}{4},\ \nu(y_3) = -\frac{9}{8},\ \nu(y_4) = -\frac{1}{2},\ \nu(y_5) = \frac{1}{8}.$$

Since $P(\sigma_1'(z),\ldots,\sigma_5'(z)) = (\sigma_1')^{m_1}\cdot\ldots\cdot(\sigma_5')^{m_5}$ is regular by assumption, we must have

$$\begin{aligned} n := \nu(P) &= -\frac{3}{8}m_1 - \frac{3}{4}m_2 - \frac{9}{8}m_3 - \frac{1}{2}m_4 + \frac{1}{8}m_5 \geq 0 \text{ and} \\ m_5 &= 8n + 3m_1 + 6m_2 + 9m_3 + 4m_4. \end{aligned}$$

Therefore,

$$P = y_1^{m_1}y_2^{m_2}y_3^{m_3}y_4^{m_4}y_5^{m_5} = y_5^{3m_1}y_1^{m_1}y_5^{4m_4}y_4^{m_4}y_5^{6m_2}y_2^{m_2}y_5^{9m_3}y_3^{m_3}y_5^{8n} = J_{16}^{m_1}J_{24}^{m_4}J_{32}^{m_2}J_{48}^{m_3}J_{40}^{n}$$

is a polynomial in $J_8, J_{16}, J_{24}, J_{32}, J_{40}$.

Next, let us show the proposition with the modified condition for a polynomial of the form

$$P(y_1,y_2,y_3,y_4,y_5) = M(y_1,y_2,y_3,y_4,y_5)\,Q_n\left(y_4^2, y_3y_5\right),$$

where M is a monomial and Q_n a homogeneous polynomial of degree n (so $Q_n\left(y_4^2, y_3y_5\right)$ regarded as a polynomial in y_1,y_2,y_3,y_4,y_5 has degree $2n$). We shall of course try to reduce this to the case already proven. In order to do so, we need the following statement:

For all $n \in \mathbb{N}^*$ we have $\nu\left(Q_n\left(y_4^2, y_3y_5\right)\right) \leq 0$. If in addition $k = \nu\left(Q_n\left(y_4^2, y_3y_5\right)\right) > -n$, then $Q_n\left(y_4^2, y_3y_5\right) = J_8^{n+k}R_k\left(y_4^2, y_3y_5\right)$, where $\nu\left(R_k\left(y_4^2, y_3y_5\right)\right) = k$.

This statement is proved by induction on n.
For $n = 1$, we must have $Q_n\left(y_4^2, y_3y_5\right) = ay_4^2 + by_3y_5$ for some $a, b \in \mathbb{C}$ and thus

$$Q_n\left(\left(\sigma_4'\right)^2, \sigma_3'\sigma_5'\right) = a\cdot 4\alpha_0^2\tau_3^2\cdot z^{-1}\left(1+\frac{3z}{4}\right)^2 + b\cdot\alpha_0^2\tau_3^2\cdot z^{-1}\left(1-\frac{z}{4}\right)^3$$

If Q_n is regular as $z \to 0$, the coefficient of z^{-1} must vanish. This is the case for all $\alpha_0 \neq 0$ and all τ_3 only if $4a + b = 0$. Hence $Q_n\left(y_4^2, y_3y_5\right)$ reduces to $ay_4^2 - 4ay_3y_5 = aJ_8^{1+0}$. So the decomposition exists for all $k > -1$ and in particular $\nu\left(Q_n\left(y_4^2, y_3y_5\right)\right) = \nu(J_8) = 0$.
For the inductive step, let the statement be proven for $n-1$. Consider a Q_n satisfying $k = \nu\left(Q_n\left(y_4^2, y_3y_5\right)\right) > -n$.
If the coefficient of y_4^{2n} in $Q_n\left(y_4^2, y_3y_5\right)$ vanishes, there is a homogeneous polynomial Q_{n-1} of degree $n-1$ such that $Q_n\left(y_4^2, y_3y_5\right) = y_3y_5Q_{n-1}\left(y_4^2, y_3y_5\right)$. In particular, $\nu\left(Q_{n-1}\left(y_4^2, y_3y_5\right)\right) = \nu\left(Q_n\left(y_4^2, y_3y_5\right)\right) - \nu(y_3y_5) = k + \frac{9}{8} - \frac{1}{8} = k+1 > -(n-1)$. By the induction hypothesis, we have $k+1 \leq 0$ and there is a homogeneous polynomial R_{k+1} such that $Q_{n-1}\left(y_4^2, y_3y_5\right) = J_8^{n-1+k+1}R_{k+1}\left(y_4^2, y_3y_5\right)$ and $\nu\left(R_{k+1}\left(y_4^2, y_3y_5\right)\right) = k+1$. Setting $R_k\left(y_4^2, y_3y_5\right) = y_3y_5R_{k+1}\left(y_4^2, y_3y_5\right)$, we have $Q_n\left(y_4^2, y_3y_5\right) = J_8^{n+k}R_k\left(y_4^2, y_3y_5\right)$ and $\nu\left(R_k\left(y_4^2, y_3y_5\right)\right) = \nu(y_3y_5) + \nu\left(R_{k+1}\left(y_4^2, y_3y_5\right)\right) = -1 + k + 1 = k$.
If the coefficient of y_4^{2n} in $Q_n\left(y_4^2, y_3y_5\right)$ is $a \neq 0$, the coefficient of y_4^{2n} vanishes in the polynomial

$$Q_n\left(y_4^2, y_3y_5\right) - aJ_8^n = Q_n\left(y_4^2, y_3y_5\right) - a\left(y_4^2 - 4y_3y_5\right).$$

From what we have just proved, we know that $k = \nu\left(Q_n\left(y_4^2, y_3y_5\right) - aJ_8^n\right) \leq -1$ or $Q_n\left(y_4^2, y_3y_5\right) = aJ_8^n$ and that there is a homogeneous polynomial $\tilde{R}_k$ of degree $-k$ with $\nu(\tilde{R}_k\left(y_4^2, y_3y_5\right)) = k$ and $Q_n\left(y_4^2, y_3y_5\right) - aJ_8^n = J_8^{n+k}\tilde{R}_k\left(y_4^2, y_3y_5\right)$. Hence, $Q_n\left(y_4^2, y_3y_5\right) = J_8^{n+k}(\tilde{R}_k\left(y_4^2, y_3y_5\right) + aJ_8^{-k})$. Setting $R_k\left(y_4^2, y_3y_5\right) := \tilde{R}_k\left(y_4^2, y_3y_5\right) +$

aJ_8^{-k}, we have $Q_n\left(y_4^2, y_3y_5\right) = J_8^{n+k}R_k\left(y_4^2, y_3y_5\right)$ and furthermore $\nu\left(R_k\left(y_4^2, y_3y_5\right)\right) = k$ since $\nu\big(\tilde{R}_k\left(y_4^2, y_3y_5\right)\big) = k$ and $\nu\big(aJ_8^{-k}\big) = 0$. This concludes the proof of the statement.

With the help of this statement, we can now show the proposition with the modified condition for a polynomial of the form

$$P\left(y_1, y_2, y_3, y_4, y_5\right) = M\left(y_1, y_2, y_3, y_4, y_5\right) Q_n\left(y_4^2, y_3y_5\right).$$

Concretely, we show that there exists a polynomial $R\left(y_1, y_2, y_3, y_4, y_5\right)$ whose monomials are all regular such that $M\left(y_1, y_2, y_3, y_4, y_5\right) Q_n\left(y_4^2, y_3y_5\right) = J_8^{n-k}R\left(y_1, y_2, y_3, y_4, y_5\right)$ for some $k \geq 0$. This gives us the desired result since we have already proven the proposition for regular monomials.

If $\nu\left(M\right) \leq 0$, then the regularity of

$$M\left(\sigma_1'\left(z\right), \sigma_2'\left(z\right), \sigma_3'\left(z\right), \sigma_4'\left(z\right), \sigma_5'\left(z\right)\right) Q_n\left(\left(\sigma_4'\left(z\right)\right)^2, \sigma_3'\left(z\right)\sigma_5'\left(z\right)\right)$$

implies that $\nu\left(Q_n\left(y_4^2, y_3y_5\right)\right) \geq -\nu\left(M\right) \geq 0$. But by the previous statement, we must have $\nu\left(Q_n\left(y_4^2, y_3y_5\right)\right) \leq 0$ and therefore $\nu\left(Q_n\left(y_4^2, y_3y_5\right)\right) = \nu\left(M\right) = 0$. Hence, $M\left(\sigma_1'\left(z\right), \sigma_2'\left(z\right), \sigma_3'\left(z\right), \sigma_4'\left(z\right), \sigma_5'\left(z\right)\right)$ and $Q_n\left(\left(\sigma_4'\left(z\right)\right)^2, \sigma_3'\left(z\right)\sigma_5'\left(z\right)\right)$ must be regular, and the same has to hold for all of their monomials.

If $0 < k = \nu\left(M\right) < n$, then the regularity of

$$M\left(\sigma_1'\left(z\right), \sigma_2'\left(z\right), \sigma_3'\left(z\right), \sigma_4'\left(z\right), \sigma_5'\left(z\right)\right) Q_n\left(\left(\sigma_4'\left(z\right)\right)^2, \sigma_3'\left(z\right)\sigma_5'\left(z\right)\right)$$

implies that $\nu\left(Q_n\left(y_4^2, y_3y_5\right)\right) \geq -k$. By the previous statement, we have $Q_n\left(y_4^2, y_3y_5\right) = J_8^{n-k}R_{-k}\left(y_4^2, y_3y_5\right)$, and $R_{-k}\left(y_4^2, y_3y_5\right)$ is homogeneous of degree $2k$ because $Q_n\left(y_4^2, y_3y_5\right)$ and J_8^{n-k} are homogeneous of degree $2n$ resp. $2\left(n-k\right)$. Since $\left(\sigma_4'\right)^2$ and $\sigma_3'\sigma_5'$ both have a pole of order 1 at $z = 0$, every monomial of $R_{-k}\left(y_4^2, y_3, y_5\right)$ has a pole of order k, thus every monomial of MR_{-k} is regular.

If $\nu\left(M\right) \geq n$, every monomial of $M\left(y_1, y_2, y_3, y_4, y_5\right) Q_n\left(y_4^2, y_3y_5\right)$ is regular, since as in the previous case every monomial of $Q_n\left(y_4^2, y_3y_5\right)$ has a pole of order n.

This shows the proposition with the modified condition for a polynomial of the form $P\left(y_1, y_2, y_3, y_4, y_5\right) = M\left(y_1, y_2, y_3, y_4, y_5\right) Q_n\left(y_4^2, y_3y_5\right)$. We can now prove it for a general polynomial.

Let $P\left(y_1, y_2, y_3, y_4, y_5\right) = \sum_{n_1,n_2} y_1^{n_1}y_2^{n_2}P_{n_1n_2}\left(y_3, y_4, y_5\right)$ with

$$P_{n_1n_2}\left(y_3, y_4, y_5\right) = \sum_{n_3,n_4,n_5} c_{n_3n_4n_5}y_3^{n_3}y_4^{n_4}y_5^{n_5}$$

be a polynomial such that $P\left(\sigma_1'\left(z\right), \ldots, \sigma_5'\left(z\right)\right)$ is regular for all $\alpha_0 \neq 0$ and τ_1, τ_2, τ_3.

Since τ_1 only occurs as a factor in σ_1', applying Lemma 6.10 to $P\left(\sigma_1', \sigma_2', \sigma_3', \sigma_4', \sigma_5'\right) = \sum_{n_1}\left(\sigma_1'\right)^{n_1}\sum_{n_2}\left(\sigma_2'\right)^{n_2}P_{n_1n_2}\left(\sigma_3', \sigma_4', \sigma_5'\right)$ with $t = \tau_1$ implies that

$$\left(\sigma_1'\right)^{n_1}\sum_{n_2}\left(\sigma_2'\right)^{n_2}P_{n_1n_2}\left(\sigma_3', \sigma_4', \sigma_5'\right)$$

is regular for all n_1. Since τ_2 only occurs as a factor in σ_2', it follows from the same argument that $(\sigma_1')^{n_1}(\sigma_2')^{n_2} P_{n_1 n_2}(\sigma_3', \sigma_4', \sigma_5')$ is regular for all n_1, n_2.

Unfortunately, this reasoning does not work for the remaining coefficients because they occur in more than one σ_i'. However, τ_3 is only a linear factor of σ_3', σ_4' and σ_5' and α_0 a linear factor of σ_4' and a quadratic factor of σ_5', so Lemma 6.10 provides at least that we can assume without loss of generality

$$n_4 + 2n_5 = \text{const} \quad \text{and} \quad n_3 + n_4 + n_5 = \text{const}.$$

Subtracting the first from the second equation, we obtain $n_3 - n_5 = \text{const}$. Consequently, $P_{n_1 n_2}$ can be assumed to be of the form $P_{n_1 n_2}(y_3, y_4, y_5) = M(y_3, y_4, y_5)\, Q_n\left(y_3 y_5, y_4^2\right)$, where M is a monomial and Q_n a homogeneous polynomial of degree n.

Thus, we are left with regular polynomials of the form $M(y_1, y_2, y_3, y_4, y_5)\, Q_n\left(y_3 y_5, y_4^2\right)$, which have already been discussed. This concludes the proof of the proposition with the modified condition that $P(\sigma_1'(z), \ldots, \sigma_5'(z))$ be regular for all $\alpha_0 \neq 0$ and τ_1, τ_2, τ_3.

We show now that $P(y_1, y_2, y_3, y_4, y_5)$ is already a polynomial in the J_n if

$$P(\sigma_1(z), \ldots, \sigma_5(z))$$

is regular for all $\alpha_0 \neq 0$ and τ_1, τ_2, τ_3. This is done by induction on the number of monomials occurring in P.
If P contains no monomial, this is trivial.
Let now P have $n > 0$ monomials. Applying again Lemma 6.10 to $P(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$ with α_0 as the graded variable t, we see that the part of $P(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$ with least degree in α_0 (the initial form of P with respect to α_0) must be regular. But since the σ_i' were the initial forms of the σ_i with respect to α_0, the initial form of $P(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$ with respect to α_0 is given by $P_{\text{in}}(\sigma_1', \sigma_2', \sigma_3', \sigma_4', \sigma_5')$ where P_{in} is the sum of the monomials $c_{n_1 n_2 n_3 n_4 n_5} y_1^{n_1} y_2^{n_2} y_3^{n_3} y_4^{n_4} y_5^{n_5}$ with $\sum_{i=1}^{5} (\deg_{\alpha_0} \sigma_i') \cdot n_i$ minimal. Thus, it follows from the proposition with the modified condition that P_{in} must be a polynomial in the J_n. For the rest of the polynomial one can use the induction hypothesis. This concludes the proof of the proposition and therefore of Step 3b.

Step 3c *Show* $\bigoplus_{k \in \mathbb{N}} \mathcal{O}(V)^G_{8k+4} = \left(\bigoplus_{k \in \mathbb{N}} \mathcal{O}(V)^G_{8k}\right)[\hat{I}_{100}]$.

This is proved completely analogously to Step 3b. First it is shown that $\hat{I}_{100}$ is regular on S' and therefore in $\mathcal{O}(V)^G$.

In order to prove the other inclusion, we use that we know from Lemma 6.7 that the Vandermonde determinant v in the coordinates λ_i of S can also be expressed in the coordinates of representatives of the same G-orbits in $S' \setminus V(\alpha_0 \alpha_1)$ by

$$v = \alpha_0 \left(3 + \frac{z}{4}\right) z^{-9/4} \left(1 - \frac{z}{4}\right)^{-9/2} \left[\prod_{j=2}^{4} z^2 \alpha_0^2 - 2\alpha_0 \alpha_j z \left(1 + \frac{3z}{4}\right) + \alpha_j^2 \left(1 - \frac{z}{4}\right)^3\right] w,$$

where w is the Vandermonde determinant in the coordinates $\alpha_2, \alpha_3, \alpha_4$ of S'. We know further that every element $P \in \bigoplus_{k \in \mathbb{N}} \mathcal{O}(V)^G_{8k+4}$ is given as the product $P = v \cdot F(\sigma_1(z), \ldots, \sigma_5(z))$ of the Vandermonde determinant and a symmetric function with monomials of degree $8k-6$, $k \in \mathbb{N}$ and must be regular on S'. As in the proof of Step 3b, it suffices to examine the necessary condition that $v \cdot F(\sigma_1(z), \ldots, \sigma_5(z))$ is regular as $z \to 0$. Since v has a pole of order $\frac{9}{4}$ in $z = 0$, $F(\sigma_1(z), \ldots, \sigma_5(z))$ must have a zero of at least that order in $z = 0$.

We prove that if $F(y_1, \ldots, y_5)$ is a polynomial such that $F(\sigma_1(z), \ldots, \sigma_5(z))$ is regular as $z \to 0$ and has a zero of order k in $z = 0$, then $F(y_1, \ldots, y_5) = y_5^{8k} R(y_1 \ldots, y_5)$, where $R(\sigma_1(z), \ldots, \sigma_5(z))$ is regular as $z \to 0$.

By the same arguments as in Step 3b, it suffices to show this for the σ_i replaced by σ_i' and F of the form $F(y_1 \ldots, y_5) = M(y_1, \ldots, y_5) Q_n(y_4^2, y_3 y_5)$, where M is a monomial and Q_n a homogeneous polynomial of degree n. But the intermediate statement from Step 3b already showed us that $Q_n(y_4^2, y_3 y_5)$ does not have a zero in $z = 0$. Hence, if F has a zero of order k, it must come from the monomial $M = y_1^{m_1} y_2^{m_2} y_3^{m_3} y_4^{m_4} y_5^{m_5}$, so

$$-\frac{3}{8}m_1 - \frac{3}{4}m_2 - \frac{9}{8}m_3 - \frac{1}{2}m_4 + \frac{1}{8}m_5 = k.$$

It follows that the function $R(\sigma_1', \ldots, \sigma_5') := \frac{M(\sigma_1', \ldots, \sigma_5')}{(\sigma_5')^{8k}}$ is regular since the order of its zero in $z = 0$ is

$$-\frac{3}{8}m_1 - \frac{3}{4}m_2 - \frac{9}{8}m_3 - \frac{1}{2}m_4 + \frac{1}{8}m_5 - k = 0.$$

We therefore have $F(y_1, \ldots, y_5) = y_5^{8k} R(y_1, \ldots, y_5)$ with $R = \frac{F}{y_5^{8k}}$ regular.

Since the zero of F must have at least order $\frac{9}{4}$, we can express the invariant $P \in \bigoplus_{k \in \mathbb{N}} \mathcal{O}(V)_{8k+4}^G$ as

$$P = v \cdot F(\sigma_1, \ldots, \sigma_5) = v \cdot \sigma_5^{18} R(\sigma_1, \ldots, \sigma_5) = I_{100} \cdot R(\sigma_1, \ldots, \sigma_5)$$

for some regular symmetric $R(\sigma_1, \ldots, \sigma_5)$, which concludes the proof of Step 3c.

We shall finish the proof of the theorem by showing that $I_8, I_{16}, I_{24}, I_{32}, I_{40}$ are algebraically independent. In fact, σ_5 is integral over $\mathbb{C}[I_8, I_{16}, I_{24}, I_{32}, I_{40}]$, because $I_{40} = \sigma_5^8$ and

$$\sigma_1 = \frac{I_{16}}{\sigma_5^3},\ \sigma_2 = \frac{I_{32}}{\sigma_5^6},\ \sigma_3 = \frac{I_{24}^2 - \sigma_5^8 I_8}{4\sigma_5^9},\ \sigma_4 = \frac{I_{24}}{\sigma_5^4}$$

are rational functions in $\sigma_5, I_8, I_{16}, I_{24}, I_{32}, I_{40}$, so $\operatorname{trdeg} \operatorname{Quot}(\mathbb{C}[I_8, I_{16}, I_{24}, I_{32}, I_{40}]) \geq \operatorname{trdeg} \operatorname{Quot}(\mathbb{C}[\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5]) = 5$. Hence, the I_n, $n = 8, 16, 24, 32, 40$ are algebraically independent. Since I_{100}^2 is a symmetric invariant, there exists a polynomial relation $I_{100}^2 = P(I_8, I_{16}, I_{24}, I_{32}, I_{40})$. This concludes the proof. $\square$

Note that we did not assume the classical result from [Sal65, Ch. XV] that the $\hat{I}_n$ are invariants as has been done in the original paper [Bek82]. This makes some arguments longer and more difficult, but has the advantage of not relying on a computation which is difficult to follow.

For the sake of compliance with the literature, the invariants $\hat{I}_n$ will from now on be denoted by I_n.

7 Some special points and hypersurfaces

This thesis will be concluded by a short overview over geometric applications arising from the theorem we have established. The examples come from [DvG07], where the interested reader will find much more material.

We may first ask ourselves which cubics might correspond to the singular points of the weighted projective space found in Proposition 5.6. We have established that the weighted projective space is the quotient of the ordinary projective space under the action of a finite group. It can be seen easily that a point which has trivial stabilizer under this action is mapped to a regular point in the quotient. So the singular points of the weighted projective space have all a non-trivial finite stabilizer group. This is also true for their counterparts in the space of cubic surfaces.

Proposition 7.1. *(i) The cubic surface corresponding to the point* $[0:0:1:0:0] \in \mathbb{P}(1,2,3,4,5)$ *is*

$$X_1^3 + \omega X_2^3 + \omega^2 X_3^3 - 3X_0^2 (X_1 + X_2 + X_3) = 0,$$

where ω *is a primitive third root of unity. It does not admit a Sylvester form and has the singular points* $\left[\pm 1 : 1 : \omega : \omega^2\right]$*. It has an automorphism of order 3 given by* $[X_0 : X_1 : X_2 : X_3] \mapsto [\omega X_0 : X_2 : X_3 : X_1]$.

(ii) The cubic surface corresponding to the point $[0:0:0:0:1]$ *has the Sylvester form*

$$\sum_{i=0}^{4} \eta^i X_i^3 = 0, \quad \sum_{i=0}^{4} X_i = 0,$$

where η *is a primitive fifth root of unity. It has the singular point* $\left[1 : \eta^2 : \eta^4 : \eta : \eta^3\right]$ *and an automorphism of order 5 given by*

$$[X_0 : X_1 : X_2 : X_3 : X_4] \mapsto [X_1 : X_2 : X_3 : X_4 : X_0]\,.$$

We know already from the proof of the theorem that the space of cubics not admitting a non-degenerate Sylvester form is the hyperplane $\hat{I}_{40} = 0$. We can use the invariants to describe further interesting subspaces of the space of cubic surfaces.

Proposition 7.2. *(i) The subvariety given by* $I_{24} = I_{40} = 0$ *consists of all surfaces of the form*

$$X_1^3 + X_2^3 + 2\lambda X_3^3 - 3X_3 \left(\mu X_1 X_3 + X_2 X_3 + X_0^2\right) = 0, \quad \lambda, \mu \in \mathbb{C}.$$

(ii) The subvariety given by $I_{24} = I_{32} = I_{40} = 0$ *consists of all surfaces of the form*

$$X_1^3 + X_2^3 + 2\lambda X_3^3 - 3X_3 \left(X_2 X_3 + X_0^2\right) = 0, \quad \lambda \in \mathbb{C}.$$

(iii) The subvariety given by $I_8 = I_{24} = I_{40} = 0$ *consists of all surfaces of the form*

$$X_1^3 + X_2^3 - 3X_3\left(\mu X_1 X_3 + X_2 X_3 + X_0^2\right) = 0, \quad \mu \in \mathbb{C}.$$

The surfaces given by $\mu = 0$ *resp.* $\mu^3 = -1$ *correspond to the singular points* $[0:1:0:0:0]$ *resp.* $[0:0:0:1:0]$ *of the weighted projective space* $\mathbb{P}(1,2,3,4,5)$.

(iv) The unique orbit of all non-stable, but semi-stable cubic surfaces given by $0 = X_0^3 - X_1 X_2 X_3$ *maps to the point* $[8:1:0:0:0] \in \mathcal{M}$.

References

[Bea09] A. Beauville. Moduli of cubic surfaces and Hodge theory (after Allcock, Carlson, Toledo). In *Géométries à courbure négative ou nulle, groupes discrets et rigidités*, volume 18 of *Sémin. Congr.*, pages 445–466. Soc. Math. France, Paris, 2009.

[Bek82] N. D. Beklemishev. Invariants of cubic forms of four variables. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (2):42–49, 116, 1982.

[Bor91] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[DD85] A. Dimca and S. Dimiev. On analytic coverings of weighted projective spaces. *Bull. London Math. Soc.*, 17(3):234–238, 1985.

[Dol82] I. Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer-Verlag, Berlin, 1982.

[Dol03] I. Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.

[Dol12] I. Dolgachev. *Classical Algebraic Geometry: A Modern View*. Cambridge University Press, Cambridge, 2012. to appear.

[DvG07] E. Dardanelli and B. van Geemen. Hessians and the moduli space of cubic surfaces. In *Algebraic geometry*, volume 422 of *Contemp. Math.*, pages 17–36. Amer. Math. Soc., Providence, RI, 2007.

[Gro61] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.

[Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2)*. Springer-Verlag, Berlin, third edition, 1994.

[Muk03] S. Mukai. *An introduction to invariants and moduli*, volume 81 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003. Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury.

[Now96] K. J. Nowak. A simple algebraic proof of Zariski's main theorem on birational transformations. *Univ. Iagel. Acta Math.*, (33):115–118, 1996.

[Rom05] M. Romagny. The fundamental theorem of alternating functions. `www.math.jussieu.fr/~romagny/notes/FTAF.pdf`, 2005. [Online; accessed 22-July-2012].

[RV04] M. Roth and R. Vakil. The affine stratification number and the moduli space of curves. In *Algebraic structures and moduli spaces*, volume 38 of *CRM Proc. Lecture Notes*, pages 213–227. Amer. Math. Soc., Providence, RI, 2004.

[Sal65] G. Salmon. *A treatise on the analytic geometry of three dimensions. Vol. II.* Fifth edition. Edited by Reginald A. P. Rogers. Chelsea Publishing Co., New York, 1965.

[TY05] P. Tauvel and R. W. T. Yu. *Lie algebras and algebraic groups.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.