Autoequivalences of twisted K3 surfaces

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1. Introduction

The Classical Global Torelli theorem asserts that a K3 surfaces is completely determined by its integral Hodge structure. More precisely, we have

Theorem (Piateckii-Shapiro, Shafarevich, Burns, Rapoport, Peters, Looijenga, Siu)**.** *Two K3 surfaces X and X' are isomorphic if and only there is a Hodge isometry $\varphi\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(X',\mathbb{Z})$. Furthermore, a Hodge isometry $\varphi\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(X',\mathbb{Z})$ is induced by an isomorphism $f\colon X \xrightarrow{\sim} X'$ if and only if φ maps at least one Kähler class in $H^2(X,\mathbb{Z})$ to a Kähler class in $H^2(X',\mathbb{Z})$.*

There is an analogous statement for the derived categories of K3 surfaces where we replace X, X' with $\mathrm{D}^{\mathrm{b}}(X)$, $\mathrm{D}^{\mathrm{b}}(X')$ and $H^2(X,\mathbb{Z})$ with the Mukai lattice $\widetilde{H}(X,\mathbb{Z})$, which is the full cohomology $H^*(X,\mathbb{Z})$ together with a certain lattice and Hodge structure. This statement is called the Derived Global Torelli theorem.

Theorem (Mukai, Orlov, Huybrechts, Macrì, Stellari)**.** *The derived categories $\mathrm{D}^{\mathrm{b}}(X)$ and $\mathrm{D}^{\mathrm{b}}(X')$ of two K3 surfaces X and X' are equivalent if and only if there is a Hodge isometry $\varphi\colon \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X',\mathbb{Z})$. Furthermore, a Hodge isometry $\varphi\colon \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X',\mathbb{Z})$ is induced by a Fourier-Mukai equivalence $\Phi_{\mathcal{P}}\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X')$ if and only if φ preserves the natural orientation of the positive four directions.*

The last condition can be explained as follows: The lattice $\widetilde{H}(X,\mathbb{Z})$ has signature $(4,20)$, hence for a holomorphic 2-form $\sigma_X \in H^{2,0}(X)$ and a Kähler class ω_X, the span $V_{\sigma_X,\omega_X} := \langle \mathrm{Re}\sigma_X, \mathrm{Im}\sigma_X, 1-\frac{\omega_X^2}{2}, \omega\rangle_{\mathbb{R}}$ forms a maximal positive definite subspace of $\widetilde{H}(X,\mathbb{R})$ with a natural orientation. We say that φ *preserves the natural orientation of the positive four directions* if the composition

$$V_{\sigma_X,\omega_X} \hookrightarrow \widetilde{H}(X,\mathbb{R}) \xrightarrow[\sim]{\varphi_{\mathbb{R}}} \widetilde{H}(X',\mathbb{R}) \twoheadrightarrow V_{\sigma_{X'},\omega_{X'}}$$

with the inclusion and the orthogonal projection is orientation preserving.

In the more general case of twisted K3 surfaces, only a weaker Torelli type theorem is known so far (see Sect. 2 for an introduction to the theory of twisted K3 surfaces (X, α_B) and twisted Hodge structures $\widetilde{H}(X, B, \mathbb{Z})$).

Theorem (Huybrechts, Stellari)**.** *If* $\varphi\colon \widetilde{H}(X, B, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X', B', \mathbb{Z})$ *is a Hodge isometry between two twisted projective K3 surfaces* (X, α_B) *and* $(X', \alpha_{B'})$ *which preserves the natural orientation of the four positive directions, there is a Fourier-Mukai equivalence* $\Phi_{\mathcal{P}}\colon \mathrm{D^b}(X, \alpha_B) \xrightarrow{\sim} \mathrm{D^b}(X', \alpha_{B'})$ *such that* $\varphi = \Phi^H_{\mathcal{P}}$*.*

It is an unsolved problem, whether we can also find a strong version of the Twisted Derived Global Torelli theorem. More concretely, the following two questions need to be answered:

1. Does there, for every twisted K3 surface (X, α_B), exist a Hodge isometry $\psi_X\colon \widetilde{H}(X, B, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X, B, \mathbb{Z})$ which reverses the natural orientation of the positive four directions? In the case of a positive answer, the composition with $\psi_{X'}$ in the previous theorem would turn an orientation reversing into an orientation preserving Hodge isometry, thus yielding a statement analogous to the first parts of the Classical and the Derived Global Torelli theorem. In the untwisted case, the role of the ψ_X is taken on by $\mathrm{id}_{(H^0 \oplus H^4)(X)} \oplus -\mathrm{id}_{H^2(X)}$ which is not a Hodge isometry in the twisted case any more.
2. Does every twisted Hodge isometry $\Phi^H_{\mathcal{P}}\colon \widetilde{H}(X, B, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X', B', \mathbb{Z})$ induced by a Fourier-Mukai equivalence $\Phi_{\mathcal{P}}\colon \mathrm{D^b}(X, \alpha_B) \xrightarrow{\sim} \mathrm{D^b}(X', \alpha_{B'})$ preserve the natural orientation of the positive four directions? A positive answer would yield a statement analogous to the second parts of the Classical and the Derived Global Torelli theorem.

In particular, an answer to the second question would give a better description of the autoequivalences of the derived category $\mathrm{D^b}(X, \alpha_B)$. The action of the Fourier-Mukai equivalences on cohomology gives a natural representation

$$\mathrm{Aut}(\mathrm{D^b}(X, \alpha_B)) \to O(\widetilde{H}(X, B, \mathbb{Z})).$$

Since the group of orientation preserving Hodge isometries $O_+(\widetilde{H}(X, B, \mathbb{Z}))$ certainly lies in the image of this representation, an answer to the second question would determine whether the image of $\mathrm{Aut}(\mathrm{D^b}(X, \alpha_B))$ in $O(\widetilde{H}(X, B, \mathbb{Z}))$ has index 1 or 2.

In this Master's thesis, we shall lay the groundwork to answering the two questions and prove some partial results.

In Section 2, we will assemble the basic notions of twisted sheaves, Hodge structures and Fourier-Mukai transforms necessary to understand the remainder of the thesis.

In Section 3, we will first give an affirmative answer to both questions in the case of big Picard numbers $\rho(X) \geq 12$. We will then turn to the first question in the general case and try to describe the Hodge isometries of an arbitrary twisted K3 surface (X, α_B). Since we will show that every lattice of small rank occurs as the generalized Picard group of some twisted K3 surface, this becomes a purely lattice-theoretical problem. We will see that the answer to the first question is negative for almost all *complex* K3 surfaces. For projective K3 surfaces, we will reduce the question to a completely elementary but cumbersome problem about the integrality of certain rational numbers.

In Section 4, we will provide the inexperienced reader with background material for the deformation theory of Fourier-Mukai kernels. Hochschild homology and cohomology as well as first and higher-order obstructions to lifting a kernel will be discussed. Although completely formal, we will generalize parts of the literature, which only exists for untwisted varieties, to the twisted case.

In Section 5, we will give partial answers to the second question. We will show that an intuitive approach via algebraic deformations of K3 surfaces does not work. We will then give a positive answer to the second question in the case that the twisted Fourier-Mukai kernel $\mathcal{P}$ is a sheaf. Finally, we will give an outlook as to how one could proceed in the general case, using the deformation theory apparatus developed in Section 4.

Notation. We will write $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$.

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2. Preliminaries

This introduction to twisted sheaves, their derived categories and Chern characters runs along the lines of [18] and [13]. The interested reader may find more material and examples there.

2.1. **Twisted sheaves.** Let X be a smooth proper variety over $\mathbb{C}$. It is a fact from elementary algebraic geometry that a sheaf $\mathcal{F}$ on X can be described by a collection of sheaves $\mathcal{F}_i$ on some open cover $\{U_i\}_{i\in I}$ and glueing functions $\varphi_{ij} \colon \mathcal{F}_i\restriction_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j\restriction_{U_{ij}}$ which satisfy $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$ and the cocycle condition $\varphi_{ij}\varphi_{jk}\varphi_{ki} = \mathrm{id}_{\mathcal{F}_i\restriction_{U_{ijk}}}$. This construction can be generalized to glueing functions fulfilling the cocycle condition only up to a twist.

Definition 2.1. The *(cohomological) Brauer group* of X is given by

$$\mathrm{Br}(X) := H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}$$

where we take the cohomology with respect to the analytic topology.

Remark 2.2. The algebraically inclined reader may also replace analytic cohomology by étale cohomology.

Remark 2.3. There is yet another way to define the Brauer group as the set of Azumaya algebras over X modulo some equivalence relation. For K3 surfaces, both notions coincide due to results by Grothendieck in the projective and Huybrechts and Schröer in the analytic case, cf. [11] and [17].

Definition 2.4. A *twisted variety* is a tuple (X, α) consisting of a proper variety X over $\mathbb{C}$ and a class $\alpha \in \mathrm{Br}(X)$. A twisted variety over some base scheme A is a twisted variety (X, α) together with a morphism $X \to A$ as part of the datum.

A morphism $f \colon (X, \alpha) \to (Y, \beta)$ is a morphism $f \colon X \to Y$ such that $f^*\beta = \alpha$.

Definition 2.5. Let (X, α) be a twisted variety and $\{\alpha_{ijk}\}$ a Čech 2-cocycle representing α for some open (analytic) cover $X = \bigcup_{i\in I} U_i$. An $\{\alpha_{ijk}\}$*-twisted sheaf* $\mathcal{F}$ is a collection of sheaves $\mathcal{F}_i$ on the U_i and glueing functions $\varphi_{ij} \colon \mathcal{F}_i\restriction_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j\restriction_{U_{ij}}$ which satisfy $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$, $\varphi_{ji} = \varphi_{ij}^{-1}$ and the cocycle condition $\varphi_{ij}\varphi_{jk}\varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}_{\mathcal{F}_i\restriction_{U_{ijk}}}$.

A morphism $f \colon \mathcal{F} \to \mathcal{G}$ of twisted sheaves $\mathcal{F}$ and $\mathcal{G}$ is a collection of morphisms $f_i \colon \mathcal{F}_i \to \mathcal{G}_i$ such that $f_j \circ \varphi_{ij}^{\mathcal{F}} = \varphi_{ij}^{\mathcal{G}} \circ f_i$ for all $i, j \in I$.

Remark 2.6. It has been shown in [6, Prop. 1.2.10] that the basic constructions from ordinary sheaf theory carry over to twisted sheaves:

- If F is an $\{\alpha_{ijk}\}$-twisted and G an $\{\alpha'_{ijk}\}$-twisted sheaf, then $F \otimes G$ is an $\{\alpha_{ijk} \cdot \alpha'_{ijk}\}$-twisted sheaf and $\mathcal{H}om(F, G)$ an $\{\alpha_{ijk}^{-1} \cdot \alpha'_{ijk}\}$-twisted sheaf.

- Let $f\colon X \to Y$ is a morphism of varieties and $\alpha \in \mathrm{Br}(X)$. If F is an $\{\alpha_{ijk}\}$-twisted sheaf on X, then f^*F is an $\{f^*\alpha_{ijk}\}$-twisted sheaf on Y. If F is an $\{f^*\alpha_{ijk}\}$-twisted sheaf on Y, then f_*F is an $\{\alpha_{ijk}\}$-twisted sheaf on X.

It is very easy to see the following

Lemma 2.7. *The $\{\alpha_{ijk}\}$-twisted sheaves on a twisted variety (X, α) form an abelian category $\mathbf{Mod}(X, \{\alpha_{ijk}\})$.*

Lemma 2.8. *Let (X, α) be a twisted variety.*

(i) If $\{\alpha_{ijk}\} \in \check{C}^2(\mathfrak{U}, \mathcal{O}_X^*)$ is a Čech 2-cocycle representing α for some open cover $\mathfrak{U}$ and $\{\alpha'_{ijk} \in \check{C}^2(\mathfrak{U}', \mathcal{O}_X^*)$ is the induced cocycle for some refinement $\mathfrak{U}'$ of $\mathfrak{U}$, we have an equivalence of categories

$$\Xi^{\alpha_{ijk}}_{\alpha'_{ijk}}\colon \mathbf{Mod}(X, \{\alpha_{ijk}\}) \simeq \mathbf{Mod}(X, \{\alpha'_{ijk}\}).$$

(ii) If $\{\alpha_{ijk}\}$ and $\{\alpha'_{ijk}\} \in \check{C}^2(\mathfrak{U}, \mathcal{O}_X^*)$ both represent the same cohomology class $\alpha \in H^2(X, \mathcal{O}_X^*)$, we have an equivalence of categories

$$\Xi^{\alpha_{ijk}}_{\alpha'_{ijk}}\colon \mathbf{Mod}(X, \{\alpha_{ijk}\}) \simeq \mathbf{Mod}(X, \{\alpha'_{ijk}\}).$$

Consequently, if $\{\alpha_{ijk}\}$ and $\{\alpha'_{ijk}\}$ are two different Čech representatives of $\alpha \in \mathrm{Br}(X)$, there is an induced equivalence of categories $\Xi^{\alpha_{ijk}}_{\alpha'_{ijk}}\colon \mathbf{Mod}(X, \{\alpha_{ijk}\}) \simeq \mathbf{Mod}(X, \{\alpha'_{ijk}\})$.

Proof. (i) Let us explain the idea of the proof. The details can be found in [6, Lemma 1.2.3]. Since every $\{\alpha_{ijk}\}$-twisted sheaf can be regarded as an $\{\alpha'_{ijk}\}$-twisted sheaf by restricting the $\mathcal{F}_{U_i}$ to the smaller open sets, we have a refinement functor

$$\Xi^{\alpha_{ijk}}_{\alpha'_{ijk}}\colon \mathbf{Mod}(X, \{\alpha_{ijk}\}) \to \mathbf{Mod}(X, \{\alpha'_{ijk}\}).$$

We need to construct a quasi-inverse functor. For $(\{\mathcal{G}_j\}, \{\psi_{jk}\}) \in \mathbf{Mod}(X, \{\alpha'_{ijk}\})$ and $U_i \in \mathfrak{U}$ we define a sheaf $\mathcal{F}_i$ on U_i by glueing the $\mathcal{G}_j\restriction_{U'_j \cap U_i}$ via the ψ_{jk}. Together with the naturally induced glueing morphisms, these $\mathcal{F}_i$ form an $\{\alpha_{ijk}\}$-twisted sheaf. This gives the wanted quasi-inverse functor $\mathbf{Mod}(X, \{\alpha'_{ijk}\}) \to \mathbf{Mod}(X, \{\alpha_{ijk}\})$.

(ii) As $\{\alpha_{ijk}\}$ and $\{\alpha'_{ijk}\}$ represent the same cohomology class, there is a Čech 1-cochain $\{\gamma_{ij}\} \in \check{C}(\mathfrak{U}, \mathcal{O}_X^*)$ such that $\alpha'_{ijk} = \alpha_{ijk} + d\gamma_{ij}$. The functor

$$\Xi^{\alpha_{ijk}}_{\alpha'_{ijk}}\colon \mathbf{Mod}(X, \alpha_{ijk}) \to \mathbf{Mod}(X, \alpha'_{ijk}),\ (\{\mathcal{F}_i\}, \{\varphi_{ij}\}) \mapsto (\{\mathcal{F}_i\}, \{\varphi_{ij} \cdot \gamma_{ij}\})$$

with the obvious operation on morphisms gives an equivalence of categories, cf. [6, Lemma 1.2.8]. □

Remark 2.9. As the choice of the cochain $\{\gamma_{ij}\}$ is non-canonical, so is the equivalence $\mathbf{Mod}(X, \{\alpha_{ijk}\}) \to \mathbf{Mod}(X, \{\alpha'_{ijk}\})$. However, this will not matter in our future considerations.

Definition 2.10. By abuse of notation, the equivalence class of the categories $\mathbf{Mod}(X, \{\alpha_{ijk}\})$ is denoted by $\mathbf{Mod}(X, \alpha)$.

We say that $\mathcal{F} \in \mathbf{Mod}(X, \{\alpha_{ijk}\})$ and $\mathcal{G} \in \mathbf{Mod}(X, \{\alpha'_{ijk}\})$ are isomorphic if $\Xi^{\alpha_{ijk}}_{\alpha'_{ijk}}(\mathcal{F})$ and $\mathcal{G}$ are.

As usual in the theory of Fourier-Mukai functors, we are only interested in a full subcategory of $\mathbf{Mod}(X, \alpha)$.

Definition 2.11. An $\{\alpha_{ijk}\}$-twisted sheaf $\mathcal{F} = \{\mathcal{F}_i, \varphi_{ij}\}$ on a twisted variety (X, α) is called *coherent* if all $\mathcal{F}_i$ are coherent (as ordinary sheaves on U_i).

The full abelian subcategory of $\mathbf{Mod}(X, \{\alpha_{ijk}\})$ spanned by all coherent sheaves is denoted by $\mathbf{Coh}(X, \{\alpha_{ijk}\})$.

In complete analogy to the above, we have

Lemma 2.12. *Let (X, α) be a twisted variety. If $\{\alpha_{ijk}\}$ and $\{\alpha'_{ijk}\}$ are two different Čech representatives of some $\alpha \in \mathrm{Br}(X)$, there is a (non-canonical) equivalence of categories $\Xi^{\alpha_{ijk}}_{\alpha'_{ijk}} \colon \mathbf{Coh}(X, \{\alpha_{ijk}\}) \simeq \mathbf{Coh}(X, \{\alpha'_{ijk}\})$.*

Proof. As for Lemma 2.8. □

Definition 2.13. By abuse of notation, the equivalence class of the categories $\mathbf{Coh}(X, \{\alpha_{ijk}\})$ is denoted by $\mathbf{Coh}(X, \alpha)$.

We say that $\mathcal{F} \in \mathbf{Coh}(X, \{\alpha_{ijk}\})$ and $\mathcal{G} \in \mathbf{Coh}(X, \{\alpha'_{ijk}\})$ are isomorphic if $\Xi^{\alpha_{ijk}}_{\alpha'_{ijk}}(\mathcal{F})$ and $\mathcal{G}$ are.

2.2. **Brauer-Severi varieties.** In this subsection, we shall explain the connection between twisted sheaves on a twisted variety (X, α) and (ordinary) sheaves on a Brauer-Severi variety over X.

We recall from [10, Sect. 8]

Definition 2.14. Let $p \colon Y \to X$ be a morphism of varieties over $\mathbb{C}$. Then Y is called a *Brauer-Severi variety of relative dimension r over X* if one of the following two conditions is satisfied:

(i) There exists an analytic open cover $X = \bigcup U_i$ together with isomorphisms $\varphi_i \colon p^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{P}^r_{\mathbb{C}}$ for all i.
(ii) The morphism f is finitely presented, proper and flat, and the fibres over the closed points of X are isomorphic to $\mathbb{P}^r_{\mathbb{C}}$.

We will be interested in

Definition 2.15. A Brauer-Severi variety $p \colon Y \to X$ is called a *projective bundle* if the transition functions $(U_i \cap U_j) \times \mathbb{P}^r_{\mathbb{C}} \xrightarrow{\varphi_i^{-1}} p^{-1}(U_i \cap U_j) \xrightarrow{\varphi_j} (U_i \cap U_j) \times \mathbb{P}^r_{\mathbb{C}}$ are given by projective linear transformations $\phi_{ij} \in H^0(U_{ij}, PGL(r+1))$ in the second coordinate.

Definition 2.16. A projective bundle Y of relative dimension r over X corresponds to a cohomology class $[Y] \in H^1(X, PGL(r+1))$. We define $\delta(Y) \in \mathrm{Br}(X)$ to be the image of $[Y]$ under the connecting homomorphism $H^1(X, PGL(r+1)) \to H^2(X, \mathcal{O}_X^*)$ induced by the short exact sequence

$$1 \to \mathcal{O}_X^* \to GL(r+1) \to PGL(r+1) \to 1.$$

Remark 2.17. It follows from the commutativity of the diagram

$$\begin{tikzcd} 1 \arrow[r] & \mu_{r+1} \arrow[r] \arrow[d, hook] & SL(r+1) \arrow[r] \arrow[d, hook] & PGL(r+1) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathcal{O}_X^* \arrow[r] & GL(r+1) \arrow[r] & PGL(r+1) \arrow[r] & 1 \end{tikzcd}$$

that $\delta(Y)$ is indeed contained in the torsion part of $H^2(X, \mathcal{O}_X^*)$.

There is a close connection between locally free twisted sheaves on X and projective bundles over X. The following Lemma is probably well known although only weaker versions seem to be present in the literature.

Lemma 2.18. *For a twisted variety (X, α), we have a one-to-one correspondence*

$$\{\text{loc. free, } \alpha\text{-twisted sheaves on } X\}\big/\sim \;\longleftrightarrow\; \{p\colon Y \to X \text{ proj. bdl.}, \delta(Y) = \alpha\}\big/\cong$$

where $\mathcal{F} \sim \mathcal{G}$ if and only if there is an $\mathcal{M} \in \operatorname{Pic} X$ such that $\mathcal{F}$ and $\mathcal{G} \otimes \mathcal{M}$ are isomorphic.

If $\delta(Y) = \alpha$, there is an (up to isomorphism and multiplication with $p^*\mathcal{M}$ unique) canonical $p^*\{\alpha_{ijk}^{-1}\}$-twisted line bundle $\mathcal{O}_p(1)$ on Y.

Proof. Let $\mathcal{E} = (\{\mathcal{E}_i\}, \{\varphi_{ij}\}) \in \mathbf{Coh}(X, \alpha)$ be a locally free sheaf of rank $r+1$ on X which is twisted with respect to the Čech cocycle $\{\alpha_{ijk}\}$ for an open cover $X = \bigcup_{i \in I} U_i$. Then we obtain projective bundles $p_i\colon \mathbb{P}(\mathcal{E}_i^\vee) \to U_i$. The glueing functions $\widetilde{\varphi}_{ij}\colon \mathbb{P}(\mathcal{E}_i^\vee)\restriction_{U_i \cap U_j} \xrightarrow{\sim} \mathbb{P}(\mathcal{E}_j^\vee)\restriction_{U_i \cap U_j}$ induced by $\varphi_{ij}\colon \mathcal{E}_i\restriction_{U_i \cap U_j} \to \mathcal{E}_j\restriction_{U_i \cap U_j}$ are locally projective linear transformations $\phi_{ij} \in H^0(U_{ij}, PGL(r+1))$ in the second coordinate. They also satisfy $\widetilde{\varphi}_{ii} = \mathrm{id}$ and the (ordinary) cocycle condition because the φ_{ij} do so up to scalars. Hence, the p_i glue to a global projective bundle $p\colon \mathbb{P}(\mathcal{E}) \to X$. It corresponds to the Čech cohomology class $\{[\phi_{ij}]\} \in H^1(X, PGL(r+1))$ which has the lift $\{[\varphi_{ij}]\} \in H^1(X, GL(r+1))$. Since $(\partial\{[\varphi_{ij}]\})_{ijk} = \varphi_{ij}\varphi_{jk}\varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id} \in H^2(X, \mathcal{O}_X^*)$, we have $\delta(\mathbb{P}(\mathcal{E})) = \alpha$ by construction of the connecting homomorphism of the short exact sequence

$$1 \to \mathcal{O}_X^* \to GL(r+1) \to PGL(r+1) \to 1.$$

Since the tautological line bundles $\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1)$ on the $\mathbb{P}(\mathcal{E}_i)$ are compatible with the glueing construction of $\mathbb{P}(\mathcal{E})$, they glue together to give a $p^*(\alpha^{-1})$-twisted line bundle $\mathcal{O}_p(1)$.

Conversely, let $p\colon Y \to X$ be a projective bundle of relative dimension r over X. Choose a trivializing open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ and trivializing functions $\psi_i\colon p^{-1}(U_i) \to U_i \times \mathbb{P}^r$. Let $\{\phi_{ij}\} \in \check{C}^1(\mathfrak{U}, PGL(r+1))$ be the projective linear transformations which form the second coordinate of the transition functions $\psi_{ij} = \psi_j \circ \psi_i^{-1}$. Refining the cover $\mathfrak{U} = \{U_i\}$ even further, we may assume that $\{\phi_{ij}\}$ has a lift $\{\varphi_{ij}\} \in \check{C}^1(\mathfrak{U}, GL(r+1))$. The choice of this lift is unique up to elements in $\check{C}^1(\mathfrak{U}, \mathcal{O}_X^*)$. Since $\partial\{\phi_{ij}\} = 0$, $\partial\{\varphi_{ij}\}$ has a (closed) lift $\{\alpha_{ijk}\} \in \check{C}^2(\mathfrak{U}, \mathcal{O}_X^*)$. Its cohomology class $\{[\alpha_{ijk}]\}$ is by definition of the connecting homomorphism the class $\delta(Y)$. Thus, the sheaves $p_*(\mathcal{O}_{\mathbb{P}^r_{U_i}}(1))^\vee$ glue via the φ_{ij} to an α-twisted sheaf $\mathcal{E}$ which is unique up to multiplication with a line bundle $\mathcal{M} \in \operatorname{Pic}(X)$ (and isomorphism).

It is straightforward to see that the two constructions are inverse to another (up to the equivalence relations). □

As there is a locally free α-twisted sheaf for all $\alpha \in \operatorname{Br}(X)$, the Lemma asserts in particular that for every α, there is a projective bundle $Y \to X$ with $\delta(Y) = \alpha$. This allows us to describe the category $\mathbf{Coh}(X, \alpha)$ as a full subcategory of the category $\mathbf{Coh}(Y)$ of (ordinary) coherent sheaves on some projective bundle Y over X. This approach is due to [34, §1].

Definition 2.19. Let $p\colon Y \to X$ be a projective bundle of relative dimension r over X with $\delta(Y) = \alpha$. Then we denote by $\mathbf{Coh}(Y/X)$ the full subcategory of $\mathbf{Coh}(Y)$ spanned by all coherent sheaves $F \in \mathbf{Coh}(Y)$ for which there exists an α-twisted sheaf $\widetilde{F} \in \mathbf{Coh}(X, \alpha)$ such that

$$F \cong p^*\widetilde{F} \otimes \mathcal{O}_p(1)$$

Lemma 2.20. *The two functors*

$$K\colon \mathbf{Coh}(X, \alpha) \to \mathbf{Coh}(Y/X),\; E \mapsto p^*E \otimes \mathcal{O}_p(1)$$
$$\Lambda\colon \mathbf{Coh}(Y/X) \to \mathbf{Coh}(X, \alpha),\; F \mapsto p_*(F \otimes \mathcal{O}_p(-1)$$

yield an equivalence of categories.

Proof. Follows direct from the projection formula and the fact that $p_*\mathcal{O}_Y = \mathcal{O}_X$ for all projective line bundles $p\colon Y \to X$. $\square$

For the purposes of deformation theory, another characterization of $\mathbf{Coh}(Y/X)$ proves to be more effective.

Definition 2.21. For all projective bundles $p\colon Y \to X$ with $\delta(X) = \alpha$, let $G_p \in \mathbf{Coh}(Y/X)$ be the sheaf given by

$$G_p := p^*(p_*(\mathcal{O}_p(1))^\vee) \otimes \mathcal{O}_p(1).$$

Lemma 2.22. *The sheaf G_p is the unique non-trivial extension of the Euler sequence*

$$0 \to \mathcal{O}_Y \to G_p \to \mathcal{T}_{Y/X} \to 0.$$

Proof. Cf. [34, Lemma 1.1]. $\square$

Lemma 2.23. *A coherent sheaf $F \in \mathbf{Coh}(Y)$ lies in $\mathbf{Coh}(Y/X)$ if and only if the adjunction counit $\varepsilon\colon p^*p_*(G_p^\vee \otimes F) \to G_p^\vee \otimes F$ is an isomorphism. In particular, $F \in \mathbf{Coh}(Y/X)$ is an open condition.*

Proof. Cf. [34, Lemma 1.5]. A similar proof is given in Lemma 2.42. $\square$

2.3. **Twisted Hodge structures.** Let at first (X, α) be a smooth projective twisted variety with $H^3(X, \mathbb{Z})_{\mathrm{tor}} = 0$. Following [18], we shall generalize Chern characters and Hodge structures to the twisted case in such a way that $\mathbf{Coh}(X, \alpha)$ interacts nicely with cohomology. To this end, we first give

Definition 2.24. The group $K(X, \alpha)$ is the Grothendieck group of the exact category $\mathbf{Coh}(X, \alpha)$.

Remark 2.25. As there is in general no natural tensor structure on $\mathbf{Coh}(X, \alpha)$, $K(X, \alpha)$ is in fact only endowed with the natural structure of an additive group and not of a ring.

In order to define a Chern character for the twisted variety (X, α), we first have to choose a rational B-field lift of α.

Definition 2.26. A *B-field* on X is a cohomology class $B \in H^2(X, \mathbb{R})$. The image of B under the morphism

$$H^2(X, \mathbb{R}) \subset H^2(X, \mathbb{C}) \cong \bigoplus_{p+q=2} H^q(X, \Omega_X^p) \twoheadrightarrow H^2(X, \mathcal{O}_X) \xrightarrow{\exp} H^2(X, \mathcal{O}_X^*)$$

is denoted by $\alpha_B := \exp(B^{0,2})$.

Remark 2.27. If $\beta \in H^2(X, \mathcal{O}_X^*)$ is a torsion class (which is the case for a twisted variety), we can always choose a *rational* B-field lift $B \in H^2(X, \mathbb{Q})$ with $\alpha_B = \beta$.

These notions enable us to give

Definition 2.28. For a rational B-field $B \in H^2(X, \mathbb{Q})$, the *B-twisted Chern character*

$$\mathrm{ch}^B\colon K(X, \alpha) \to H^*(X, \mathbb{Q})$$

is defined as follows, cf. [18, Prop. 1.2]:

It suffices to give ch^B on α-twisted sheaves such that it is additive for short exact sequences. As the sheaf $\mathcal{C}_X^\infty$ of smooth functions on X is acylic, we can (after passing to a sufficiently fine covering $\mathfrak{U} = \{U_i\}$) choose a Čech cocycle $\{a_{ij}\} \in \check{C}^1(\mathfrak{U}, \mathcal{C}_X^\infty)$ such that $-\partial\{a_{ij}\} = \{B_{ijk}\} = B$. Then for all α-twisted sheaves $E = (\{E_i\}, \{\varphi_{ij}\})$, we can define

$$E_B := (\{E_i\}, \{\varphi_{ij} \cdot \exp(a_{ij})\}).$$

It is easy to check that E_B is an untwisted sheaf which does not depend on the choice of the Čech cocycle $\{a_{ij}\}$. We can therefore set $\mathrm{ch}^B(E) := \mathrm{ch}(E_B)$.

With this construction, the important properties of the Chern character are preserved in the twisted case.

Proposition 2.29. *For all rational B-fields $B, B_1, B_2 \in H^2(X, \mathcal{O}_X)$, the twisted Chern character satisfies*

(i) ch^B *is additive, i.e.* $\mathrm{ch}^B(E_1 \oplus E_2) = \mathrm{ch}^B(E_1) + \mathrm{ch}^B(E_2)$

(ii) If $B = c_1(L) \in H^2(X, \mathbb{Z})$, *we have* $\mathrm{ch}^B(E) = \exp(c_1(L)) \cdot \mathrm{ch}(E)$ *where* $\exp(c_1(L)) = \sum_i \frac{1}{i!}(c_1(L))^i$ *is the formal exponential of* $c_1(L)$.

(iii) $\mathrm{ch}^{B_1}(E_1) \cdot \mathrm{ch}^{B_2}(E_2) = \mathrm{ch}^{B_1+B_2}(E_1 \otimes E_2)$ *and*

(iv) $\mathrm{ch}^B(E) \in \exp(B)\left(\bigoplus H^{p,p}(X)\right)$ *for all* $E \in K(X, \alpha_B)$.

Proof. Cf. [18, Prop. 1.2]. □

Remark 2.30. In particular, (ii) and (iii) of the previous Proposition (and the Lefschetz theorem on $(1,1)$-classes) imply that for two different B-field lifts B_1, B_2 of α, the twisted Chern characters $\mathrm{ch}^{B_1}, \mathrm{ch}^{B_2}$ differ only by the vector space isomorphism $\exp(B_1 - B_2)$.

As in the untwisted case, one further defines a twisted Mukai vector.

Definition 2.31. The *Mukai vector* of a class $E \in K(X, \alpha_B)$ is given by

$$v^B(E) := \mathrm{ch}^B(E) \cdot \sqrt{\mathrm{td}(X)}.$$

We now restrict our attention to the case of K3 surfaces. Recall that for all K3 surfaces X, the intersection pairing (.) endows $H^2(X, \mathbb{Z})$ with a lattice structure and that there is a lattice isometry $H^2(X, \mathbb{Z}) \cong \Lambda$ to the *K3 lattice* $\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. Here E_8 denotes the usual E_8-lattice and U the hyperbolic plane. The full cohomology group $H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ receives a lattice structure from the *Mukai pairing*

$$\langle\ ,\ \rangle \colon H^*(X, \mathbb{Z}) \times H^*(X, \mathbb{Z}) \to \mathbb{Z},\ \langle \varphi, \psi \rangle := (\varphi_2.\psi_2) - (\varphi_0.\psi_4) - (\varphi_4.\psi_0).$$

This lattice structure with the grading suppressed is commonly denoted by $\widetilde{H}(X, \mathbb{Z})$. It carries a natural weight-two Hodge structure given by

$$\widetilde{H}^{2,0}(X) := H^{2,0}(X),\ \widetilde{H}^{1,1}(X) := H^0(X) \oplus H^{1,1}(X) \oplus H^4(X),\ \widetilde{H}^{0,2}(X) := H^{0,2}(X).$$

As an abstract lattice, $\widetilde{H}(X, \mathbb{Z})$ is isomorphic to the *extended K3 lattice* $\widetilde{\Lambda} := \Lambda \oplus U = E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}$.

Prop. 2.29 (iv) suggests a definition of a twisted Hodge structure which is well suited for the interaction with twisted Chern characters.

Lemma 2.32. *The multiplication with the formal exponential* $\exp B$ *of a B-field* $B \in H^2(X, \mathbb{R})$ *is an isometry of* $\widetilde{H}(X, \mathbb{R})$. *If* $B \in H^2(X, \mathbb{Q})$ *resp.* $H^2(X, \mathbb{Z})$, $\exp B$ *is an isometry of* $H^2(X, \mathbb{Q})$ *resp.* $H^2(X, \mathbb{Z})$.

Proof. It follows from $\sum_{j=0}^n (-1)^j \binom{n}{j} = 0$ that $\exp B$ is a vector space (resp. $\mathbb{Z}$-module) isomorphism. A straightforward calculation shows that $\exp B$ preserves the Mukai pairing. □

Definition 2.33. For $B \in H^2(X, \mathbb{R})$, the *B-twisted Hodge structure* $\widetilde{H}(X, B, \mathbb{Z})$ is the Hodge structure of K3 type given by

$$\widetilde{H}^{p,q}(X, B) := \exp(B) \cdot \left(\widetilde{H}^{p,q}(X)\right).$$

Remark 2.34. The Hodge structure $\widetilde{H}(X, B, \mathbb{Z})$ of K3 type is uniquely determined by its $(2,0)$-part, i.e. by the *generalized Calabi-Yau form* $\exp B \cdot \sigma = \sigma + B \wedge \sigma$.

If $B_1, B_2 \in H^2(X, \mathbb{R})$ are B-field lifts of the same Brauer class $\alpha \in \operatorname{Br}(X)$, we have $B_1 - B_2 \in H^2(X, \mathbb{Z})$ and therefore obtain an integral Hodge isometry $\exp B \colon \widetilde{H}(X, B_1, \mathbb{Z}) \cong \widetilde{H}(X, B_2, \mathbb{Z})$ by Lemma 2.32. Thus, it makes sense to give

Definition 2.35. The (abstract) Hodge structure $\widetilde{H}(X, \alpha, \mathbb{Z})$ is the Hodge isometry type of $\widetilde{H}(X, B, \mathbb{Z})$ for some B-field lift $B \in H^2(X, \mathbb{R})$ of α.

Remark 2.36. Note however that this is only an abstract Hodge structure which can only be realized after the choice of a B-field.

2.4. **Twisted Fourier-Mukai transforms.** The notion of a Fourier-Mukai transform can be generalized to twisted derived categories. In what follows, (X, α), (Y, β) and (Z, γ) will be twisted varieties over some base A. All constructions shall be taken over A, e.g. $X \times Y = X \times_A Y$ or $\omega_X = \omega_{X/A}$. For the sake of clarity, we will denote derived functors by their non-derived symbols, e.g. p_* instead of Rp_* unless it is not apparent from the context which functor is meant.

Definition 2.37. The *derived category* of the twisted variety (X, α) is the bounded derived category

$$\mathrm{D^b}(X, \alpha) := \mathrm{D^b}(\mathbf{Coh}(X, \alpha))$$

of the abelian category $\mathbf{Coh}(X, \alpha)$ of α-twisted coherent sheaves on X.

Căldăraru shows in [6] that all basic constructions, such as derived pullback, pushforward, tensor product and Hom complexes, and results carry over from the untwisted case.

It has been alluded to in [34, Sect. 4] and proven rigorously in [2, Sect. 4.1] that the derived category of a twisted variety (X, α) is again equivalent to the (ordinary) derived category of a projective bundle Y over X. As in Def. 2.19, we have

Definition 2.38. Let $p \colon Y \to X$ be a projective bundle of relative dimension r over X with $\delta(Y) = \alpha$. Then we denote by $\mathrm{D^b}(Y/X)$ the full subcategory of $\mathrm{D^b}(Y)$ spanned by all $\mathcal{F} \in \mathrm{D^b}(Y)$ for which there exist an $\widetilde{\mathcal{F}} \in \mathrm{D^b}(X, \alpha)$ such that

$$\mathcal{F} \simeq p^* \widetilde{\mathcal{F}} \otimes \mathcal{O}_p(1).$$

Remark 2.39. Since p is flat, we can take the ordinary pullback p^*.

Remark 2.40. It is a priori not clear whether $\mathrm{D^b}(Y/X) = \mathrm{D^b}(\mathbf{Coh}(Y/X))$ although it will turn out that the two categories are equivalent. One should rather think of $\mathrm{D^b}(Y/X)$ as the derived equivalent of $\mathbf{Coh}(Y/X)$.

One finds again

Lemma 2.41. *The two functors*

$$K \colon \mathrm{D^b}(X, \alpha) \to \mathrm{D^b}(Y/X),\ \mathcal{E} \mapsto p^* \mathcal{E} \otimes \mathcal{O}_p(1)$$
$$\Lambda \colon \mathrm{D^b}(Y/X) \to \mathrm{D^b}(X, \alpha),\ \mathcal{F} \mapsto p_*(\mathcal{F} \otimes \mathcal{O}_p(-1)$$

yield an equivalence of categories.

Proof. Similar to the proof of Lemma 2.20. See [2, Thm. 4.4] for details (note however that there is a sign issue because Bernardara constructs the projective bundle Y by glueing bundles $\mathbb{P}(E_i) \to U_i$ and not $\mathbb{P}(E_i^\vee) \to U_i$). □

In addition, we can prove by methods similar to those in [34, Lemma 1.5]

Lemma 2.42. *A complex $\mathcal{F} \in \mathrm{D^b}(Y)$ lies in $\mathrm{D^b}(Y/X)$ if and only if the adjunction counit $\varepsilon \colon p^* Rp_*(G_p^\vee \otimes \mathcal{F}) \to G_p^\vee \otimes \mathcal{F}$ is a quasi-isomorphism (i.e. an isomorphism in $\mathrm{D^b}(Y)$). In particular, $\mathcal{F} \in \mathbf{Coh}(Y/X)$ is an open condition.*

Proof. Since $p\colon Y \to X$ is a projective bundle, we have $Rp_*\mathcal{O}_Y = \mathcal{O}_X$. In particular, it follows that
$$p^*Rp_*p^*(p_*(\mathcal{O}_p(1))^\vee) \simeq p^*(p_*(\mathcal{O}_p(1))^\vee)$$
because $p^*(p_*(\mathcal{O}_p(1))^\vee)$ is locally free. As $G_p = p^*(p_*(\mathcal{O}_p(1))^\vee) \otimes \mathcal{O}_p(1)$, this yields
$$p^*Rp_*(G_p^\vee \otimes \mathcal{F}) \simeq p^*(p_*(\mathcal{O}_p(1))^\vee) \otimes p^*Rp_*(\mathcal{F} \otimes \mathcal{O}_p(-1)).$$
Hence, ε is a quasi-isomorphism if and only if the adjunction counit
$$\widetilde{\varepsilon}\colon p^*Rp_*(\mathcal{F} \otimes \mathcal{O}_p(-1)) \to \mathcal{F} \otimes \mathcal{O}_p(-1)$$
is.

If $\widetilde{\varepsilon}$ is a quasi-isomorphism, then $\mathcal{F} \simeq p^*(Rp_*(\mathcal{F} \otimes \mathcal{O}_p(-1))) \otimes \mathcal{O}_p(1)$ immediately gives $\mathcal{F} \in \mathrm{D^b}(Y/X)$. For the converse, let $\mathcal{D}$ be the full subcategory of $\mathrm{D^b}(Y, p^*\alpha)$ spanned by all elements $\mathcal{G} \in \mathrm{D^b}(Y, p^*\alpha)$ for which there exists a $\widetilde{\mathcal{G}} \in \mathrm{D^b}(X, \alpha)$ such that $\mathcal{G} \simeq p^*\widetilde{\mathcal{G}}$. As $p\colon Y \to X$ is proper, this is well-defined. We have to show that the natural functor transformation
$$\epsilon\colon p^*Rp_* \Rightarrow \mathrm{id}_\mathcal{D}$$
of functors $\mathcal{D} \to \mathcal{D}$ given by the counit of the adjunction is a natural isomorphism. By [21, Lemma A1.1.1], this is the case if there is *any* natural functor isomorphism $p^*Rp_* \Rightarrow \mathrm{id}_\mathcal{D}$. But the projection formula implies that for every $\mathcal{G} \simeq p^*\widetilde{\mathcal{G}} \in \mathcal{D}$, there is a natural quasi-isomorphism
$$p^*Rp_*(\mathcal{G}) \simeq p^*Rp_*p^*\widetilde{\mathcal{G}} \simeq p^*(\widetilde{\mathcal{G}} \otimes Rp_*\mathcal{O}_Y) \simeq p^*(\widetilde{\mathcal{G}} \otimes \mathcal{O}_X) \simeq \mathcal{G}. \qquad \square$$

Having generalized derived categories to the twisted case, we can now proceed to Fourier-Mukai functors.

Definition 2.43. Let $\mathcal{P} \in \mathrm{D^b}(X \times Y, \alpha^{-1} \boxtimes \beta)$. Then the functor
$$\Phi_\mathcal{P}\colon\ \mathrm{D^b}(X, \alpha) \to \mathrm{D^b}(Y, \beta), \quad \mathcal{E}^\bullet \mapsto \pi_{2*}(\mathcal{P} \otimes \pi_1^*\mathcal{E}^\bullet)$$
(where π_i denotes the projection from $X \times Y$ to the i-th factor) is called the *twisted Fourier-Mukai transform* with *Fourier-Mukai kernel* $\mathcal{P}$.

Examples for twisted Fourier-Mukai transforms can be found in [19, §1].

As in the untwisted case, it is easy to see that composition of two twisted Fourier-Mukai transforms is again a Fourier-Mukai transform.

Definition 2.44. Let $\mathcal{E} \in \mathrm{D^b}(X \times Y, \alpha^{-1} \boxtimes \beta)$ and $\mathcal{F} \in \mathrm{D^b}(Y \times Z, \beta^{-1} \boxtimes \gamma)$. Then the *convolution* of $\mathcal{E}$ and $\mathcal{F}$ is
$$\mathcal{E} * \mathcal{F} := \pi_{13*}(\pi_{12}^*\mathcal{E} \otimes \pi_{23}^*\mathcal{F}) \in \mathrm{D^b}(X \times Z, \alpha^{-1} \boxtimes \gamma)$$
(where again the π_{ij} denote the obvious projections from $X \times Y \times Z$).

Lemma 2.45. *Let $\mathcal{P} \in \mathrm{D^b}(X \times Y, \alpha^{-1} \boxtimes \beta)$ and $\mathcal{Q} \in \mathrm{D^b}(Y \times Z, \beta^{-1} \boxtimes \gamma)$. Then the functors $\Phi_\mathcal{Q} \circ \Phi_\mathcal{P}$ and $\Phi_{\mathcal{P}*\mathcal{Q}}$ are naturally isomorphic. In particular, the composition of two twisted Fourier-Mukai transforms is again a twisted Fourier-Mukai transform.*

Proof. Exactly as in the untwisted case [28, Prop. 1.3], taking into account that the projection formula also holds in twisted derived categories (cf. [6, Prop. 2.3.5]). $\square$

We will also need

Definition 2.46. Let $\tau\colon X \times Y \to Y \times X$ denote the transposition of the two factors. Then we define for all $\mathcal{E} \in \mathrm{D^b}(X \times Y, \alpha^{-1} \boxtimes \beta)$
$$\mathcal{E}_L := \tau^*(\mathcal{E}^\vee \otimes \pi_2^*\omega_Y[\dim Y]) \in \mathrm{D^b}(Y \times X, \beta^{-1} \boxtimes \alpha) \quad \text{and}$$
$$\mathcal{E}_R := \tau^*(\mathcal{E}^\vee \otimes \pi_1^*\omega_X[\dim X]) \in \mathrm{D^b}(Y \times X, \beta^{-1} \boxtimes \alpha).$$

Lemma 2.47. *Let* $\mathcal{P} \in \mathrm{D^b}(X \times Y, \alpha^{-1} \boxtimes \beta)$*. Then* $\Phi_{\mathcal{P}_L}\colon \mathrm{D^b}(Y,\beta) \to \mathrm{D^b}(X,\alpha)$ *resp.* $\Phi_{\mathcal{P}_R}\colon \mathrm{D^b}(Y,\beta) \to \mathrm{D^b}(X,\alpha)$ *is right resp. left adjoint to* $\Phi_{\mathcal{P}}\colon \mathrm{D^b}(X,\alpha) \to \mathrm{D^b}(Y,\beta)$.

Proof. The proof can again be given as in the untwisted case [12, Prop. 5.9] because Grothendieck-Verdier duality and basic properties of duals and direct and inverse images still hold, cf. [6, Thm. 2.4.1, Prop. 2.3.14 and Cor. 2.3.9]. $\square$

Due to the existence of an analogue of Orlov's result, twisted Fourier-Mukai functors retain their importance for the study of the derived categories of twisted varieties.

Proposition 2.48 (Canonaco, Stellari)**.** *Let* (X,α) *and* (Y,β) *be twisted smooth projective varieties and* $F\colon \mathrm{D^b}(X,\alpha) \to \mathrm{D^b}(Y,\beta)$ *an exact full functor. Then there is a (up to isomorphism unique)* $\mathcal{P} \in \mathrm{D^b}(X \times Y, \alpha^{-1} \boxtimes \beta)$ *such that* $F \cong \Phi_{\mathcal{P}}$.

Proof. [8, Thm. 1.1] $\square$

Concerning the compatibility of Fourier-Mukai transforms with the projective bundle construction, let us finally add the following

Remark 2.49. Fourier-Mukai transforms can also be naturally defined for the category $\mathrm{D^b}(Y/X)$ of a projective bundle. Let $p\colon Y \to X$, $p'\colon Y' \to X'$ be two projective bundles with $\delta(Y) = \alpha$, $\delta(Y') = \alpha'$. Then we have again an equivalence of categories

$$\mathrm{D^b}(X \times X', \alpha^{-1} \boxtimes \alpha') \simeq \mathrm{D^b}(Y \times Y'/(X, X'))$$

where $\mathrm{D^b}(Y \times Y'/(X,X'))$ is the full subcategory of $\mathrm{D^b}(Y \times Y')$ spanned by all $\mathcal{F}$ for which the natural morphism $(p' \times p)^*(p' \times p)_*(G_{p'} \otimes \mathcal{F} \otimes G_p^\vee) \to (G_{p'} \otimes \mathcal{F} \otimes G_p^\vee)$ is an isomorphism. The Fourier-Mukai transform associated to some $\mathcal{P} \in \mathrm{D^b}(Y \times Y'/(X,X'))$ is given by

$$\Phi_{\mathcal{P}}\colon \mathrm{D^b}(Y/X) \to \mathrm{D^b}(Y'/X'),\ \mathcal{E} \mapsto \left(\pi_2^{Y' \times Y}\right)_*\left(\left(\pi_1^{Y' \times Y}\right)^* \mathcal{E} \otimes \mathcal{P}\right).$$

3. Hodge theory of twisted K3 surfaces

In this chapter, we shall investigate cohomological properties and invariants of twisted K3 surfaces.

Let us first recall the necessary notation. We denote by M the Fermat quartic $V(x_0^4+x_1^4+x_2^4+x_3^4) \subset \mathbb{P}^3$, by $\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ the K3 lattice and by $\widetilde{\Lambda} := \Lambda \oplus U$ the extended K3 lattice. If X is a K3 surface and $B \in H^2(X,\mathbb{Q})$ a rational B-field on it, we can put $\alpha_B := \exp(B^{0,2})$ to obtain a twisted K3 surface (X, α_B).

An important concept in the study of $\widetilde{H}(X,B,\mathbb{Z})$ is the generalized Picard group.

3.1. **The generalized Picard group.**

Definition 3.1. Let X be a K3 surface with holomorphic two-form $\sigma \in H^{2,0}(X)$ and $B \in H^2(X,\mathbb{Q})$ a rational B-field. Then the *generalized Picard group* of the twisted K3 surface (X,α_B) is the sublattice

$$\mathrm{Pic}(X,B) := \{\delta \in \widetilde{H}(X,B,\mathbb{Z}) \mid \langle \exp(B) \cdot \sigma, \delta \rangle = 0\}$$

and the *generalized transcendental lattice* of (X,α_B) is

$$T(X,B) := \mathrm{Pic}(X,B)^{\perp} \subset \widetilde{H}(X,B,\mathbb{Z})$$

with the natural Hodge structure induced from $\widetilde{H}(X,B,\mathbb{Z})$.

Remark 3.2. In the untwisted case $B = 0$, we have $\mathrm{Pic}(X,0) = H^0(X,\mathbb{Z}) \oplus NS(X) \oplus H^4(X,\mathbb{Z})$.

Remark 3.3. If (X, α) is a twisted K3 surface, then we have $\mathrm{Pic}(X, B) \cong \mathrm{Pic}(X, B')$ resp. $T(X, B) \cong T(X, B')$ for different choices of B-field lifts $B, B' \in H^2(X, \mathbb{Q})$ of α. We can therefore define $\mathrm{Pic}(X, \alpha)$ resp. $T(X, \alpha)$ to be the isomorphism class of the lattices $\mathrm{Pic}(X, B)$ resp. $T(X, B)$.

Remark 3.4. More generally, we always have a finite index immersion

$$\mathbb{Z} \cdot (a, a \cdot B, 0) \oplus NS(X) \oplus H^4(X, \mathbb{Z}) \hookrightarrow \mathrm{Pic}(X, B).$$

Hence, $\mathrm{Pic}(X, B)$ has signature $(2, n)$ if and only if $NS(X)$ has signature $(1, n)$ if and only if X is algebraic.

The first question that arises is which lattices can occur as the generalized Picard group of a twisted K3 surface.

Proposition 3.5. *Let Γ be an even lattice of rank $3 \leq \mathrm{rk}\,\Gamma \leq 12$ and signature $(2, \mathrm{rk}\,\Gamma - 2)$ which contains a primitive $x \in \Gamma$ with $(x.x) = 0$. Then there exists a twisted algebraic K3 surface (X, α_B) with $\mathrm{Pic}(X, B) \cong \Gamma$.*

Although the proof for the untwisted case was given in [27, Cor. 1.9] and carries over with minor changes, we shall give the complete argument because the twisted case does not seem to be treated in the literature.

Proof of Prop. 3.5. Recall that $\widetilde{\Lambda} = \Lambda \oplus U$ is the extended Mukai lattice equipped with the Mukai pairing $\langle\, , \rangle$. We denote the standard basis vectors of $U = (H^0 \oplus H^4)(M, \mathbb{Z})$ by e and f where e resp. f is the generator of the cohomology in degree 0 resp. 4.

We have $\mathrm{rk}\,\Gamma \leq 12 = \frac{1}{2} \cdot \mathrm{rk}\,\widetilde{\Lambda}$, so there exists a primitive embedding $\Gamma \hookrightarrow \widetilde{\Lambda}$ by [29, Thm. 1.12.4] and [26, Thm. II.5.3]. Since x is primitive with $\langle x, x \rangle = 0$, we may, after composing with an automorphism of $\widetilde{\Lambda}$ if necessary, assume that $x = f$ by [33, Thm. 3].

Let us now define a Hodge structure of K3 type on $\widetilde{\Lambda}$. As the Mukai pairing restricted to $T := \Gamma^{\perp}$ is non-degenerate, it defines a smooth quadric $X \subset \mathbb{P}(T_{\mathbb{C}})$, which is not contained in any hyperplane. In particular, for all $\alpha \in T$, the intersection $X \cap \alpha^{\perp} \subset X$ with the hyperplane $\alpha^{\perp} \subset \mathbb{P}(T_{\mathbb{C}})$ is a proper closed subset. By the Baire category theorem, the set $G := \bigcap_{\alpha \in T} X \setminus \alpha^{\perp}$ of all isotropic vectors in T who are not orthogonal to any $\alpha \in T$ is dense in X (with the complex topology). Thus, the intersection of G with the open (again in the complex topology) subset $\{x \in X \mid \langle x, \bar{x} \rangle > 0\} \subset X$ is non-empty. In other words, there exists a $p \in T_{\mathbb{C}}$ such that

$$\langle p, p \rangle = 0, \quad \langle p, \bar{p} \rangle > 0 \quad \text{and} \quad p^{\perp} \cap T = 0.$$

We can now define a Hodge structure on $\widetilde{\Lambda}$ by setting

$$\widetilde{\Lambda}^{2,0} := \mathbb{C} \cdot p, \quad \widetilde{\Lambda}^{0,2} := \mathbb{C} \cdot \bar{p} \quad \text{and} \quad \widetilde{\Lambda}^{1,1} := (\widetilde{\Lambda}^{2,0} \oplus \widetilde{\Lambda}^{0,2})^{\perp}.$$

As $T \oplus \Gamma \subset \widetilde{\Lambda}$ is of finite index, this Hodge structure satisfies $\widetilde{\Lambda}^{1,1} \cap \widetilde{\Lambda} = \Gamma$.

It is left to show that this is the Hodge structure of a twisted K3 surface. Since $x = f \in \Gamma$ and Γ is non-degenerate, there exists a $\delta_0 e + \delta_2 + \delta_4 f \in \Gamma$ with $\delta_2 \in \Lambda$, $\delta_0, \delta_4 \in \mathbb{Z}$ and $\delta_0 = \langle \delta_0 e + \delta_2 + \delta_4 f, f \rangle \neq 0$. Let $B := [(1/\delta_0) \cdot \delta_2] \in H^2(M, \mathbb{Q})$. Then we have $\exp(B) \in \Gamma_{\mathbb{Q}}$ and therefore

$$\exp(-B) \cdot p \in \exp(-B) \cdot T_{\mathbb{C}} \subseteq \exp(-B) \cdot \Gamma_{\mathbb{C}}^{\perp} \subset H^2(M, \mathbb{C}).$$

By the surjectivity of the period map, there exists a K3 surface X whose Calabi-Yau form $\sigma \in \mathcal{A}^2_{\mathbb{C}}(M)$ satisfies $[\sigma] = \exp(-B)p$. The the twisted K3 surface (X, α_B) is equipped with the generalized Calabi-Yau structure $\varphi := \exp(B)\sigma$ with period p and has therefore the Hodge structure defined above. In particular, we have

$$\mathrm{Pic}(X, B) = \widetilde{H}^{1,1}(X, B) \cap H^*(X, \mathbb{Z}) \cong \widetilde{\Lambda}^{1,1} \cap \widetilde{\Lambda} = \Gamma.$$

As the signature of $\mathrm{Pic}(X,B)$ is $(2, \mathrm{rk}\,\Gamma - 2)$, the K3 surface X is algebraic by Rem. 3.4. $\square$

Remark 3.6. By Meyer's theorem, an even lattice Γ of rank $5 \leq \mathrm{rk}\,\Gamma \leq 12$ and signature $(2, \mathrm{rk}\,\Gamma - 2)$ satisfies the assumptions of the proposition.

The situation is very different for high Picard rank, where comparatively few Picard groups can appear.

Proposition 3.7. *Let (X, α_B) be a twisted K3 surface. Then there exists a K3 surface Y with $\mathrm{D^b}(Y) \simeq \mathrm{D^b}(X, \alpha_B)$ if and only if $\mathrm{Pic}(X,B)$ contains a hyperbolic plane $U \subset \mathrm{Pic}(X,B)$.*

Proof. Every equivalence $\mathrm{D^b}(Y) \simeq \mathrm{D^b}(X, \alpha_B)$ induces a Hodge isometry $\widetilde{H}(Y,\mathbb{Z}) \cong \widetilde{H}(X,B,\mathbb{Z})$. Under this isometry $(H^0 \oplus H^4)(Y,\mathbb{Z}) \subset \widetilde{H}^{1,1}(Y,\mathbb{Z})$ maps to a hyperbolic plane $U \subset \mathrm{Pic}(X,B)$.

For the other direction, we assume that there is an embedding $U \hookrightarrow \mathrm{Pic}(X,B)$. Then there is a sublattice $\Gamma \subset \mathrm{Pic}(X,B)$ with $\mathrm{Pic}(X,B) = U \oplus \Gamma$. Since $T(X,B) = \mathrm{Pic}(X,B)^\perp \subset \widetilde{H}(X,B,\mathbb{Z})$ is primitive and $\widetilde{H}(X,B,\mathbb{Z})$ is unimodular, we have $\ell(T(X,B)) = \ell(\mathrm{Pic}(X,B))$ by [29, Prop. 1.6.1]. It follows that

$$\begin{aligned}\ell(T(X,B)) + 2 &= \ell(\mathrm{Pic}(X,B)) + 2 = \ell(\Gamma) + 2 \leq \mathrm{rk}\,\Gamma + 2 \\ &= \mathrm{rk}\,\mathrm{Pic}(X,B) = \mathrm{rk}\,\widetilde{\Lambda} - \mathrm{rk}\,T(X,B).\end{aligned}$$

Thus, we can conclude from [29, Thm. 1.14.4] that there exists a primitive embedding $\iota\colon T(X,B) \hookrightarrow \widetilde{\Lambda}$.

This embedding induces on $\widetilde{\Lambda}$ a Hodge structure of K3 type with

$$\widetilde{\Lambda}^{2,0} = \iota_{\mathbb{C}}(T^{2,0}) \quad \text{and} \quad \widetilde{\Lambda}^{1,1} = (\widetilde{\Lambda}^{2,0} \oplus \widetilde{\Lambda}^{0,2})^\perp$$

whose transcendental lattice is again $\iota(T(X,B)) \subset \widetilde{\Lambda}$. By the surjectivity of the period map, this is the Hodge structure $\widetilde{H}(Y,\mathbb{Z})$ of some K3 surface Y.

We have now two primitive embeddings

$$T(X,B) \hookrightarrow \widetilde{H}(X,B,\mathbb{Z}) \quad \text{and} \quad T(X,B) \cong T(Y) \hookrightarrow \widetilde{H}(Y,\mathbb{Z})$$

into Hodge structures with the same underlying lattice $\widetilde{\Lambda}$. But by [29, Thm. 1.14.4], the embedding $T(X,B) \hookrightarrow \widetilde{\Lambda}$ is unique up to automorphisms of $\widetilde{\Lambda}$ (Note that we need the existence of a hyperbolic plane $U \subset T(X,B)^\perp$ in order to draw this conclusion!). Hence, we can follow that there exists a lattice isometry

$$\varphi\colon \widetilde{H}(X,B,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\mathbb{Z}) \quad \text{with} \quad \varphi\restriction_{T(X,B)} = \iota.$$

This property of φ together with the choice of Y guarantees that φ is in fact a Hodge isometry. Composing with $\mathrm{id}_{H^0} \oplus -\mathrm{id}_{H^2} \oplus \mathrm{id}_{H^4}$ if necessary, we may in addition assume that φ preserves the natural orientation of the four positive directions.

Căldăraru's conjecture, proven in [19], now implies that φ can be lifted to a Fourier-Mukai equivalence $\Phi\colon \mathrm{D^b}(X, \alpha_B) \cong \mathrm{D^b}(Y)$. $\square$

In particular, this shows that the strong version of the Twisted Derived Global Torelli theorem is true for the set of all (X, α_B) with $U \subset \mathrm{Pic}(X,B)$.

Corollary 3.8. *Let (X, α_B) be a twisted K3 surface with Picard number $\rho(X) \geq 12$. Then there exists an K3 surface Y with $\mathrm{D^b}(Y) \cong \mathrm{D^b}(x, \alpha_B)$.*

Proof. If $\rho(X) \geq 12$, $\mathrm{NS}(X) \subset \mathrm{Pic}(X,B)$ contains a hyperbolic plane by [23, Lem. 4.1]. $\square$

Remark 3.9. Note that Cor. 3.8 has already been shown in [18, Prop. 7.3]. However, the use of Căldăraru's conjecture, which had not been verified then, simplifies the proof significantly.

3.2. **The spinor norm of Hodge isometries.** In this subsection, we investigate which values the spinor norm can take on a Hodge isometry $\varphi\colon \widetilde{H}(X,B,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X,B,\mathbb{Z})$.

Definition 3.10. Let V be a real vector space equipped with a non-degenerate symmetric bilinear form $\beta\colon V\times V\to\mathbb{R}$. We define the *real spinor norm*

$$\theta\colon O(V,\beta)\to\{\pm1\}$$

on the orthogonal group $O(V,\beta)$ of (V,β) as follows:

For the reflection $\tau_v\colon w\mapsto w-2\frac{\beta(w,v)}{\beta(v,v)}v$ along a vector $v\in V$ with $\beta(v,v)\neq 0$, we put

$$\theta(\tau_v) := \begin{cases} 1 & \text{if } \beta(v,v)<0,\\ -1 & \text{if } \beta(v,v)>0.\end{cases} \tag{1}$$

By the Cartan-Dieudonné theorem, every $g\in O(V,\beta)$ is a composition of reflections. Hence, we can extend the spinor norm θ to the whole automorphism group of (V,β).

If Λ is a lattice, we have a canonical inclusion $O(\Lambda)\subset O(\Lambda_{\mathbb{R}})$. We can therefore define the spinor norm $\theta\colon O(\Lambda)\to\{\pm1\}$ for Λ as the restriction of the real spinor norm for $\Lambda_{\mathbb{R}}$ to $O(\Lambda)$.

Remark 3.11. Note that there is another definition of the spinor norm in the literature in which the signs in (1) are switched. However, our definition will be more appropriate for our approach.

Restricting the attention to the Mukai lattice $\widetilde{H}(X,B,\mathbb{Z})$ of a twisted K3 surface (X,α_B), one can ask in which cases there exist a Hodge isometry $\varphi\colon \widetilde{H}(X,B,\mathbb{Z})$ with $\theta(\varphi)=-1$. This is the first question from the Introduction because φ has spinor norm -1 if and only if it reverses the natural orientation of the four positive directions.

In the untwisted case $B\in H^2(X,\mathbb{Z})$, such a g always exists and is given by $\varphi=-\mathrm{id}_{H^2(X,\mathbb{Z})}\oplus\mathrm{id}_{(H^0\oplus H^4)(X,\mathbb{Z})}$. Although φ is in general not a Hodge isometry of a twisted Hodge structure $\widetilde{H}(X,B,\mathbb{Z})$, we have

Lemma 3.12. *Let (X,α_B) be a twisted K3 surface. If its generalized Picard group contains a hyperbolic plane $U\subset\mathrm{Pic}(X,B)$, there exists a Hodge isometry $\varphi\colon \widetilde{H}(X,B,\mathbb{Z})\xrightarrow{\sim}\widetilde{H}(X,B,\mathbb{Z})$ with spinor norm $\theta(\varphi)=-1$.*

Proof. By Prop. 3.7, there is a K3 surface Y with $\widetilde{H}(X,B,\mathbb{Z})\cong\widetilde{H}(Y,\mathbb{Z})$. But $-\mathrm{id}_{H^2(Y,\mathbb{Z})}$ is a Hodge isometry of $\widetilde{H}(Y,\mathbb{Z})$ with non-trivial spinor norm. □

Corollary 3.13. *Let (X,α_B) be a twisted K3 surface with Picard number $\rho(X)\geq 12$. Then there exists a Hodge isometry $\varphi\colon\widetilde{H}(X,B,\mathbb{Z})\xrightarrow{\sim}\widetilde{H}(X,B,\mathbb{Z})$ with spinor norm $\theta(\varphi)=-1$.*

Proof. If $\rho(X)\geq 12$, $\mathrm{NS}(X)\subset\mathrm{Pic}(X,B)$ contains a hyperbolic plane by [23, Lem. 4.1]. □

Remark 3.14. The Hodge isometry with non-trivial spinor norm can also be constructed explicitly once the existence of a hyperbolic plane $U\subset\mathrm{NS}(X)$ has been established.

One may ask oneself whether the condition in Lemma 3.12 is not only sufficient, but also necessary. As the following counterexample shows, this is not the case.

Example 3.15. By Prop. 3.5, there is a twisted K3 surface (X,α_B) with generalized Picard group $\mathrm{Pic}(X,B)=\langle 4\rangle\oplus U(2)$. Then $\langle x,x\rangle\in 4\mathbb{Z}$ for all $x\in\mathrm{Pic}(X,B)$. In particular, $\mathrm{Pic}(X,B)$ contains no $+2$-classes and there is no primitive embedding

$U \hookrightarrow \mathrm{Pic}(X,B)$. Nevertheless, the orientation reversing isometry $\mathrm{id}_{\langle 4\rangle} \oplus -\mathrm{id}_{U(2)}$ of the generalized Picard group $\mathrm{Pic}(X,B)$ acts trivially on $A_{\mathrm{Pic}(X,B)}$ and therefore induces a Hodge isometry $\varphi\colon \widetilde{H}(X,B,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X,B,\mathbb{Z})$ of spinor norm $\theta(\varphi) = -1$, which does not arise as the reflection at some $+2$-class.

Now we want to show that for almost all B and almost all α_B-twisted K3 surfaces (X, α_B), there does not exist a Hodge isometry $\varphi\colon \widetilde{H}(X,B,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X,B,\mathbb{Z})$ of spinor norm $\theta(\varphi) = -1$. To make this precise, note that for fixed $B \in \Lambda_{\mathbb{Q}}$, every marked K3 surfaces (X,ϕ) can be made in a unique way into a twisted K3 surfaces by choosing the Brauer class $\alpha_B := \exp((\phi^{-1}(B))^{0,2})$. In this way, we obtain a moduli space N^B of marked, "B-twisted" K3 surfaces. By the Local and Global Torelli Theorem, the global period map $\mathcal{P}^B\colon N^B \to D^B$ from the moduli space N^B of marked, B-twisted K3 surfaces to the period domain $D^B := \exp B\left(\{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid (x.x) = 0, (x.\bar{x}) > 0\}\right)$ is surjective and a local isomorphism.

Definition 3.16. A rational B-field $B \in \Lambda_{\mathbb{Q}}$ is *spinor trivial* if there is some $A \subset N^B$ such that $N^B \smallsetminus A$ is a countable union of closed submanifolds of positive codimension and $\widetilde{H}(X,B,\mathbb{Z})$ does not have a Hodge isometry of spinor norm -1 for all $(X,\phi) \in A$.

It is *d-polarized spinor trivial* if the same holds true for N^B replaced by N^B_d.

Theorem 3.17. *Let $B \in \Lambda_{\mathbb{Q}}$ be a rational B-field and $a := \min\{n \in \mathbb{N}^* \mid aB \in \Lambda\}$. Then B is spinor trivial if and only if neither $\frac{2}{a} \in \mathbb{Z}$ nor $\frac{(aB.aB)}{2} \equiv 1 \mod a$.*

Proof. As before we denote by N^B the moduli space of marked B-twisted K3 surfaces and by $D^B := \exp B(D) = \exp B\left(\{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid (x.x) = 0, (x.\bar{x}) > 0\}\right)$ the B-twisted period domain. Since the period map $\mathcal{P}^B\colon N^B \to D^B$ is surjective and a local isomorphism, it suffices to show the corresponding lattice theoretic statement for D^B: Let $B \in \Lambda_{\mathbb{Q}}$ and $a := \min\{n \in \mathbb{N}^* \mid aB \in \Lambda\}$. Then there is a subset $A \subset D^B$ whose complement is a countable union of linear sections of positive codimension such that $x + B \wedge x \notin \mathrm{Fix}(\bar{g})$ for all $x + B \wedge x \in A$ and all isometries $g \in O(\widetilde{\Lambda})$ of spinor norm -1 if and only if neither $\frac{2}{a} \in \mathbb{Z}$ nor $\frac{(aB.aB)}{2} \equiv 1 \mod a$. Here, $\bar{g}$ denotes of course the linear map on $\mathbb{P}(\Lambda_{\mathbb{C}})$ induced by $g \in O(\Lambda)$ and $\mathrm{Fix}(g)$ the set of its fixed points.

We first give the elementary

Lemma 3.18. *Let t be an automorphism of a vector space V over a field k, $\sigma(t)$ the set of its eigenvalues, $\mathrm{Eig}(t;\lambda)$ the eigenspace of t corresponding to $\lambda \in \sigma(t)$ and $\bar{t}$ the projective linear map on $\mathbb{P}(V)$ induced by t. Then*

$$\mathrm{Fix}(\bar{t}) = \bigsqcup_{\lambda_i \in \sigma(t)} \mathbb{P}(\mathrm{Eig}(t;\lambda_i)).$$

Proof. Follows directly from the definition of eigenspaces. □

Continuing with the proof of Theorem 3.17, we use Lemma 3.18 to obtain yet another equivalent version of the statement. Let $\Lambda^B_{\mathbb{C}} := \exp B(\Lambda_{\mathbb{C}})$. For all $\lambda \in \sigma(g_{\mathbb{C}})$, the subset $D^B \cap \mathbb{P}(\mathrm{Eig}(g_{\mathbb{C}};\lambda)) \subseteq D^B$ is a linear section of positive codimension if and only if $\Lambda^B_{\mathbb{C}} \nsubseteq \mathrm{Eig}(g_{\mathbb{C}};\lambda)$ because $D^B_{\mathbb{C}} \subset \Lambda^B_{\mathbb{C}}$ is a non-degenerate quadric. As there are only countably many isometries $g \in O(\widetilde{\Lambda})$, it suffices to prove the following: There exists an isometry $g \in O(\widetilde{\Lambda})$ of spinor norm -1 with $\Lambda^B_{\mathbb{C}} \subseteq \mathrm{Eig}(g_{\mathbb{C}},\lambda)$ for some $\lambda \in \sigma(g_{\mathbb{C}})$ if and only if either $\frac{2}{a} \in \mathbb{Z}$ or $\frac{(aB.aB)}{2} \equiv 1 \mod a$.

On the other hand, the signature of the lattice $\widetilde{\Lambda}_{\mathbb{C}}$ is $(4,20)$ and $\dim \Lambda^B_{\mathbb{C}} = 20 \geq 4 = \min(4,20)$. In particular, $\Lambda^B_{\mathbb{C}}$ contains a vector $v \in \Lambda^B_{\mathbb{C}}$ with $(v.v) \neq 0$. Thus if an isometry $g \in O(\widetilde{\Lambda})$ has an eigenvalue $\lambda \in \sigma(g_{\mathbb{C}})$ with $\Lambda_{\mathbb{C}}^B \subseteq \operatorname{Eig}(g_{\mathbb{C}}, \lambda)$, it follows from

$$\lambda^2(v.v) = (\lambda v.\lambda v) = (gv.gv) = (v.v)$$

that $\lambda = \pm 1$, i.e. that $g_{\mathbb{C}}\restriction_{\Lambda_{\mathbb{C}}^B} = \pm \operatorname{id}$.

Let $\Gamma := (\Lambda_{\mathbb{C}}^B)^{\perp} \cap \widetilde{\Lambda}$. Then by [14, Lemma 2.4, Prop. 2.5], there is a one-to-one correspondence

$$\{g \in O(\widetilde{\Lambda}) \mid g_{\mathbb{C}} \restriction_{\Lambda_{\mathbb{C}}^B} = \pm \operatorname{id}\} \longleftrightarrow \{g \in O(\Gamma) \mid \bar{g} = \pm \operatorname{id} \in O(A_{\Gamma})\}$$

where A_{Γ} denotes the discriminant group of Γ. Since $\theta(\pm \operatorname{id}_{\Lambda_{\mathbb{C}}^B}) = \pm 1$ and the spinor norm is multiplicative, we have furthermore

$$\{g \in O(\widetilde{\Lambda}) \mid g_{\mathbb{C}}\restriction_{\Lambda_{\mathbb{C}}^B} = \pm \operatorname{id}, \theta(g) = -1\} \longleftrightarrow \{g \in O(\Gamma) \mid \bar{g} = \pm \operatorname{id}_{A_{\Gamma}}, \theta(g) = \mp 1\}.$$

Hence, the theorem follows from Prop. 3.19. □

Proposition 3.19. *Let $B \in \Lambda_{\mathbb{Q}}$, $a := \min\{n \in \mathbb{N}^* \mid aB \in \Lambda\}$ and $\Gamma := (\Lambda_{\mathbb{C}}^B)^{\perp} \cap \widetilde{\Lambda}$. Then there exists a $g \in O(\Gamma)$ with $\bar{g} = \pm \operatorname{id} \in O(A_{\Gamma})$ and $\theta(g) = \mp 1$ if and only if $\frac{2}{a} \in \mathbb{Z}$ or $\frac{(aB.aB)}{2} \equiv 1 \mod a$.*

Proof. As we can always compose g with $-\operatorname{id}_{\Gamma}$, which satisfies $\theta(-\operatorname{id}) = -1$, we may prove the following equivalent statement: There exists a $g \in O(\Gamma)$ with $\bar{g} = -\operatorname{id} \in O(A_{\Gamma})$ and $\theta(g) = 1$ if and only if $\frac{2}{a} \in \mathbb{Z}$ or $\frac{(aB.aB)}{2} \equiv 1 \mod a$.

We have $\widetilde{\Lambda} = \Lambda \oplus U$ and denote the standard basis vectors of U again by e and f. Then

$$\begin{aligned}(\Lambda_{\mathbb{C}}^B)^{\perp} &= (\exp B(\Lambda_{\mathbb{C}}))^{\perp} = \exp B(\Lambda_{\mathbb{C}}^{\perp}) = \exp B(\langle e, f\rangle_{\mathbb{C}}) = \langle (\exp B \cdot e), (\exp B \cdot f)\rangle_{\mathbb{C}} \\ &= \left\langle \left(e + B + \tfrac{(B.B)}{2} f\right), f \right\rangle_{\mathbb{C}} = \langle (e + B), f\rangle_{\mathbb{C}}.\end{aligned}$$

Defining $b := \frac{(aB.aB)}{2}$, we conclude that

$$\Gamma = (\Lambda_{\mathbb{C}}^B)^{\perp} \cap \widetilde{\Lambda} = \langle (e + B), f\rangle_{\mathbb{C}} \cap \widetilde{\Lambda} = \langle (ae + aB), f\rangle_{\mathbb{Z}}$$

has intersection matrix $\left(\begin{smallmatrix} 2b & -a \\ -a & 0 \end{smallmatrix}\right)$.

Let us first find a necessary and sufficient condition for the existence of a $g \in O(\Gamma)$ with $\bar{g} = -\operatorname{id}_{A_{\Gamma}}$, $\theta(g) = 1$ and $\det g = 1$. Since Γ is isotropic (i.e. there exists an $x \in \Gamma$ with $(x.x) = 0$), it follows from the classical theory of isotropic binary quadratic forms that the only $g \in O(\Gamma)$ with $\det g = 1$ are $\pm \operatorname{id}_{\Gamma}$, cf. [9, Lemma 13.3.2]. But $\theta(-\operatorname{id}) = -1$, so we are left with the question when id, which has spinor norm 1, acts on the discriminant group A_{Γ} as $-\operatorname{id}$.

This is the case if and only if the finite abelian group A_{Γ} is of the form $A_{\Gamma} = (\mathbb{Z}/2)^k$, $k \in \mathbb{Z}_{\geq 0}$. As Γ is of rank 2, A_{Γ} has at most two generators. Hence, $\operatorname{id} = -\operatorname{id} \in O(A_{\Gamma})$ if and only if $A_{\Gamma} = (\mathbb{Z}/2)^k$, $k = 0, 1, 2$. On the other hand, we have $|A_{\Gamma}| = |\operatorname{disc} \Gamma| = a^2$, so this is only possible if $a = 1$ or $a = 2$. For $a = 1$, one immediately obtains $A_{\Gamma} \cong 1$, and in case $a = 2$, one sees that

$$A_{\Gamma} = \left\langle \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \right\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$

We conclude that there exists an $g \in O(\Gamma)$ with $\bar{g} = -\operatorname{id}_{A_{\Gamma}}$, $\theta(g) = 1$ and $\det g = 1$ if and only if $a = 1$ or $a = 2$, i.e. $\frac{2}{a} \in \mathbb{Z}$.

We now look at the case $\bar{g} = -\operatorname{id}_{A_{\Gamma}}$, $\theta(g) = 1$ and $\det g = -1$. Let us assume that there exists such a $g \in O(\Gamma)$. We shall use the following

Lemma 3.20. *If a (non-degenerate) lattice Γ of rank 2 possesses an isometry $g \in O(\Gamma)$ of determinant $\det g = -1$, its intersection matrix is, after some base change, given by*

$$\textit{(i)}\ \begin{pmatrix} 2b' & 0 \\ 0 & 2c' \end{pmatrix} \ \textit{or (ii)}\ \begin{pmatrix} 2b' & b' \\ b' & 2c' \end{pmatrix}.$$

In the first case, g is given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. In the second case, we have $g = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$.

Proof. Cf. [9, Lemma 3.1, Thm. 3.2, Lemma 3.4]. □

We can thus conclude the proof of Prop. 3.19 by treating the two cases of Lemma 3.20 separately.

In case (i), we have $c' < 0$ and $b' > 0$ because $\theta(g) = 1$, and then $b' = 1$ because $\bar{g} = -\operatorname{id}_{A_\Gamma}$. Therefore, Γ contains a 2-class, i.e. an element $\delta \in \Gamma$ with $(\delta.\delta) = 2$, and g is the reflection τ_δ along δ. Conversely, every 2-class $\delta \in \Gamma$ gives, via the reflection τ_δ, rise to an orthogonal decomposition $\Gamma = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2c' \end{smallmatrix}\right)$, $c' < 0$, and thus to a $g \in O(\Gamma)$ with $\bar{g} = -\operatorname{id}_{A_\Gamma}$, $\theta(g) = 1$ and $\det g = -1$.

Case (ii) cannot occur: Since Γ is isotropic, one finds $x, y \in \mathbb{Z}$ such that

$$0 = \begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} 2b' & b' \\ b' & 2c' \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2b'(x^2 + xy) + 2c'y^2.$$

It follows that $v := \left(\frac{2x^2+2xy}{b'(2x+y)^2}, \frac{y^2}{b'(2x+y)^2} \right) \in \Gamma^* \subset \Gamma_{\mathbb{Q}}$. Let $\bar{v} \in A_\Gamma$ be the induced class in the discriminant group. As $\bar{g} = -\operatorname{id} \in O(A_\Gamma)$, we must have

$$v + g(v) = \left(\frac{1}{b'}, 0 \right) \in \Gamma,$$

i.e. $|b'| = 1$ and $\Gamma = \left(\begin{smallmatrix} \pm 2 & \pm 1 \\ \pm 1 & 2c' \end{smallmatrix}\right)$. In particular, this means that $\operatorname{disc} \Gamma = \pm 4c' - 1 \equiv 3 \mod 4$. But Γ is isotropic, hence $\operatorname{disc} \Gamma$ has to be a square number, a contradiction.

We have shown that there exists a $g \in O(\Gamma)$ with $\bar{g} = -\operatorname{id}_{A_\Gamma}$, $\theta(g) = 1$ and $\det g = -1$ if and only if there exists a 2-class $\delta \in \Gamma$, i.e. some $x, y \in \mathbb{Z}$ with

$$2 = \begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} 2b & -a \\ -a & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2x(bx - ay).$$

This is equivalent to $x = bx - ay = \pm 1$, or in other words, to $b \equiv 1 \mod a$. As b is by definition given by $b = \frac{(aB.aB)}{2}$, this concludes the proof of Prop. 3.19. □

We now look at the situation of primitively polarized marked K3 surfaces of degree $2d$. We may assume without loss of generality that the polarization is with respect to a class $e' + df'$ where e', f' form the basis of some hyperbolic plane $U \subset \Lambda \subset \widetilde{\Lambda}$.

Let $B \in \Lambda_{\mathbb{Q}}$. As in the complex setting, every d-polarized marked K3 surface (X, L, ϕ) gives rise to the (unique) twisted, d-polarized marked K3 surface $(X, \alpha_{\phi^{-1}(B)}, L, \phi)$. This makes sense because $\operatorname{Pic}(X) \subset \operatorname{Pic}(X, B)$. That procedure gives again rise to a moduli space N_d^B of B-twisted, d-polarized marked K3 surfaces and a B-twisted polarized period map $\mathcal{P}_d^B : N_d^B \to D_d^B$ from N_d^B to the B-twisted polarized period domain $D_d^B := \exp B(D_d) = \exp B(\{x \in \mathbb{P}(\Lambda_{d\mathbb{C}}) \mid (x.x) = 0, (x.\bar{x}) > 0\})$. As in the untwisted case, $\mathcal{P}_d^B$ is a local isomorphism, injective and its image is dense in D_d^B. We are immediately left to an

Open Question. *Let $B \in \Lambda_{\mathbb{Q}}$ and $d \in \mathbb{N}^*$. What are necessary and sufficient conditions for B to be d-polarized spinor trivial?*

In the following, we shall describe a possible approach to solving this problem. Since $\mathcal{P}_d^B$ is a local isomorphism with image dense in D_d^B, it suffices again to answer the corresponding lattice theoretic question: For which $B \in \Lambda_{\mathbb{Q}}$ is there a subset $A \subset D_d^B$ whose complement is a countable union of linear sections of positive codimension such that $x + B \wedge x \notin \operatorname{Fix}(\bar{g_{\mathbb{C}}})$ for all $x + B \wedge x \in A$ and all isometries $g \in O(\widetilde{\Lambda})$ with $\theta(g) = -1$.

As in the proof of Thm. 3.17, one sees that such an A exists if and only if there is no $g \in O(\widetilde{\Lambda})$ with $g_{\mathbb{C}}\restriction_{\Lambda^B_{d\mathbb{C}}} = \pm\,\mathrm{id}$ and $\theta(g) = -1$. Defining $\Gamma_d := (\Lambda^B_{d\mathbb{C}})^{\perp} \cap \widetilde{\Lambda}$, we have furthermore a one-to-one correspondence

$$\{g \in O(\widetilde{\Lambda}) \mid g_{\mathbb{C}}\restriction_{\Lambda^B_{d\mathbb{C}}} = \pm\,\mathrm{id},\ \theta(g) = -1\} \longleftrightarrow \{g \in O(\Gamma_d) \mid \bar{g} = \pm\,\mathrm{id}_{A_{\Gamma_d}},\ \theta(g) = -1\}.$$

Thus, we are interested in the following

Open Question. *Let $B \in \Lambda_{\mathbb{Q}}$, $d \in \mathbb{N}^*$ and $\Gamma_d := (\Lambda^B_{d\mathbb{C}})^{\perp} \cap \widetilde{\Lambda}$. When does there exist a $g \in O(\Gamma_d)$ with $\bar{g} = \pm\,\mathrm{id} \in O(A_{\Gamma_d})$ and $\theta(g) = -1$?*

Composing with $-\,\mathrm{id}_{\Gamma_d}$ if necessary, one may additionally assume that $\det g = 1$.

Recall that $\widetilde{\Lambda} = \Lambda \oplus U$ and we denoted the canonical basis of U by e, f. The basis of a further hyperbolic plane $U \subset \Lambda$ was denoted by e', f' and our polarization was with respect to the integral class $\ell = e' + df'$. Then B is of the form $\widetilde{B} + \eta_1 e' + \eta_2 f'$ with $(\widetilde{B}.e') = (\widetilde{B}.f') = 0$ and $\eta_1, \eta_2 \in \mathbb{Q}$. If we additionally put $B' := B - \eta_1 \ell$, $a := \min\{n \in \mathbb{N}^* \mid aB' \in \Lambda\}$, $b := \frac{(aB'.aB')}{2}$ and $c := (aB'.\ell) = a(\eta_2 - d\eta_1)$, we have

$$\begin{aligned}
\Gamma_d &= (\Lambda^B_{d\mathbb{C}})^{\perp} \cap \widetilde{\Lambda} = \exp B(\Lambda^{\perp}_{d\mathbb{C}}) \cap \widetilde{\Lambda} = \exp B(\langle e, \ell, f\rangle_{\mathbb{C}}) \cap \widetilde{\Lambda} \\
&= \left\langle \left(e + B + \tfrac{(B.B)}{2} f\right), \ell, f \right\rangle_{\mathbb{C}} \cap \widetilde{\Lambda} = \langle e + B, \ell, f\rangle_{\mathbb{C}} \cap \widetilde{\Lambda} = \langle e + B', \ell, f\rangle_{\mathbb{C}} \cap \widetilde{\Lambda} \\
&= \langle (ae + aB'), \ell, f\rangle_{\mathbb{Z}}.
\end{aligned}$$

Hence, Γ_d is given by the intersection matrix

$$\begin{pmatrix} 2b & c & -a \\ c & 2d & 0 \\ -a & 0 & 0 \end{pmatrix}.$$

Let us put $PSL^{\pm}(2, \mathbb{R}) := \{M \in GL(2, \mathbb{R}) \mid \det M = \pm 1\}/\langle -\mathbb{I}_2\rangle$. It is known that the group of real isometries of determinant 1 of the lattice

$$\Gamma_0 := \begin{pmatrix} 0 & 0 & -d \\ 0 & 2d & 0 \\ -d & 0 & 0 \end{pmatrix}$$

can be described by the isomorphism

$$\rho\colon PSL^{\pm}(2, \mathbb{R}) \xrightarrow{\sim} O^+_{\mathbb{R}}(\Gamma_0), \ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta + \beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\delta & \delta^2 \end{pmatrix}$$

and that $\rho(PSL^+(2, \mathbb{R})) = \{g \in O^+_{\mathbb{R}}(\Gamma_0) \mid \theta(g) = 1\}$, $\rho(PSL^-(2, \mathbb{R})) = \{g \in O^+_{\mathbb{R}}(\Gamma_0) \mid \theta(g) = -1\}$, cf. [31, Thm. 2.I.1] and [9, Lemma 5.1, Lemma 5.2]. Since Γ_0 and Γ_d are rationally equivalent via

$$\begin{pmatrix} 1 & 0 & -b/d \\ 0 & 1 & -c/d \\ 0 & 0 & a/d \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -d \\ 0 & 2d & 0 \\ -d & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b/d & -c/d & a/d \end{pmatrix} = \begin{pmatrix} 2b & c & -a \\ c & 2d & 0 \\ -a & 0 & 0 \end{pmatrix}$$

and $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b/d & -c/d & a/d \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b/a & c/a & d/a \end{smallmatrix}\right)$, the set $O^+_{\mathbb{R}}(\Gamma_d) \cap \theta^{-1}(-1)$ of real isometries of Γ_d of determinant 1 and spinor norm -1 consists of all elements of the form

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b/a & c/a & d/a \end{pmatrix} \cdot \begin{pmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta + \beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\delta & \delta^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b/d & -c/d & a/d \end{pmatrix} \\
&= \begin{pmatrix} \alpha^2 - \frac{b\gamma^2}{d} & 2\alpha\gamma - \frac{c\gamma^2}{d} & \frac{a\gamma^2}{d} \\ \alpha\beta - \frac{b\gamma\delta}{d} & \alpha\delta + \beta\gamma - \frac{c\gamma\delta}{d} & \frac{a\gamma\delta}{d} \\ M & N & L \end{pmatrix}, \quad \text{where} \qquad (2)
\end{aligned}$$

$$M = \frac{b\alpha^2 + c\alpha\beta + d\beta^2 - b\delta^2}{a} - \frac{b^2\gamma^2 + bc\gamma\delta}{ad},$$
$$N = \frac{2b\alpha\gamma + c\alpha\delta + c\beta\gamma + 2d\beta\delta - c\delta^2}{a} - \frac{bc\gamma^2 + c^2\gamma\delta}{ad} \quad \text{and}$$
$$L = \delta^2 + \frac{b\gamma^2 + c\gamma\delta}{d}$$

and $\alpha\delta - \beta\gamma = -1$.

We want to find all $g \in O^+_{\mathbb{Z}}(\Gamma_d) \cap \theta^{-1}(-1) \subset O^+_{\mathbb{R}}(\Gamma_d) \cap \theta^{-1}(-1)$ with $\bar{g} = \pm\,\mathrm{id} \in O(A_{\Gamma_d})$. The property $g \in O^+_{\mathbb{Z}}(\Gamma_d)$ means that all entries of the matrix in (2) have to be integral. Since $\left[\frac{1}{a}, -\frac{c}{2da}, \frac{2b}{a^2} - \frac{c^2}{2a^2d}\right] \in A_{\Gamma_d} = \Gamma_d^*/\Gamma_d$ has order $2a^2d$ and $|\operatorname{disc}\Gamma_d| = 2a^2d$, the discriminant group of Γ_d is the cyclic group $A_{\Gamma_d} = \left\langle \left[\frac{1}{a}, -\frac{c}{2da}, \frac{2b}{a^2} - \frac{c^2}{2a^2d}\right] \right\rangle$ generated by this class. In particular, the second condition $\bar{g} = \pm\,\mathrm{id} \in O(A_{\Gamma_d})$ implies that $\frac{\gamma^2}{d}, \frac{\alpha\gamma}{d}, \frac{\gamma\delta}{d} \in \mathbb{Z}$ because $\left[0, 0, \frac{1}{a}\right], \left[0, \frac{1}{2d}, \frac{c}{2ad}\right] \in A_{\Gamma_d}$.

It remains an open question whether we can find conditions on B which are equivalent to the existence of such a $g \in O^+_{\mathbb{Z}}(\Gamma_d) \cap \theta^{-1}(-1)$.

4. Deformation of Fourier-Mukai kernels

We generalize the deformation theory of [1], which is itself based on the works [16] and [32], to twisted Fourier-Mukai kernels.

4.1. Hochschild homology and cohomology.

The notions of Hochschild homology and cohomology have been introduced for twisted sheaves by Căldăraru and Willerton in [7]. Let us, for the convenience of the reader, recall the basic definitions and properties. In the following, we shall always work with smooth projective varieties (X, α) and (X', α') of dimension m resp. m' over some base A. All relative constructions are taken over A if not noted otherwise, e.g. $X \times X = X \times_A X$. As $\mathcal{O}_\Delta$ may be regarded as an $\alpha^{-1} \boxtimes \alpha$-twisted sheaf on $X \times X$, we can give

Definition 4.1. The *N-th (relative) Hochschild cohomology* of (X, α) is

$$\mathrm{HH}^N(X, \alpha) := \mathrm{Hom}_{\mathrm{D^b}(X\times X, \alpha^{-1}\boxtimes\alpha)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[N]).$$

Although Hochschild cohomology is in general not functorial, we have the following result.

Lemma 4.2. *Let $\mathcal{P} \in \mathrm{D^b}(X \times X', \alpha^{-1} \boxtimes \alpha')$ such that the induced twisted Fourier-Mukai transform $\Phi_{\mathcal{P}} \colon \mathrm{D^b}(X, \alpha) \to \mathrm{D^b}(X', \alpha')$ is fully faithful. Then $\Phi_{\mathcal{P}}$ gives rise to a natural map*

$$\Phi_{\mathcal{P}}^{\mathrm{HH}^N} \colon \mathrm{HH}^N(X', \alpha') \to \mathrm{HH}^N(X, \alpha).$$

Proof. As $\Phi_{\mathcal{P}*\mathcal{O}_{\Delta_{X'}}} = \Phi_{\mathcal{O}_{\Delta_{X'}}} \circ \Phi_{\mathcal{P}} = \mathrm{id}_{\mathrm{D^b}(X',\alpha')} \circ \Phi_{\mathcal{P}} = \Phi_{\mathcal{P}}$, it follows from Prop. 2.48 that $\mathcal{P} * \mathcal{O}_{\Delta_{X'}} \cong \mathcal{P}$. Hence, convolution with $\mathcal{P}$ induces a morphism

$$\mathcal{P}* \colon \mathrm{HH}^N(X', \alpha') = \mathrm{Hom}(\mathcal{O}_{\Delta_{X'}}, \mathcal{O}_{\Delta_{X'}}[N]) \to \mathrm{Hom}(\mathcal{P}, \mathcal{P}[N]),\ f \mapsto \mathrm{id}_{\mathcal{P}} * f.$$

Similarly, one has

$$*\mathcal{P} \colon \mathrm{HH}^N(X, \alpha) = \mathrm{Hom}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}[N]) \to \mathrm{Hom}(\mathcal{P}, \mathcal{P}[N]),\ f \mapsto f * \mathrm{id}_{\mathcal{P}}.$$

As $\Phi_{\mathcal{P}}$ is fully faithful, composition with the left adjoint yields $\Phi_{\mathcal{P}_L} \circ \Phi_{\mathcal{P}} \cong \mathrm{id}_{\mathrm{D^b}(X,\alpha)}$ and therefore $\mathcal{P} * \mathcal{P}_L \cong \mathcal{O}_{\Delta_X}$. Since $*\mathcal{P}_L$ is left adjoint to $*\mathcal{P}$,

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}[N]) \arrow[r, "\sim"] \arrow[d, "*\mathcal{P}"] & \mathrm{Hom}(\mathcal{P} * \mathcal{P}_L, \mathcal{O}_{\Delta_X}[N]) \arrow[dl, "\sim"] \\
\mathrm{Hom}(\mathcal{P}, \mathcal{P}[N]) &
\end{tikzcd}$$

shows that the vertical map $*\mathcal{P}$ is an isomorphism. Thus, we can define the natural morphism

$$\Phi_{\mathcal{P}}^{\mathrm{HH}^N}\colon \mathrm{HH}^N(X',\alpha') \xrightarrow{\mathcal{P}*} \mathrm{Hom}(\mathcal{P},\mathcal{P}[N]) \xrightarrow{(*\mathcal{P})^{-1}} \mathrm{HH}^N(X,\alpha). \qquad \square$$

Definition 4.3. The *N-th (relative) Hochschild homology* of (X,α) is

$$\mathrm{HH}_N(X,\alpha) := \mathrm{Hom}_{\mathrm{D^b}(X\times X,\alpha^{-1}\boxtimes\alpha)}(\Delta_{X*}\omega_X^{-1}[N-m],\mathcal{O}_{\Delta_X}).$$

It turns out that Hochschild homology is in fact functorial.

Lemma 4.4. *Let $\mathcal{P}\in\mathrm{D^b}(X\times X',\alpha^{-1}\boxtimes\alpha')$. Then there is a natural map*

$$\Phi_{\mathcal{P}}^{\mathrm{HH}_N}\colon \mathrm{HH}_N(X,\alpha)\to\mathrm{HH}_N(X',\alpha').$$

Proof. One has $\Delta_{X*}\omega_X^{-1}*\mathcal{P}\cong\mathcal{P}\otimes\left(\pi_1^{X\times X'}\right)^*\omega_X^{-1}$. Furthermore, it follows from Def. 2.46 that $\mathcal{P}_R=\mathcal{P}_L\otimes\left(\pi_1^{X'\times X}\right)^*\omega_{X'}^{-1}\otimes\left(\pi_2^{X'\times X}\right)^*\omega_X[m-m']$. Hence,

$$\begin{aligned}
&\mathcal{P}_R*\Big(\mathcal{P}\otimes\left(\pi_1^{X\times X'}\right)^*\omega_X^{-1}\Big)\\
&=\Big(\mathcal{P}_L\otimes\left(\pi_1^{X'\times X}\right)^*\omega_{X'}^{-1}\otimes\left(\pi_2^{X'\times X}\right)^*\omega_X[m-m']\Big)*\Big(\mathcal{P}\otimes\left(\pi_1^{X\times X'}\right)^*\omega_X^{-1}\Big)\\
&=\left(\pi_{13}^{X'\times X\times X'}\right)_*\Big(\left(\pi_{12}^{X\times X'\times X}\right)^*\mathcal{P}_L\otimes\left(\pi_2^{X'\times X\times X'}\right)^*\omega_X\\
&\quad\otimes\left(\pi_{13}^{X'\times X\times X'}\right)^*\left(\pi_1^{X'\times X'}\right)^*\omega_{X'}^{-1}[m-m']\otimes\left(\pi_{23}^{X'\times X\times X'}\right)^*\mathcal{P}\otimes\left(\pi_2^{X'\times X\times X'}\right)^*\omega_X^{-1}\Big)\\
&\cong(\mathcal{P}_L*\mathcal{P})\otimes\left(\pi_1^{X'\times X'}\right)^*\omega_{X'}^{-1}[m-m'].
\end{aligned}$$

As above, convolution from the left with $\mathcal{P}_R$ and from the right with $\mathcal{P}$ gives a map

$$\mathrm{HH}_N(X,\alpha)=\mathrm{Hom}(\Delta_{X*}\omega_X^{-1}[N-m],\mathcal{O}_{\Delta_X})\to\mathrm{Hom}(\mathcal{P}_L*\mathcal{P}\otimes\pi_1^*\omega_{X'}^{-1}[N-m'],\mathcal{P}_R*\mathcal{P})$$

sending f to $\mathrm{id}_{\mathcal{P}_R}*f*\mathrm{id}_{\mathcal{P}}$. Since $*\mathcal{P}_L\dashv*\mathcal{P}$ and $*\mathcal{P}\dashv*\mathcal{P}_R$, we have a unit $\eta\colon\mathcal{O}_{\Delta_{X'}}\to\mathcal{P}_L*\mathcal{P}$ and a counit $\varepsilon\colon\mathcal{P}_R*\mathcal{P}\to\mathcal{O}_{\Delta_{X'}}$. Together with the identity $\pi_1^*\omega_{X'}^{-1}\otimes\mathcal{O}_{\Delta_{X'}}\cong\Delta_{X'*}\omega_{X'}^{-1}$, these give rise to a natural map

$$\begin{array}{rccl}
\Phi_{\mathcal{P}}^{\mathrm{HH}_N}: \mathrm{HH}_N(X,\alpha) & \to & \mathrm{Hom}(\mathcal{O}_{\Delta_{X'}}\otimes\pi_1^*\omega_{X'}^{-1}[N-m'],\mathcal{O}_{\Delta_{X'}}) & \cong\mathrm{HH}_N(X',\alpha')\\
f & \mapsto & \varepsilon\circ(\mathrm{id}_{\mathcal{P}_R}*f*\mathrm{id}_{\mathcal{P}})\circ\eta. &
\end{array} \qquad \square$$

Definition 4.5. A (right) action of $\mathrm{HH}^*(X,\alpha)$ on $\mathrm{HH}_*(X,\alpha)$ is given by the map

$$\begin{array}{ccccc}
\mathrm{Hom}(\Delta_{X*}\omega_X^{-1}[i-m],\mathcal{O}_{\Delta_X}) & \otimes\mathrm{Hom}(\mathcal{O}_\Delta,\mathcal{O}_\Delta[j]) & \to & \mathrm{Hom}(\Delta_{X*}\omega_X^{-1}[i-j-m],\mathcal{O}_{\Delta_X})\\
f & \otimes \quad g & \mapsto & (g\circ f)[-i].
\end{array}$$

This action is compatible with the action of $\Phi_{\mathcal{P}}^{\mathrm{HH}^*}$ and $\Phi_{\mathcal{P}}^{\mathrm{HH}_*}$ in the sense of

Proposition 4.6. *Let $\mathcal{P}\in\mathrm{D^b}(X\times X',\alpha^{-1}\boxtimes\alpha')$. Then for all $c\in\mathrm{HH}^j(X',\alpha')$, the following diagram commutes*

$$\begin{tikzcd}
\mathrm{HH}_i(X,\alpha) \arrow[r, "\Phi_{\mathcal{P}}^{\mathrm{HH}_*}"] \arrow[d, "\Phi_{\mathcal{P}}^{\mathrm{HH}^*}(c)"] & \mathrm{HH}_i(X',\alpha') \arrow[d, "c"] \\
\mathrm{HH}_{i-j}(X,\alpha) \arrow[r, "\Phi_{\mathcal{P}}^{\mathrm{HH}_*}"] & \mathrm{HH}_{i-j}(X',\alpha').
\end{tikzcd}$$

Proof. The proof merely consists of plugging in the definitions of $\Phi_{\mathcal{P}}^{\mathrm{HH}^*}$ and $\Phi_{\mathcal{P}}^{\mathrm{HH}_*}$ from the previous lemmata and is completely analogous to the untwisted case. For details (in the untwisted case) and a wonderful diagram see e.g. [1, Prop. 6.1]. $\square$

4.2. **Atiyah classes.** We continue to generalize the deformation theory of [1] to the twisted case.

Definition 4.7. Let (Y, β) be a projective variety, $\mathcal{I}_{\Delta_Y}$ the ideal sheaf of the diagonal $\Delta_Y \subset Y \times Y$ and $2\Delta_Y \subset Y \times Y$ the first-order thickening of Δ_Y with ideal sheaf $\mathcal{I}^2_{\Delta_Y}$. Then the *universal Atiyah class* of Y is the class $\mathcal{A}_{(Y,\beta)} \in \mathrm{Ext}^1_{\mathrm{D^b}(Y\times Y,\beta^{-1}\boxtimes\beta)}(\mathcal{O}_{\Delta_Y}, \Delta_{Y*}\Omega_Y)$ induced by the boundary map $\mathcal{O}_{\Delta_Y} \to \Delta_{Y*}\Omega_Y[1]$ of the obvious short exact sequence (in $\mathrm{D^b}(Y \times Y, \beta^{-1} \boxtimes \beta)$)

$$0 \to \Delta_{Y*}\Omega_Y \to \mathcal{O}_{2\Delta_Y} \to \mathcal{O}_{\Delta_Y} \to 0. \tag{3}$$

Definition 4.8. With the same definition as in the previous definition, let $\mathcal{E} \in \mathrm{D^b}(Y, \beta)$. Then the *Atiyah class* $A(\mathcal{E}) \in \mathrm{Ext}^1_{\mathrm{D^b}(Y,\beta)}(\mathcal{E}, \mathcal{E}\otimes\Omega_Y)$ of $\mathcal{E}$ is the morphism $\mathcal{E} \to \mathcal{E} \otimes \Omega_Y[1]$ obtained by applying the natural transformation of Fourier-Mukai functors $\Phi_{\mathcal{O}_{\Delta_Y}} \to \Phi_{\Delta_{Y*}\Omega_Y[1]}$ induced by $\mathcal{A}_{(Y,\beta)}$ to $\mathcal{E}$.

Remark 4.9. The Atiyah class $A(\mathcal{E})$ can also be described as the connecting homomorphism of the exact triangle

$$\mathcal{E} \otimes \Omega_Y \to J^1(\mathcal{E}) \to \mathcal{E}$$

which is obtained by applying the short exact sequence (3) of Fourier-Mukai kernels to $\mathcal{E}$. Here $J^1(\mathcal{E}) := \pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{O}_{2\Delta_Y})$ denotes the *first jet space* of $\mathcal{E}$.

We know look at the case of later interest where $(Y, \beta) = (A \times B, \beta_A \boxtimes \beta_B)$ is a product.

Definition 4.10. Assume that $(Y, \beta) = (A \times B, \beta_A \boxtimes \beta_B)$ and let $\mathcal{E} \in \mathrm{D^b}(Y, \beta)$. Then the isomorphism $\Omega_Y \xrightarrow{\sim} \pi_1^*\Omega_A \oplus \pi_2^*\Omega_B$ induces a splitting

$$\theta\colon \mathrm{Ext}^1_{\mathrm{D^b}(Y,\beta)}(\mathcal{E}, \mathcal{E}\otimes\Omega_Y) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{D^b}(Y,\beta)}(\mathcal{E}, \mathcal{E}\otimes\pi_1^*\Omega_A) \oplus \mathrm{Ext}^1_{\mathrm{D^b}(Y,\beta)}(\mathcal{E}, \mathcal{E}\otimes\pi_2^*\Omega_B).$$

The *partial Atiyah classes* $A_A(\mathcal{E})$ and $A_B(\mathcal{E})$ are the components of the image $\theta(A(\mathcal{E})) = (A_A(\mathcal{E}), A_B(\mathcal{E}))$ of $A(\mathcal{E})$ under this isomorphism.

Lemma 4.11. *Let $\mathcal{P} \in \mathrm{D^b}(X \times X', \alpha^{-1} \boxtimes \alpha')$. Applying the exact Fourier-Mukai functor $*\mathcal{P}$ resp. $\mathcal{P}*$ to the exact triangle*

$$\Delta_{X*}\Omega_X \to \mathcal{O}_{2\Delta_X} \to \mathcal{O}_{\Delta_X} \quad \textit{resp.} \quad \Delta_{X'*}\Omega_{X'} \to \mathcal{O}_{2\Delta_{X'}} \to \mathcal{O}_{\Delta_{X'}}$$

in $\mathrm{D^b}(X \times X, \alpha^{-1} \boxtimes \alpha)$ resp. $\mathrm{D^b}(X' \times X', \alpha'^{-1} \boxtimes \alpha')$ yields an exact triangle

$$\mathcal{P} \otimes \Omega_X \to \mathcal{O}_{2\Delta_X} * \mathcal{P} \to \mathcal{P} \quad \textit{resp.} \quad \mathcal{P} \otimes \Omega_{X'} \to \mathcal{P} * \mathcal{O}_{2\Delta_{X'}} \to \mathcal{P}$$

in $\mathrm{D^b}(X \times X'\alpha^{-1} \boxtimes \alpha')$. The connecting homomorphism of this triangle is the relative Atiyah class $A_X(\mathcal{P})$ resp. $A_{X'}(\mathcal{P})$.

Proof. Once more, the proof is entirely similar to the untwisted one in [1, Lemma 7.2, Lemma 7.3] (Note that the short exact sequence (7.10) has to be considered in $\mathrm{D^b}(A \times B \times A, \beta_A^{-1} \boxtimes 1 \boxtimes \beta_B)$. □

Lemma 4.12. *The two partial Atiyah classes $A_1(\mathcal{O}_{\Delta_X})$ and $A_2(\mathcal{O}_{\Delta_X})$ of $\mathcal{O}_{\Delta_X} \in \mathrm{D^b}(X \times X, \alpha^{-1} \boxtimes \alpha)$ satisfy the relation*

$$A_1(\mathcal{O}_{\Delta_X}) = \mathcal{A}_{(X,\alpha)} = -A_2(\mathcal{O}_{\Delta_X}) \in \mathrm{Ext}^1_{\mathrm{D^b}(X\times X,\alpha^{-1}\boxtimes\alpha)}(\mathcal{O}_{\Delta_X}, \Delta_{X*}\Omega_X)$$

where we canonically identify $\mathcal{O}_{\Delta_X} \otimes \pi_i^*\Omega_X \cong \Delta_{X*}(\Delta_X^*\pi_i^*\Omega_X) \cong \Delta_{X*}\Omega_X$ via the projection formula.

Proof. We indicate the caveats when transferring the untwisted proof [1, Lemma 7.4, Cor. 7.5] to the twisted one.

Note that Lemma 7.4 only makes sense if $Y = (X \times X, \alpha^{-1} \boxtimes \alpha)$ and $Z = \Delta_X$ which is the case we will need in the progress of the proof. As usual, we can then regard (7.15) as a sequence in $\mathrm{D^b}(X \times X, \alpha^{-1} \boxtimes \alpha)$ and the short exact sequence

$$0 \to \Omega_Y \otimes \mathcal{O}_{\Delta_Y} \to \mathcal{O}_{2\Delta_Y} \to \mathcal{O}_{\Delta_Y} \to 0$$

in $\mathrm{D^b}(X \times X \times X \times X, (\alpha^{-1} \boxtimes \alpha)^{-1} \boxtimes (\alpha^{-1} \boxtimes \alpha)))$.

Later, we again have to regard the conormal sheaf $\mathcal{I}_{\Delta_X}/\mathcal{I}^2_{\Delta_X}$ and the cotangent sheaf Ω_{Δ_X} as $\alpha^{-1} \boxtimes \alpha$-twisted sheaves on $X \times X$. □

Remark 4.13. The minus sign in the Lemma comes from choosing the canonical identification of $\mathcal{I}_{\Delta_X}/\mathcal{I}^2_{\Delta_X}$ with $\Delta_{X*}\Omega_X$ via π_1. If we chose the other canonical identification via π_2, the minus sign would occur before $A_1(\mathcal{O}_{\Delta_X})$.

4.3. **HKR isomophism and decomposition theorem.** Hochschild homology and cohomology draws its significance from its close relation with deformation theory. This is due to the (relative) *Hochschild-Kostant-Rosenberg isomorphism* which leads to a direct sum decomposition of Hochschild homology and cohomology similar to the Hodge decomposition.

We start by constructing the HKR-isomorphism $I\colon \Delta_Y^*\mathcal{O}_{\Delta_Y} \to \bigoplus_i \Omega^i_Y[i]$.

Definition 4.14. Let (Y, β) be a smooth proper twisted variety over some base A. The universal Atiyah class $\mathcal{A}_{(Y,\beta)} \in \mathrm{Ext}^1_{\mathrm{D^b}(Y\times Y,\beta^{-1}\boxtimes\beta)}(\mathcal{O}_{\Delta_Y}, \Delta_{Y*}\Omega_Y)$ gives, via its exponential, rise to a morphism

$$\exp(\mathcal{A}_{(Y,\beta)})\colon \mathcal{O}_{\Delta_Y} \to \bigoplus_i \Delta_{Y*}\Omega^i_Y[i]$$

in $\mathrm{D^b}(Y \times Y, \beta^{-1} \boxtimes \beta)$. The image of this morphism under the adjunction isomorphism

$$\mathrm{Hom}(\mathcal{O}_{\Delta_Y}, \bigoplus_i \Delta_{Y*}\Omega^i_Y[i]) \cong \mathrm{Hom}(\Delta_Y^*\mathcal{O}_{\Delta_Y}, \bigoplus_i \Omega^i_Y[i])$$

is called the *HKR-isomorphism*

$$I\colon \Delta_Y^*\mathcal{O}_{\Delta_Y} \to \bigoplus_i \Omega^i_Y[i].$$

Lemma 4.15. *This naming is justified, i.e. I is really an isomorphism.*

Proof. Cf. [3, Thm. 3.1.3]. □

This Lemma provides us with the promised Hodge-like decomposition.

Definition 4.16. Let Y be a smooth proper variety over some base A of dimension m. Then the isomorphisms

$$\begin{aligned} I^{HKR}\colon \mathrm{HH}^*(Y,\beta) &\cong \mathrm{Ext}^*_{\mathrm{D^b}(Y)}(\Delta_Y^*\mathcal{O}_\Delta, \mathcal{O}_Y) \xrightarrow{\circ I^{-1}} \mathrm{Ext}^*_{\mathrm{D^b}(Y)}(\bigoplus_i \Omega^i_Y[i], \mathcal{O}_Y) \cong \\ &\cong \bigoplus_{i+j=*} H^j(Y, \bigwedge^i T_Y) \xrightarrow[\sim]{\mathrm{td}(X)^{-1/2}\lrcorner} \bigoplus_{i+j=*} H^j(Y, \bigwedge^i T_Y) \\ I_{HKR}\colon \mathrm{HH}_*(Y,\beta) &= \mathrm{Ext}^{-*}_{\mathrm{D^b}(Y\times Y,\beta^{-1}\boxtimes\beta)}(\Delta_{Y*}\omega^{-}1_Y[-m], \mathcal{O}_{\Delta_Y}) \cong \\ &\cong \mathrm{Ext}^{-*}_{\mathrm{D^b}(Y)}(\mathcal{O}_Y, \Delta_Y^*\mathcal{O}_{\Delta_Y}) \xrightarrow{I\circ} \mathrm{Ext}^{-*}_Y(\mathcal{O}_Y, \bigoplus_i \Omega_Y[i]) \cong \bigoplus_{i-j=*} H^j(\Omega^i_Y) \\ &\xrightarrow[\sim]{\mathrm{td}(X)^{1/2}\wedge} \bigoplus_{i-j=*} H^j(\Omega^i_Y) \end{aligned}$$

are called the *HKR-isomorphisms for Hochschild cohomology resp. homology.*

Inclusions such as $H^1(T_Y) \subset \mathrm{HH}^2(Y)$ will always be understood under these isomorphisms.

Remark 4.17. The fact that I^{HKR} resp. I_{HKR} is an isomorphism is sometimes called the *decomposition theorem for Hochschild cohomology resp. homology.*

The contraction with $\mathrm{td}(X)^{-1/2}$ resp. the exterior product with $\mathrm{td}(X)^{1/2}$ are added in order to make firstly the action of $\mathrm{HH}^*(Y,\beta)$ on $\mathrm{HH}_*(Y,\beta)$ compatible with the action of $H^*(\bigwedge^* T_Y)$ on $H^*(\Omega_Y^*)$, and secondly the map $\Phi_{\mathcal{P}}^{\mathrm{HH}_*}$ compatible with $\Phi_{\mathcal{P}}^H$. Let us make this precise. As the proofs of the relevant statements are quite technical, we shall henceforth restrict ourselves to the untwisted case (i.e. the Brauer class β will be trivial and suppressed in the notation) which is completely sufficient for our treatment and has been discussed exhaustively in the literature. Nevertheless, it is expected that everything can be generalized to the twisted world in the obvious ways.

Proposition 4.18. *Let Y be a smooth proper variety. Then the HKR-isomorphism I^{HKR} for Hochschild cohomology is a ring isomorphism (with the ring structure on $\mathrm{HH}^*(Y)$ given by composition of maps). Furthermore, the module structures of $\mathrm{HH}_*(Y)$ over $\mathrm{HH}^*(Y)$ and $\bigoplus_{i-j=*} H^j(\Omega_Y^i)$ over $\bigoplus_{i+j=*} H^j(Y, \bigwedge^i T_Y)$ are compatible, i.e. for all $c \in \mathrm{HH}^*(Y)$, the following diagram commutes:*

$$\begin{tikzcd} \mathrm{HH}_*(Y) \arrow[r, "I_{HKR}"] \arrow[d, "c"] & \bigoplus_{i-j=*} H^j(\Omega_Y^i) \arrow[d, "I^{HKR}(c)\lrcorner"] \\ \mathrm{HH}_*(Y) \arrow[r, "I_{HKR}"] & \bigoplus_{i-j=*} H^j(\Omega_Y^i). \end{tikzcd}$$

Proof. Cf. [5, Cor. 1.5] and [4, Thm. 1.4]. □

Lemma 4.19. *For all smooth projective varieties X, X' and $\mathcal{P} \in \mathrm{D^b}(X \times X')$, the following diagram commutes:*

$$\begin{tikzcd} \mathrm{HH}_*(X) \arrow[r, "\Phi_{\mathcal{P}}^{\mathrm{HH}_*}"] \arrow[d, "I_{HKR}"] & \mathrm{HH}_*(X') \arrow[d, "I_{HKR}"] \\ H^*(X,\mathbb{C}) \arrow[r, "\Phi_{\mathcal{P}}^H"] & H^*(X',\mathbb{C}). \end{tikzcd}$$

Proof. Cf. [25, Thm. 1.2]. □

4.4. **First-order liftings of Fourier-Mukai kernels.** In this subsection, we will use the theory built up so far to give a criterion for the liftability of Fourier-Mukai kernels to first order. We shall again stick to the untwisted case, which has been developed in [16] and [32], and refrain from giving a generalization to twisted Fourier-Mukai kernels. We will once more follow the simplified account in [1].

Let $R_n := \mathbb{C}[t]/(t^{n+1})$ and $A_n := \operatorname{Spec} R_n$. Let Y be a smooth proper variety. If $f_1 \colon Y_1 \to A_1$ is a smooth proper first-order deformation of Y, it has been shown in [16] that the relative Kodaira-Spencer class which parametrizes this deformation provides an obstruction for complexes $E \in \mathrm{D^b}(Y)$ to be liftable to $E_1 \in \mathrm{D^b}(Y_1)$. Recall

Definition 4.20. Let $f_{n+1} \colon Y_{n+1} \to A_{n+1}$ be a smooth proper $(n+1)$-th-order deformation of Y. The *(relative) Kodaira-Spencer class* $\kappa_{Y_n} \in H^1(Y_n, T_{Y_n}) \cong \mathrm{Ext}^1_{Y_n}(\Omega_{Y_n}, \mathcal{O}_{Y_n})$ of order n is the extension class induced by the boundary map $\Omega_{Y_n} \to \mathcal{O}_{Y_n}[1]$ of the natural short exact sequence

$$0 \to \mathcal{O}_{Y_n} \to \Omega_{Y_{n+1}/\mathbb{C}}\restriction_{Y_n} \to \Omega_{Y_n} \to 0.$$

Note that as always, all relative constructions are taken over the base if not noted otherwise, so $\Omega_{Y_n} = \Omega_{Y_n/A_n}$.

Proposition 4.21. *Let $f_1\colon Y_1 \to A_1$ be a smooth proper first-order deformation of a smooth proper variety Y with Kodaira-Spencer class κ_{Y_1} and $i_n\colon Y \hookrightarrow Y_1$ the canonical inclusion. Let $E \in \mathrm{D}^{\mathrm{b}}(Y)$ be a (perfect) complex with Atiyah class $A(E)\colon E \to E \otimes \Omega_Y[1]$. Then there exists an $E_1 \in \mathrm{D}^{\mathrm{b}}(Y_1)$ with derived restriction $i^*E_1 \simeq E$ if and only if the relative obstruction class $o(E) \in \mathrm{Ext}^2_Y(E,E)$ corresponding to the map*

$$(\mathrm{id}_E \otimes \kappa_{Y_1}) \circ A(E)\colon E \to E \otimes \Omega_Y[1] \to E \otimes \mathcal{O}_Y[2] = E[2]$$

is trivial.

Proof. Cf. [20, Cor. 3.4]. □

In [16], [1, Thm. 7.1], this criterion is used to give a sufficient condition for Fourier-Mukai kernels to be liftable to first order.

Theorem 4.22. *Let X and X' be smooth proper varieties and $X_1 \to A_1$ resp. $X'_1 \to A_1$ smooth proper deformations of X resp. X'. If $P \in \mathrm{D}^{\mathrm{b}}(X \times X')$ is a Fourier-Mukai kernel such that $\Phi_P^{\mathrm{HH}^*}(\kappa_{X_1}) = \kappa_{X'_1}$, it deforms to a kernel $P_1 \in \mathrm{D}^{\mathrm{b}}(X_1 \times_{A_1} X'_1)$.*

Proof. By the previous Proposition, we only have to show that

(4) $$0 = \big(\mathrm{id}_P \otimes \kappa_{X_1 \times_{A_1} X'_1}\big) \circ A(P) = (\mathrm{id}_P \otimes \pi_1^* \kappa_X) \circ A_X(P) + (\mathrm{id}_P \otimes \pi_2^* \kappa_{X'}) \circ A_{X'}(P)$$

in $\mathrm{Ext}^2_{X \times X'}(P,P)$. The strategy is to explicitly compute the images of κ_{X_1}, $\kappa_{X'_1}$ under the inclusions $H^1(X,T_X) \subset \mathrm{HH}^2(X)$, $H^1(X,T_X) \subset \mathrm{HH}^2(X)$ via the cohomological HKR-isomorphisms, compare them to the summands in (4) and finally use $\Phi_P^{\mathrm{HH}^*}(\kappa_{X_1}) = \kappa_{X'_1}$.

Since $\mathrm{td}(X)^{-1/2} \lrcorner$ acts as the identity on $H^1(X,T_X)$ and the component of the HKR-isomorphism I in degree 1 is given by the adjoint of $\mathcal{A}_X$ under Δ_Y, we have

$$(I^{HKR})^{-1}(\kappa_{X_1}) = (\Delta_{X*}\kappa_{X_1}) \circ \mathcal{A}_X \colon \mathcal{O}_{\Delta_X} \to \mathcal{O}_{\Delta_X}[2].$$

Using the natural identification $\mathcal{O}_{\Delta_X} \otimes \pi_i^* \Omega_X \cong \Delta_{X*}\Omega_X$ from Lemma 4.12, we obtain $(I^{HKR})^{-1}(\kappa_{X_1}) = (\mathrm{id}_{\mathcal{O}_{\Delta_X}} \otimes \pi_1^* \kappa_{X_1}) \circ A_1(\mathcal{O}_{\Delta_X})$. Similarly, $(I^{HKR})^{-1}(\kappa_{X'_1}) = -(\mathrm{id}_{\mathcal{O}_{\Delta_{X'}}} \otimes \pi_2^* \kappa_{X'_1}) \circ A_2(\mathcal{O}_{\Delta_{X'}})$.

Now $\Phi_P^{\mathrm{HH}^*}(\kappa_{X_1}) = \kappa_{X'_1}$ means by the definition of Fourier-Mukai transforms in Hochschild cohomology that $(P*)(\kappa_{X_1}) = (*P)(\kappa_{X'_1})$. But by Lemma 4.11, we have

$$(P*)(\kappa_{X_1}) = (P*)((\mathrm{id}_{\mathcal{O}_{\Delta_X}} \otimes \pi_1^* \kappa_{X_1}) \circ A_1(\mathcal{O}_{\Delta_X})) = (\mathrm{id}_P \otimes \pi_1^* \kappa_{X_1}) \circ A_X(P) \text{ and}$$
$$(*P)(\kappa_{X'_1}) = (*P)(-(\mathrm{id}_{\mathcal{O}_{\Delta_{X'}}} \otimes \pi_2^* \kappa_{X'_1}) \circ A_2(\mathcal{O}_{\Delta_{X'}})) = -(\mathrm{id}_P \otimes \pi_2^* \kappa_{X'_1}) \circ A_{X'}(P).$$

This proves equation (4). □

4.5. **Higher-order liftings of Fourier-Mukai kernels.** We shall finish our general analysis of infinitesimal deformations of Fourier-Mukai kernels by considering deformations to higher order. The problem of deforming complexes of sheaves to higher order can be reduced to the problem of deforming to first order by T^1-lifting methods. Since the arguments are rather formal, we shall refrain from developing the theory and instead merely state the (sole) result needed for our purposes. For introductions to the theory, see [30] and [22], or again [1, Sect. 7.2].

We shall consider families over the Artinian spaces

$$A_n := \operatorname{Spec} \mathbb{C}[t]/(t^{n+1}) \text{ and}$$
$$B_n := \operatorname{Spec} \mathbb{C}[x,y]/(x^{n+1}, y^2) = A_n \times A_1.$$

We have inclusions $i_n\colon A_n \hookrightarrow A_{n+1}$ and the map $t \mapsto x+y$ induces a natural surjection $q_n\colon B_n \twoheadrightarrow A_{n+1}$. Then every inductive system $f_n\colon X_n \to A_n$ of n-th-order-deformations of some smooth proper variety X induces via base change with the p_n an inductive system $g_n\colon \widetilde{X}_n := X_{n+1} \times_{A_{n+1}} B_n \to B_n$. There are again inclusions $j_n\colon \widetilde{X}_n \hookrightarrow \widetilde{X}_{n+1}$ and canonical morphisms $\widetilde{X}_n \to X_{n+1}$, which we by abuse of notation call again q_n. Now n-th-order deformations $X_{n-1} \hookrightarrow X_n$ can be reduced to (already understood) first-order deformations $X_{n-1} \hookrightarrow \widetilde{X}_{n-1}$. In fact, one can show

Proposition 4.23. *Let $X_n \to A_n$ and $\widetilde{X}_n \to B_n$ be inductive systems of deformations as above and $P_n \in \mathrm{D}^{\mathrm{b}}(X_n)$. If there is some first-order deformation $\widetilde{P}_{n+1} \in \mathrm{D}^{\mathrm{b}}(\widetilde{X}_n)$ of P_n with $j_{n-1}^*\widetilde{P}_{n+1} = q_{n-1}^*P_n \in \mathrm{D}^{\mathrm{b}}(\widetilde{X}_{n-1})$, there exists a $P_{n+1} \in \mathrm{D}^{\mathrm{b}}(X_{n+1})$ with $i_n^*P_{n+1} = P_n \in \mathrm{D}^{\mathrm{b}}(X_n)$ and $q_n^*P_{n+1} = \widetilde{P}_{n+1} \in \mathrm{D}^{\mathrm{b}}(\widetilde{X}_n)$.*

Proof. Cf. [1, Prop. 7.6]. ☐

5. A strong version of the Twisted Derived Global Torelli Theorem

In [16], Huybrechts, Macrì and Stellari show the following strong version of the Derived Global Torelli theorem.

Theorem 5.1 (Huybrechts, Macrì, Stellari)**.** *Two projective K3 surfaces X and X' are derived equivalent if and only if there exists a Hodge isometry $\varphi\colon \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X',\mathbb{Z})$. Furthermore, a Hodge isometry $\varphi\colon \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X,\mathbb{Z})$ is induced by a Fourier-Mukai equivalence $\Phi_{\mathcal{P}}\colon \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X')$ if and only if φ preserves the natural orientation of the four positive directions.*

Proof. Cf. [16, Cor. 4.10]. ☐

In the twisted case, only a weak version is known so far.

Theorem 5.2. *If $\varphi\colon \widetilde{H}(X,B,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X',B',\mathbb{Z})$ is a Hodge isometry between two twisted projective K3 surfaces (X,α_B) and $(X',\alpha_{B'})$ which preserves the natural orientation of the four positive directions, there is a Fourier-Mukai equivalence $\Phi_{\mathcal{P}}\colon \mathrm{D}^{\mathrm{b}}(X,\alpha_B) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X',\alpha_{B'})$ such that $\varphi = \Phi_{\mathcal{P}}^H$.*

Proof. Cf. [19, Thm. 0.1]. ☐

In this section, we shall try to use the deformation theory developed before to prove parts of a strong version of the Twisted Derived Global Torelli theorem.

Conjecture 5.3. *Two twisted projective K3 surfaces (X,α_B) and $(X',\alpha_{B'})$ are derived equivalent if and only if there exists a Hodge isometry $\varphi\colon \widetilde{H}(X,B,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X',B',\mathbb{Z})$ which preserves the orientation of the four natural directions. Furthermore, a Hodge isometry $\varphi\colon \widetilde{H}(X,B,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X,B',\mathbb{Z})$ is induced by a Fourier-Mukai equivalence $\Phi_{\mathcal{P}}\colon \mathrm{D}^{\mathrm{b}}(X,\alpha_B) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X',\alpha_{B'})$ if and only if φ preserves the natural orientation of the four positive directions.*

Remark 5.4. The derived equivalence is here only equivalent to the existence of an *orientation preserving* Hodge isometry because it is not clear whether arbitrary twisted projective K3 surfaces are equipped with an orientation reversing Hodge isometry, cf. the open question from Sect. 3. For untwisted K3 surfaces, such an orientation reversing Hodge isometry is given by $\mathrm{id}_{H^0 \oplus H^4} \oplus - \mathrm{id}_{H^2}$.

5.1. **Algebraic families.** We use the description of twisted sheaves as ordinary sheaves on Brauer-Severi varieties to obtain a good deformation theory of twisted K3 surfaces.

By results of Yoshioka, every family $(\mathcal{X}, \mathcal{H}) \to S$ of polarized K3 surfaces with base point s_0 can be made into a family of twisted polarized K3 surfaces which reduces over the base point s_0 to $(\mathcal{X}_{s_0}, \alpha_0)$ for some given Brauer class $\alpha_0 \in \mathrm{Br}(\mathcal{X}_{s_0})$. Let us make this precise.

Proposition 5.5. *Let $f\colon (\mathcal{X}, \mathcal{H}) \to S$ be a family of polarized K3 surfaces and $\alpha_0 \in \mathrm{Br}(\mathcal{X}_{s_0})$. Then there exists a smooth morphism $U \to S$ whose image contains s_0 and a projective bundle $\mathcal{Y} \to \mathcal{X} \times_S U$ such that the Brauer class associated to $\mathcal{Y}_{s_0}$ is $\delta(\mathcal{Y}_{s_0}) = \alpha_0$.*

Proof. By [34, Prop. 3.15] there is a projective bundle $p\colon Y \to \mathcal{X}_{s_0}$ such that $\delta(Y) = \alpha_0$ and G_p is μ-stable. The claim now follows from [34, Prop. 3.17]. □

On the other hand, we have

Proposition 5.6. *The periods of all twisted projective K3 surfaces (X, α_B) whose derived categories $\mathrm{D^b}(X, \alpha_B)$ do not contain any spherical objects (i.e. any $\mathcal{E} \in \mathrm{D^b}(X, \alpha_B)$ with $\mathrm{Hom}(\mathcal{E}, \mathcal{E}[*]) = H^*(S^2, \mathbb{C})$) are dense in the period domain*

$$\widetilde{D} = \exp(\Lambda_{\mathbb{R}})(\{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid (x.x) = 0,\ (x.\bar{x}) > 0,\ \exists\, h \in \Lambda \text{ with } h^2 > 0,\ (x.h) = 0\})$$

of all twisted algebraic K3 surfaces. For two such (X, α_B), (X', α'_B), Conj. 5.3 holds true.

Proof. This is [15, Cor. 3.21, Cor. 3.23]. □

The obvious question which springs to mind is whether these two results go together. In other words, does there, for every twisted projective K3 surface (X, α_B), exist a family $f\colon (\mathcal{X}, \mathcal{H}) \to S$ together with a projective bundle $p\colon \mathcal{Y} \to \mathcal{X}$ such that $\delta(\mathcal{Y}_{s_0}) = \alpha_B$ and $\mathrm{D^b}(\mathcal{X}_t, \delta(\mathcal{Y}_t))$ contains no spherical object for some dense subset $T \subset S$? If a Hodge isometry $\varphi\colon \widetilde{H}(X, B, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X', B', \mathbb{Z})$ which does not preserve the natural orientation of the positive four directions can be lifted to a Fourier-Mukai equivalence $\Phi_{\mathcal{P}}\colon \mathrm{D^b}(X, \alpha_B) \xrightarrow{\sim} \mathrm{D^b}(X', \alpha_{B'})$, the corresponding twisted Fourier-Mukai kernel $\mathcal{P} \in \mathrm{D^b}(Y \times Y'/(X, X'))$ deforms to any order by the results from the previous section. Using algebraicity, it even deforms over some open subset of $\mathcal{Y} \times \mathcal{Y}'$. Hence, if there were dense subsets $T \subset S$ with the above properties, we would obtain a contradiction to Prop. 5.6 (because the induced cohomological Fourier-Mukai transforms stay constant under this deformation), and therefore a proof of the Conjecture. Unfortunately, it turns out that the answer to the above question is negative.

Recall that in Def. 2.16, we defined the Brauer class $\delta(Y)$ associated to a projective bundle $Y \to X$ as the image of the cohomology class $[Y] \in H^1(X, PGL(r+1))$ of Y under the connecting homomorphism of the short exact sequence

$$1 \to \mathcal{O}_X^* \to GL(r+1) \to PGL(r+1) \to 1.$$

The diagram

$$\begin{tikzcd}
1 \arrow[r] & \mu_{r+1} \arrow[r] \arrow[d, hook] & SL(r+1) \arrow[r] \arrow[d, hook] & PGL(r+1) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathcal{O}_X^* \arrow[r] & GL(r+1) \arrow[r] & PGL(r+1) \arrow[r] & 1
\end{tikzcd}$$

of Rem. 2.17 leads us to

Definition 5.7. For a projective bundle $p\colon Y \to X$ which corresponds to $[Y] \in H^1(X, PGL(r+1))$, the class $\delta'(Y) \in H^2(X, \mu_{r+1})$ is the image of $[Y]$ under the connecting homomorphism induced by the short exact sequence

$$1 \to \mu_{r+1} \to SL(r+1) \to PGL(r+1) \to 1.$$

Furthermore, $\kappa\colon H^2(X, \mu_{r+1}) \to H^2(X, \mathcal{O}_X^*)$ is the homomorphism induced by the inclusion of sheaves $\mu_{r+1} \hookrightarrow \mathcal{O}_X^*$.

This definition allows us to render more precisely how Brauer classes stay constant in families of twisted K3 surfaces.

Lemma 5.8. *Let $f\colon (\mathcal{X}, \mathcal{H}) \to S$ be a flat family of polarized K3 surfaces over a simply connected base S together with a projective bundle $p\colon \mathcal{Y} \to \mathcal{X}$. Then the cohomology classes $\delta'(\mathcal{Y}_s) \in H^2(\mathcal{X}_s, \mu_{r+1})$ stay constant (under the canonical identifications $H^2(\mathcal{X}_s, \mu_{r+1}) \cong H^2(\mathcal{X}_s, \mathbb{Z}) \otimes \mu_{r+1} \cong H^2(\mathcal{X}_{s_0}, \mu_{r+1})$ with the cohomology group corresponding the base point s_0).*

Proof. By [34, Thm. 3.16, Step 4], we know that we can find a vector bundle $\mathcal{G}$ on $\mathcal{Y}$ whose fibres satisfy $\mathcal{G}_s \cong G_{p_s}$ for the old extensions G_{p_s} of the bundles $p_s\colon \mathcal{Y}_s \to \mathcal{X}_s$ from Def. 2.21. Now we can use

Lemma 5.9. *(i) For a projective bundle $p\colon Y \to X$ of relative dimension r over a K3 surface X, the pullback of $\delta'(Y) \in H^2(X, \mu_{r+1})$ under p is given by $p^*(\delta'(Y)) = [c_1(G_p)] \in H^2(Y, \mu_{r+1})$.*

(ii) The canonical maps $p^*\colon H^2(X, \mu_{r'}) \to H^2(Y, \mu_{r'})$ are injective for all $r' \in \mathbb{N}^*$. In particular, the class $\delta'(Y) \in H^2(X, \mu_{r+1})$ is uniquely determined by (i).

Proof. Cf. [34, Lemma 1.3, Lemma 1.6]. □

Bearing in mind this result, we can easily finish the proof of Lemma 5.8: Since $f \circ p\colon \mathcal{Y} \to \mathcal{X} \to S$ is a flat family and $\mathcal{G}$ is locally free, the first Chern classes $c_1(\mathcal{G}_s) \in H^2(\mathcal{Y}_s, \mathbb{Z})$ of the fibres $\mathcal{G}_s$ stay constant under the canonical identifications $H^2(\mathcal{Y}_s, \mathbb{Z}) \cong H^2(\mathcal{Y}_{s_0}, \mathbb{Z})$. In particular, this is true for the $[c_1(\mathcal{G}_s)] \in H^2(\mathcal{Y}_s, \mu_{r+1})$, and then by Lemma 5.9 also for the family $\delta'(\mathcal{Y}_s)$. □

We can now use this Lemma to explain why for a twisted projective K3 surface (X, α_B), there does in general not exist an algebraic deformation $f\colon (\mathcal{X}, \mathcal{H}) \to S$ together with a projective bundle $p\colon \mathcal{Y} \to \mathcal{X}$ such that $\delta(\mathcal{Y}_{s_0}) = \alpha_B$ and $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_t, \delta(\mathcal{Y}_t))$ contains no spherical object for some dense subset $T \subset S$.

Proposition 5.10. *Let (X, α) be a twisted projective K3 surface of Picard rank $\rho(X) = 1$. Let furthermore $f\colon (\mathcal{X}, \mathcal{H}) \to S$ be a (flat) family of polarized K3 surfaces with $\mathcal{X}_{s_0} = X$ and $\mathcal{Y} \to \mathcal{X}$ be a projective bundle of relative dimension r such that $\delta(\mathcal{Y}_{s_0}) = \alpha$. Then if $\mathrm{Pic}(\mathcal{X}_{s_0}, \alpha)$ contains a class $x \in \mathrm{Pic}(\mathcal{X}_{s_0}, \alpha)$ with $(x.x) = -2$, the same holds true for all s in some open neighbourhood $U \subset S$ of s_0.*

Proof. By passing to a smaller open subset if necessary, we may assume without loss of generality that S is simply connected. Choose a marking $\phi_{s_0}\colon H^2(\mathcal{X}_{s_0}, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ such that $\phi_{s_0}(\mathcal{H}_{s_0}) = \ell := e' + df'$ with e', f' the canonical basis of some hyperbolic plane $U \subset \Lambda$ and $d \in \mathbb{N}^*$. Since S is simply connected, ϕ_{s_0} induces canonical markings $\phi_s\colon H^2(\mathcal{X}_s, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ which satisfy $\phi_s(\mathcal{H}_s) = \ell$ for all $s \in S$.

Let now $B \in \Lambda$ such that $[\phi_{s_0}^{-1}(B)] = \delta'(\mathcal{Y}_{s_0}) \in H^2(X, \mu_{r+1})$. From the choice of the markings ϕ_s, it follows that $[\phi_s^{-1}(B)] = \delta'(\mathcal{Y}_s) \in H^2(\mathcal{X}_s, \mu_{r+1})$ for all $s \in S$. Hence, $\phi_s^{-1}(B)$ is a B-field lift for all twisted K3 surfaces $(\mathcal{X}_s, \delta(\mathcal{Y}_s))$. As $\rho(X) = 1$, we have $\mathrm{Pic}(X, 0) \cong_{\phi_{s_0}} (U \oplus \langle \ell \rangle) \cap \widetilde{\Lambda}$, and therefore

$$\mathrm{Pic}(X, \phi_{s_0}^{-1}(B)) \cong_{\phi_{s_0}} \exp B(U \oplus \langle \ell \rangle) \cap \widetilde{\Lambda} = (\Lambda_{d\mathbb{C}}^B)^\perp \cap \widetilde{\Lambda} =: \Gamma_d.$$

Since we have $\phi_s(\mathcal{H}_s) = \ell$ for all $s \in S$, we deduce furthermore that there is an embedding of lattices

$$\mathrm{Pic}(X, \alpha) \cong \mathrm{Pic}(X, B) \cong \Gamma_d \hookrightarrow \mathrm{Pic}(\mathcal{X}_s, \phi_s^{-1}(B)) \cong \mathrm{Pic}(\mathcal{X}_s, \delta(\mathcal{Y}_s)).$$

In particular, if $\mathrm{Pic}(X, \alpha)$ contains a (-2)-class, so do all the $\mathrm{Pic}(\mathcal{X}_s, \delta(\mathcal{Y}_s))$. □

On the other hand, one can in general only exclude the existence of a spherical object in $\mathrm{D^b}(\mathcal{X}_t, \delta(\mathcal{Y}_t))$ if $\mathrm{Pic}(\mathcal{X}_t, \delta(\mathcal{Y}_t))$ contains no (-2)-class. As a result, we cannot necessarily find a dense subset $T \subset S$ with the property that $\mathrm{D^b}(\mathcal{X}_t, \delta(\mathcal{Y}_t))$ contains no spherical object for all $t \in T$ because the lattice Γ_d might represent -2.

5.2. **The strong Twisted Derived Global Torelli theorem for sheaf kernels.** By the previous section, we cannot pass from generic to arbitrary twisted K3 surfaces by algebraic deformations in order to prove Conj. 5.3. We therefore try to adapt the approach taken in the proof of strong version of the (untwisted) Global Derived Torelli theorem in [16] to the twisted case. As a weak Twisted Derived Global Torelli theorem has already been shown in [19] (cf. Thm. 5.2, we have to prove the following

Conjecture 5.11. *Let $\Phi_{\mathcal{P}} \colon \mathrm{D^b}(X, \alpha_B) \xrightarrow{\sim} \mathrm{D^b}(X', \alpha_{B'})$ be a Fourier-Mukai equivalence between to twisted K3 surfaces (X, α_B) and $(X', \alpha_{B'})$. Then the induced Hodge isometry $\varphi := \Phi_{\mathcal{P}}^H \colon \widetilde{H}(X, B, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X', B', \mathbb{Z})$ preserves the natural orientation of the four positive directions.*

In order to transfer the geometric arguments from [16] to our situations, we need

Lemma 5.12. *In the situation of Conj. 5.11, we may assume that $\varphi((1,0,0)) = (1,0,0)$, $\varphi((0,0,1)) = (0,0,1)$, that $\varphi(B) = B'$ and that φ maps the ample cone $\mathrm{Amp}(X)$ isomorphically onto $\mathrm{Amp}(X')$ or $-\mathrm{Amp}(X')$, depending on whether φ preserves the natural orientation of the positive four directions or not.*

Remark 5.13. In particular, φ can be assumed to preserve the grading of $\widetilde{H}$.

Proof. The proof can be taken almost word for word from the proof of Thm. 5.2 in [19, Thm. 0.1]. The idea is to compose φ with standard (cohomological) Fourier-Mukai isometries who are known preserve the natural orientation of the positive four directions. Using the theory of moduli spaces of α_B-twisted sheaves on X with some Mukai vector v and the shift functor, one can first reduce to the case $\varphi((0,0,1)) = \pm(0,0,1)$. Composing with Fourier-Mukai equivalences which come from integral B-fields, on arrives at $\varphi((1,0,0)) = (1,0,0)$, $\varphi((0,0,1)) = (0,0,1)$. Changing B' by a $(1,1)$-class yields $\varphi(B) = B'$. Lastly, as φ preserves the grading of $\widetilde{H}$, composing with reflections at (-2)-classes guarantees $\varphi(\mathrm{Amp}(X)) = \mathrm{Amp}(X')$ or $\varphi(\mathrm{Amp}(X)) = -\mathrm{Amp}(X')$, depending on whether φ preserves the orientation of the positive four directions or not. For details see [19]. □

Using these assumptions, we can now generalize [19, Lemma 4.8] and obtain a proof of Conj. 5.11 in the case that the Fourier-Mukai kernel $\mathcal{P}$ is a sheaf.

Proposition 5.14. *Let (X, α_B) and $(X', \alpha_{B'})$ be twisted K3 surfaces and $\mathcal{P} \in$ $\mathbf{Coh}(X \times X', \alpha_B^{-1} \boxtimes \alpha_{B'})$ be a twisted sheaf such that the corresponding Fourier-Mukai transform $\Phi_{\mathcal{P}} \colon \mathrm{D^b}(X, \alpha_B) \xrightarrow{\sim} \mathrm{D^b}(X', \alpha_{B'})$ is an equivalence. Then the induced Hodge isometry $\varphi := \Phi_{\mathcal{P}}^H \colon \widetilde{H}(X, B, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X', B', \mathbb{Z})$ preserves the natural orientation of the four positive directions.*

Proof. Assume that φ reverses the natural orientation of the positive four directions, i.e. $\varphi(\mathrm{Amp}(X)) = -\mathrm{Amp}(X')$. Choose an α_B-twisted, locally free sheaf E of rank $r \in \mathbb{N}^*$ on X and an $\alpha_{B'}^{-1}$-twisted, locally free sheaf E' of rank $r' \in \mathbb{N}^*$ on X'. Then

$\widetilde{\mathcal{P}} := \pi_1^* E \otimes \mathcal{P} \otimes \pi_2^* E'$ is an untwisted sheaf on $X \times X'$. Choose furthermore ample line bundles L and L' on X resp. X'.

Since $\pi_1^* L$ is π_2-relatively ample, we have $R^i \pi_{2*}(\widetilde{\mathcal{P}} \otimes \pi_1^* L^n) = 0$ for all $i > 0$ and $n \gg 0$. Furthermore, $\pi_1^* L \otimes \pi_2^* L'^{n'}$ is ample for all $n' \gg 0$. Therefore, we can choose $n, n' \gg 0$ such that $\widetilde{\mathcal{P}}_{n,n'} := \widetilde{\mathcal{P}} \otimes \pi_1^* L^n \otimes \pi_2^* L'^{n'}$ is globally generated and $\pi_{2*}(\widetilde{\mathcal{P}}_{n,n'}) = \pi_{2*}\big(\widetilde{\mathcal{P}} \otimes \pi_1^* L^n \otimes \pi_2^* L'^{n'}\big) = \pi_{2*}(\widetilde{\mathcal{P}} \otimes \pi_1^* L^n) \otimes L'^{n'}$ is a sheaf.

As $\widetilde{\mathcal{P}}_{n,n'}$ is globally generated, there is a short exact sequence $0 \to \mathcal{K} \to \mathcal{O}_{X \times X'}^N \to \widetilde{\mathcal{P}}_{n,n'} \to 0$. We have again $R^i \pi_{2*}(\mathcal{K} \otimes \pi_1^* L^m) = 0$ for all $i > 0$ and $m \gg 0$ because $\pi_1^* L$ is π_2-ample. This implies that the map $\pi_{2*}(\mathcal{O}_{X \times X'}^N \otimes \pi_1^* L^m) \twoheadrightarrow \pi_{2*}(\widetilde{\mathcal{P}}_{n+m,n'})$ is surjective. Since $\pi_{2*}(\mathcal{O}_{X \times X'}^N \otimes \pi_1^* L^m) = \mathcal{O}_{X'} \otimes H^0(X, L^m)$, the direct image $\pi_{2*}(\widetilde{\mathcal{P}}_{n+m,n'})$ is also globally generated for all $m \gg 0$.

On the other hand, as $\varphi((1,0,0)) = (1,0,0)$ by Lemma 5.12, it follows that

$$\operatorname{rk}(\pi_{2*}(\mathcal{P} \otimes \pi_1^*(E \otimes L^{n+m}))) = \operatorname{rk}(\Phi_{\mathcal{P}}(E \otimes L^{n+m})) = \operatorname{rk}(E \otimes L^{n+m}) = r.$$

The assumption that $\varphi(\operatorname{Amp}(X)) = -\operatorname{Amp}(X')$ implies that there is an ample line bundle M on X' with $\varphi(c_1(L)) = -c_1(M)$. Hence, we have

$$\begin{aligned}
c_1(\pi_{2*}(\widetilde{\mathcal{P}}_{n+m,n'})) &= c_1\big(\pi_{2*}\big(\mathcal{P} \otimes \pi_1^*(E \otimes L^{n+m}) \otimes \pi_2^*(E' \otimes L'^{n'})\big)\big)\\
&= c_1\big(\pi_{2*}(\mathcal{P} \otimes \pi_1^*(E \otimes L^{n+m})) \otimes \big(E' \otimes L'^{n'}\big)\big)\\
&= r' \cdot c_1^{B'}(\Phi_{\mathcal{P}}(E \otimes L^{n+m})) + r \cdot c_1^{-B'}\big(E' \otimes L'^{n'}\big)\\
&= r' \cdot \big(\Phi_{\mathcal{P}}^H\big(v^B(E \otimes L^{n+m})\big)\big)_2 + r \cdot c_1^{-B'}\big(E' \otimes L'^{n'}\big)\\
&= r' \cdot \big(\varphi\big(v^B(E) \cdot \operatorname{ch}(L^{n+m})\big)\big)_2 + r \cdot c_1^{-B'}\big(E' \otimes L'^{n'}\big)\\
&= r' \cdot (s - r(n+m) \cdot c_1(M)) + r \cdot s'
\end{aligned}$$

for certain $s \in H^{1,1}(X', B', \mathbb{Q})$, $s' \in H^{1,1}(X', -B', \mathbb{Q})$. Therefore, for all $m \gg 0$, $\pi_{2*}(\widetilde{\mathcal{P}}_{n+m,n'})$ is a globally generated coherent sheaf of rank $r \cdot r'$ whose first Chern class

$$c_1(\pi_{2*}(\widetilde{\mathcal{P}}_{n+m,n'})) = r' \cdot (s - r(n+m) \cdot c_1(M)) + r \cdot s' =: t - rr'(n+m) \cdot c_1(M)$$

lies in $-\operatorname{Amp}(X')$: Indeed, since the ample cone $\operatorname{Amp}(X')$ is open, there is an $\varepsilon > 0$ such that $\varepsilon t - c_1(M) \in -\operatorname{Amp}(X')$, or equivalently $t - \frac{1}{\varepsilon} \cdot c_1(M) \in -\operatorname{Amp}(X')$. Using the convexity of $\operatorname{Amp}(X')$, we obtain $t - rr'(n+m) \cdot c_1(M) = (t - \frac{1}{\varepsilon} \cdot c_1(M)) - (rr'(n+m) - \frac{1}{\varepsilon}) \cdot c_1(M) \in -\operatorname{Amp}(X')$ for all $m \gg 0$. But by the following lemma, the first Chern class of a globally generated sheaf is nef, a contradiction. $\square$

Lemma 5.15. *The first Chern class $c_1(G)$ of a globally generated coherent sheaf G on a smooth complex surface X is nef.*

Proof. Let us first assume that G is torsion-free. Since X is smooth, there are finitely many points $x_1, \ldots, x_n$ such that $G\restriction_U$ is locally free on the subset $U := X \smallsetminus \{x_1, \ldots, x_n\}$. The determinant bundle $\det(G\restriction_U)$ of the locally free, globally generated sheaf $G\restriction_U$ is still globally generated, hence it has a global section and is either trivial or of the form $\mathcal{O}(C)$ for some curve $C \subset U$. As X is normal and $X \smallsetminus U$ has codimension 2, the extension of the line bundle $\det(G\restriction_U) = (\det G)\restriction_U$ is unique. Thus, we have either $\det G = \mathcal{O}_X$ or $\det G = \mathcal{O}(\bar{C})$ where $\bar{C}$ is the closure of C in X. In particular, $c_1(G) = c_1(\det G)$ is nef.

In the general case, let $T(G)$ be the torsion part of G, $T_0(G)$ its maximal subsheaf of dimension 0 and D the (one-dimensional) support of $T(G)/T_0(G)$. As the determinant of a zero-dimensional sheaf is trivial, $\det(T(G)) = \det(T(G)/T_0(G)) = \mathcal{O}(D)$ is nef. By passing to the globally generated quotient $G/T(G)$, we can therefore reduce to the torsion-free case. $\square$

5.3. **Where to go from here.** One can now apply the deformation theory developed in Sect. 4 in order to pass from the case that $\mathcal{P} \in \mathrm{D^b}(X \times X', \alpha_B^{-1} \boxtimes \alpha_{B'})$ is a sheaf to the general case. Unfortunately, due to the time limitations for this thesis, we were not able to check this approach rigorously. Let us therefore only give an outlook how one could proceed. We hope to be able to make this into a substantial proof in the near future.

Let (X, α_B) and $(X', \alpha_{B'})$ be twisted K3 surfaces and $P \in \mathrm{D^b}(X \times X', \alpha_B^{-1} \boxtimes \alpha_{B'})$ be the kernel of a Fourier-Mukai equivalence. We would like to show that the induced Hodge isometry $\varphi\colon \Phi_P^H\colon \widetilde{H}(X, B, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X', B', \mathbb{Z})$ preserves the natural orientation of the positive four directions. Assume that this is not the case. By Lemma 5.12, we may then without loss of generality restrict to the case $\varphi(\mathrm{Amp}(X)) = -\mathrm{Amp}(X')$. By [34, Prop. 3.15], we can furthermore find simple α_B- resp. $\alpha_{B'}$-twisted vector bundles E resp. E' on X resp. X'. These correspond to projective bundles $p\colon Y := \mathbb{P}(E) \to X$ resp. $p'\colon Y' := \mathbb{P}(E') \to X'$ with $\delta(Y) = \alpha_B$ and $\delta(Y') = \alpha_{B'}$, cf. Lemma 2.18.

The idea of [16] in the untwisted case is to find a formal twistor deformation $\mathcal{X} \times \mathcal{X}'$ with special fibre $X \times X'$ such that P deforms to a complex $\mathcal{P} \in \mathrm{D^b}(\mathcal{X} \times \mathcal{X}')$. Its general fibre $\mathcal{P}_K$ can be shown to be a sheaf, which lifts to a *sheaf* $\widetilde{\mathcal{P}}$ on $\mathcal{X} \times \mathcal{X}'$. Then $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ agree up to torsion complexes, so the special fibres $P = \mathcal{P}_0$ and $\widetilde{\mathcal{P}}_0$ induce the same action on cohomology. The assertion follows from the untwisted analogue of Prop. 5.14.

Let us discuss how we could adapt this strategy to twisted sheaves, using the theory developed so far. As in the untwisted case, we start by choosing a very general real ample class $\omega \in \mathrm{Amp}(X)$, i.e. $\omega^\perp \cap H^{1,1}(X, \mathbb{Z})$. Then the associated twistor space $\mathbb{X}(\omega) \to \mathbb{P}(\omega)$ with base point $0 \in \mathbb{P}(\omega)$ gives a non-algebraic deformation of $X = \mathbb{X}_0$. Choosing a local parameter t around 0, we obtain an algebraic (!) formal deformation $\mathcal{X} \to \mathcal{A}$ where $\mathcal{A} := \varprojlim A_n = \operatorname{Spec} \mathbb{C}[\![t]\!]$. Our first aim is to find successively for all $n \in \mathbb{N}$ deformations X'_n of X', $p_n\colon Y_n \to X_n$ of $p\colon Y \to X$ and $p'_n\colon Y'_n \to X'_n$ of $p'\colon Y' \to X'$ such that the complex $Q \in \mathrm{D^b}(Y \times Y'/(X, X'))$ corresponding to $P \in \mathrm{D^b}(X \times X', \alpha_B^{-1} \boxtimes \alpha_{B'})$ (cf. Rem. 2.49) can be lifted to some $Q_n \in \mathrm{D^b}(Y_n \times Y'_n/(X_n, X'_n))$.

Assume that such deformations have already been found for $n \in \mathbb{N}$, with $n = 0$ being the trivial case $X'_n = X'$, $Y_0 = Y$ etc. We want to construct the respective deformations for $n+1$. In [16, Sect. 4.1, 4.2], it is shown that if Φ_P^H preserves the Kähler cone up to sign, the class $\Phi_{P_n}^{\mathrm{HH}^*}(\kappa_{X_{n+1}})$ lies in $H^1(X'_n, T_{X'_n})$ and that we can further find an infinitesimal twistor deformation X'_{n+1} of X'_n such that $\kappa_{X'_{n+1}} = \Phi_{P_n}^{\mathrm{HH}^*}(\kappa_{X_{n+1}})$. With the reductions from Lemma 5.12, all arguments are still valid in the twisted setting, i.e. we can find a deformation X'_n with $\Phi_{P_n}^{\mathrm{HH}^*}(\kappa_{X_{n+1}}) = \kappa_{X'_{n+1}}$.

Since E and E' are simple, we can find deformations $p_{n+1}\colon Y_{n+1} \to X_{n+1}$ of $p_n\colon Y_n \to X_n$ and $p'_{n+1}\colon Y'_{n+1} \to X'_{n+1}$ of $p'_n\colon Y'_n \to X'_n$. Although we have no rigorous proof, we expect that p_{n+1} and p'_{n+1} can be chosen in such a way that still $\Phi_{Q_n}^{\mathrm{HH}^*}(\kappa_{Y_{n+1}}) = \kappa_{Y'_{n+1}}$ where $Q_n \in \mathrm{D^b}(Y_n \times Y'_n/(X_n, X'_n))$ is again the complex corresponding to P_n. In that case, let $\widetilde{Y_n} \times_{B_n} \widetilde{Y'_n} \to B_n$ be the family obtained from $Y_{n+1} \times_{A_{n+1}} Y'_{n+1} \to A_{n+1}$ via the base change $B_n \twoheadrightarrow A_{n+1}$ as in Prop. 4.23. Following [1, Thm. 7.7], we could then argue that by (a relative version of) Thm. 4.22 that there is a complex $\widetilde{Q}_{n+1} \in \mathrm{D^b}(\widetilde{Y}_n \times \widetilde{Y}'_n)$ whose restriction to $\mathrm{D^b}(Y_n \times Y'_n)$ is Q_n. Via the isomorphism $*P_n$ from Lemma 4.2, we have

$$\mathrm{Ext}^1(P_n, P_n) \cong \mathrm{HH}^1(Y_n) \cong H^1(Y_n, \mathcal{O}_{Y_n}) \oplus H^0(Y_n, T_{Y_n}) = 0,$$

hence the deformation $\widetilde{Q}_{n+1}$ is unique and restricts to $q_{n-1}^* Q_n$ on $\mathrm{D^b}(\widetilde{Y}_{n-1} \times \widetilde{Y}'_{n-1})$. By Prop. 4.23, we therefore obtain a deformation $Q_{n+1} \in \mathrm{D^b}(Y_{n+1} \times Y'_{n+1})$ of Q_n.

Since the condition that

$$(p'_{n+1} \times p_{n+1})^*(p'_{n+1} \times p_{n+1})_*(G_{p'_{n+1}} \otimes F \otimes G^\vee_{p_{n+1}}) \simeq G_{p'_{n+1}} \otimes F \otimes G^\vee_{p_{n+1}}$$

in $\mathrm{D^b}(Y_{n+1} \times Y'_{n+1})$ is open, we still have $Q_{n+1} \in \mathrm{D^b}(Y_{n+1} \times Y'_{n+1}/(X_{n+1}, X'_{n+1}))$.

Arguments of these kind would show that we can find the deformations X'_n, Y_n, Y'_n and Q_n for all $n \in \mathbb{N}$. Putting $\mathcal{X}' := \varprojlim X'_n$, $\mathcal{Y} := \varprojlim Y_n$ and $\mathcal{Y}' := \varprojlim Y'_n$, we would obtain a formal deformation $\mathcal{X} \times \mathcal{X}'$ of $X \times X'$ with projective bundles $\mathcal{Y} \to \mathcal{X}$, $\mathcal{Y}' \to \mathcal{X}'$. By results of Lieblich [24, Sect. 3.6], we would furthermore have a complex $\mathcal{Q} \in \mathrm{D^b}((\mathcal{Y} \times \mathcal{Y}'), (\mathcal{X}, \mathcal{X}')$ which restricts to the Q_n on $\mathrm{D^b}(Y_n \times Y'_n)$. Provided the general fibre $\mathcal{P}_K$ was still a sheaf in twisted case, we would then be able to deduce the strong version of the Twisted Derived Global Torelli theorem from Prop. 5.14.

References

[1] Nicolas Addington and Richard Thomas. Hodge theory and derived categories of cubic fourfolds. *Duke Math. J.*, 163(10):1885–1927, 2014.

[2] Marcello Bernardara. A semiorthogonal decomposition for Brauer-Severi schemes. *Math. Nachr.*, 282(10):1406–1413, 2009.

[3] Ragnar-Olaf Buchweitz and Hubert Flenner. The global decomposition theorem for Hochschild (co-)homology of singular spaces via the Atiyah-Chern character. *Adv. Math.*, 217(1):243–281, 2008.

[4] Damien Calaque, Carlo A. Rossi, and Michel Van den Bergh. Căldăraru's conjecture and Tsygan's formality. *Ann. of Math. (2)*, 176(2):865–923, 2012.

[5] Damien Calaque and Michel Van den Bergh. Hochschild cohomology and Atiyah classes. *Adv. Math.*, 224(5):1839–1889, 2010.

[6] Andrei Căldăraru. *Derived categories of twisted sheaves on Calabi-Yau manifolds*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–Cornell University.

[7] Andrei Căldăraru and Simon Willerton. The Mukai pairing. I. A categorical approach. *New York J. Math.*, 16:61–98, 2010.

[8] Alberto Canonaco and Paolo Stellari. Twisted Fourier-Mukai functors. *Adv. Math.*, 212(2):484–503, 2007.

[9] John W. S. Cassels. *Rational quadratic forms*, volume 13 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1978.

[10] Alexander Grothendieck. Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses [MR0244269 (39 #5586a)]. In *Séminaire Bourbaki, Vol. 9*, pages Exp. No. 290, 199–219. Soc. Math. France, Paris, 1995.

[11] Alexander Grothendieck. Le groupe de Brauer. II. Théorie cohomologique [MR0244270 (39 #5586b)]. In *Séminaire Bourbaki, Vol. 9*, pages Exp. No. 297, 287–307. Soc. Math. France, Paris, 1995.

[12] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[13] Daniel Huybrechts. The global Torelli theorem: classical, derived, twisted. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 235–258. Amer. Math. Soc., Providence, RI, 2009.

[14] Daniel Huybrechts. Lectures on K3 surfaces [accessed 02/20/2014], 2014. Available at http://www.math.uni-bonn.de/people/huybrech/K3Global.pdf.

[15] Daniel Huybrechts, Emanuele Macrì, and Paolo Stellari. Stability conditions for generic $K3$ categories. *Compos. Math.*, 144(1):134–162, 2008.

[16] Daniel Huybrechts, Emanuele Macrì, and Paolo Stellari. Derived equivalences of $K3$ surfaces and orientation. *Duke Math. J.*, 149(3):461–507, 2009.

[17] Daniel Huybrechts and Stefan Schröer. The Brauer group of analytic $K3$ surfaces. *Int. Math. Res. Not.*, (50):2687–2698, 2003.

[18] Daniel Huybrechts and Paolo Stellari. Equivalences of twisted $K3$ surfaces. *Math. Ann.*, 332(4):901–936, 2005.

[19] Daniel Huybrechts and Paolo Stellari. Proof of Căldăraru's conjecture. Appendix: "Moduli spaces of twisted sheaves on a projective variety" [in *moduli spaces and arithmetic geometry*, 1–30, Math. Soc. Japan, Tokyo, 2006; see 2306170] by K. Yoshioka. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 31–42. Math. Soc. Japan, Tokyo, 2006.

[20] Daniel Huybrechts and Richard P. Thomas. Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes. *Math. Ann.*, 346(3):545–569, 2010.

[21] Peter T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 1*, volume 43 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 2002.

[22] Yujiro Kawamata. Unobstructed deformations. A remark on a paper of Z. Ran: "Deformations of manifolds with torsion or negative canonical bundle" [J. Algebraic Geom. **1** (1992), no. 2, 279–291; MR1144440 (93e:14015)]. *J. Algebraic Geom.*, 1(2):183–190, 1992.

[23] Sándor J. Kovács. The cone of curves of a $K3$ surface. *Math. Ann.*, 300(4):681–691, 1994.

[24] Max Lieblich. Moduli of complexes on a proper morphism. *J. Algebraic Geom.*, 15(1):175–206, 2006.

[25] Emanuele Macrì and Paolo Stellari. Infinitesimal derived Torelli theorem for $K3$ surfaces. *Int. Math. Res. Not. IMRN*, (17):3190–3220, 2009. With an appendix by Sukhendu Mehrotra.

[26] John Milnor and Dale Husemoller. *Symmetric bilinear forms*. Springer-Verlag, New York, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

[27] David R. Morrison. On $K3$ surfaces with large Picard number. *Invent. Math.*, 75(1):105–121, 1984.

[28] Shigeru Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Math. J.*, 81:153–175, 1981.

[29] Viacheslav V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):111–177, 238, 1979.

[30] Ziv Ran. Deformations of manifolds with torsion or negative canonical bundle. *J. Algebraic Geom.*, 1(2):279–291, 1992.

[31] Olga Taussky, editor. *Ternary quadratic forms and norms*, volume 79 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1982.

[32] Yukinobu Toda. Deformations and Fourier-Mukai transforms. *J. Diff. Geom.*, 81(1):197–224, 2009.

[33] Charles T. C. Wall. On the orthogonal groups of unimodular quadratic forms. *Math. Ann.*, 147:328–338, 1962.

[34] Kōta Yoshioka. Moduli spaces of twisted sheaves on a projective variety. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 1–30. Math. Soc. Japan, Tokyo, 2006.