Green's conjecture

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Introduction

Clifford's theorem is a classical result in algebraic geometry. It proves an upper bound of the dimension of $|D|$ for a special effective divisor D on a curve.

Theorem 0.1 (Clifford)**.** *[Har77, IV Thm. 5.4] Let C be a smooth projective curve and let D be a special effective divisor on C. Then the dimension of $|D|$ is bounded from above:*

$$\dim|D| \leq \frac{1}{2}\deg D.$$

Furthermore, equality holds if and only if D is trivial, D is the canonical divisor or C is hyperelliptic and D corresponds to some $f^*\mathcal{O}_{\mathbb{P}^1}(n)$ where f is the map that makes C hyperelliptic.

This leads to the question of how good the upper bound is. The Clifford index is a measurement that answers this question.

Definition 0.1. [Eis05, 9A] Let C be a smooth projective curve and let D be a special effective divisor on C. The *Clifford index* of D is defined as

$$\mathrm{Cliff}(D) := \deg D - 2\dim|D|$$

and the *Clifford index* of C is defined as

$$\mathrm{Cliff}(C) := \min\{d - 2r \mid \exists \mathcal{L} \in \mathrm{Pic}(C), \deg \mathcal{L} = d, h^0(\mathcal{L}) = r + 1 \geq 2, h^1(\mathcal{L}) \geq 2\},$$

if the genus is larger than three and defined as one, if the genus equals three and C is not hyperelliptic. Otherwise it is defined as zero.

Note that the curve C is hyperelliptic if and only if $\mathrm{Cliff}(C) = 0$. Hence, the Clifford index can be viewed as a measurement of how far a curve fails to be hyperelliptic. In [Gre84] Mark Green made the following conjecture:

Conjecture 0.2. *Let C be a smooth projective curve over a field of characteristic zero. Then*

$$\mathrm{Cliff}(C) > l \Leftrightarrow K_{p,2}(C, \omega_C) = 0 \ \forall p \leq l.$$

Here, $K_{p,2}(C, \omega_C)$ denotes the Koszul cohomology and ω_C is the canonical bundle.

This is very spectacular since the Clifford index on the one hand combines information of many different line bundles while the vanishing of the Koszul cohomology on the other hand depends only on the canonical bundle. A reformulation will make this apparent:

Conjecture 0.3 (Reformulation)**.** *Let C be a smooth projective curve over a field of characteristic zero. Then*

$$\mathrm{Cliff}(C) = \min\{p \mid K_{p,2}(C, \omega_C) \neq 0\}.$$

In other words, the Clifford index of a curve can be determined by calculating its Koszul cohomology. The direction "$\Leftarrow$" was proven by Green and Lazarsfeld in an appendix to Green's paper. The direction "$\Rightarrow$" has not yet been proven. Nevertheless, there are partial results such as Voisin's theorem. Kemeny's proof of this theorem will be explained in this bachelor thesis.

Another interesting point of view is to consider Green's conjecture as a generalization of other classical results such as Max Noether's theorem and Petri's theorem.

Theorem 0.4 (Max Noether)**.** *[Ara+85, III §2] Let C be a smooth projective curve which is not hyperelliptic. Then the canonical ring of C is generated in degree one.*

Later we will see that in the language of Koszul cohomology this assertion translates to $K_{0,2}(C, \omega_C) = 0$. Hence, Noether's theorem is just the direction "$\Rightarrow$" in Green's conjecture in case of $l = 0$.

Theorem 0.5 (Petri)**.** *[Ara+85, III §3] Let C be a canonical curve of genus $g \geq 4$. Then the ideal sheaf $\mathcal{I}$ is generated by elements of degree two, unless C is trigonal or a plane quintic.*

By translating this theorem into the language of Koszul cohomology we will see again that it is equivalent to the vanishing of $K_{1,2}(C, \omega_C)$. Furthermore, the considered curves are those curves that satisfy $\mathrm{Cliff}(C) > 1$. Thus, it corresponds to the direction "$\Rightarrow$" in Green's conjecture in the case of $l = 1$.

The main result of this thesis is Voisin's theorem, that implies Green's conjecture for curves of even genus which lie in K3 surfaces and generate their Picard group. The proof we will consider is due to Kemeny [Kem20]. It requires some knowledge of K3 surfaces which will be recalled, and Lazarsfeld–Mukai bundles which will be introduced. Furthermore, we will define Koszul cohomology, relate it to sheaf cohomology and explore the connection between Green's conjecture, Noether's theorem, and Petri's theorem. In addition, the proof of the direction "$\Leftarrow$" will be given. After that we will construct the secant bundles Γ and $\mathcal{S}$, that will fit into the exact sequence

$$0 \to \mathcal{S} \to \pi^*\mathcal{M} \to \Gamma \to 0.$$

This exact sequence among others will be used to compute multiple (sheaf) cohomology groups, which will lead to the proof of Voisin's theorem. Finally, we will apply the developed techniques to prove a theorem that implies a special case of the geometric syzygy conjecture, a different conjecture about Koszul cohomology.

Einleitung

Ein klassisches Resultat der algebraischen Geometrie ist der Satz von Clifford, welcher eine obere Abschätzung für die Dimension $|D|$ eines speziellen effektiven Divisors D auf einer Kurve liefert.

Satz 0.1 (Clifford)**.** *[Har77, IV Thm. 5.4] Sei C eine glatte projektive Kurve und sei D ein spezieller effektiver Divisor auf C. Dann ist die Dimension von $|D|$ von oben beschränkt durch:*

$$\dim |D| \leq \frac{1}{2} \deg D.$$

Desweiteren gilt Gleichheit genau dann, wenn D trivial ist, D der kanonische Divisor ist oder wenn C hyperelliptisch ist und D zu $f^*\mathcal{O}_{\mathbb{P}^1}(n)$ korrespondiert, wobei f die Abbildung ist, die C hyperelliptisch macht.

Daraus ergibt sich die Frage, wie genau diese Abschätzung ist. Dies wird vom Clifford Index gemessen.

Definition 0.1. [Eis05, 9A] Sei C eine glatte projektive Kurve und sei D ein spezieller effektiver Divisor auf C. Dann ist der *Clifford Index* von D definiert als

$$\operatorname{Cliff}(D) := \deg D - 2 \dim |D|$$

und der *Clifford Index* von C ist definiert als

$$\operatorname{Cliff}(C) := \min\{d - 2r \mid \exists \mathcal{L} \in \operatorname{Pic}(C), \deg \mathcal{L} = d, h^0(\mathcal{L}) = r + 1 \geq 2, h^1(\mathcal{L}) \geq 2\},$$

falls das Geschlecht größer als Drei ist und ist definiert als Eins, falls das Geschlecht gleich Drei ist und C nicht hyperelliptisch ist. Sonst ist er als Null definiert.

Es gilt, dass C hyperelliptisch ist, genau dann wenn $\operatorname{Cliff}(C) = 0$. Daher kann der Clifford Index als eine Größe gesehen werden, die misst, wie stark eine Kurve nicht hyperelliptisch ist. In [Gre84] stellte Mark Green die folgende Vermutung auf:

Vermutung 0.2. *Sei C eine glatte projektive Kurve über einem Körper der Charakteristik Null. Dann gilt*

$$\operatorname{Cliff}(C) > l \Leftrightarrow K_{p,2}(C, \omega_C) = 0 \ \forall p \leq l.$$

Hier ist $K_{p,2}(C, \omega_C)$ die Koszul Kohomologie und ω_C das kanonische Bündel.

Diese Vermutung ist sehr spektakulär, da der Clifford Index auf der einen Seite Informationen von sehr vielen verschiedenen Geradenbündeln kombiniert, während das Verschwinden der Koszul Kohomologie auf der anderen Seite nur vom kanonischen Bündel abhängt. Eine Umformulierung verdeutlicht dies:

Vermutung 0.3 (Umformulierung)**.** *Sei C eine glatte projektive Kurve über einem Körper der Charakteristik Null. Dann gilt*

$$\operatorname{Cliff}(C) = \min\{p \mid K_{p,2}(C, \omega_C) \neq 0\}.$$

Der Clifford Index einer Kurve kann also durch die Koszul Kohomologie berechnet werden. Die Richtung „$\Leftarrow$“ wurde von Green und Lazarsfeld in einem Anhang zu Greens Artikel über Koszul Kohomologie bewiesen. Die Richtung „$\Rightarrow$“ ist bis heute offen. Trotzdem gibt es Teil–Resultate, wie den Satz von Voisin. Kemenys Beweis dieses Satzes wird in dieser Bachelorarbeit erläutert.

Ein weiterer interessanter Aspekt ist, dass Greens Vermutung als eine Verallgemeinerung von klassischen Resultaten, wie dem Satz von Max Noether oder dem Satz von Petri, betrachtet werden kann.

Satz 0.4 (Max Noether)**.** *[Ara+85, III §2] Sei C eine glatte projektive Kurve, welche nicht hyperelliptisch ist. Dann wird der kanonische Ring von C in Grad Eins erzeugt.*

Später wird festgestellt, dass dieser Satz zu der Bedingung $K_{0,2}(C, \omega_C) = 0$ äquivalent ist. Daher entspricht Noethers Satz die Richtung „$\Rightarrow$“ in Greens Vermutung für $l = 0$.

Satz 0.5 (Petri)**.** *[Ara+85, III §3] Sei C eine kanonische Kurve des Geschlechts $g \geq 4$. Dann wird die Idealgarbe $\mathcal{I}$ von Elementen von Grad Zwei erzeugt, außer C ist trigonal oder planar und quintisch.*

Die Aussage, dass $\mathcal{I}$ von Elementen von Grad zwei erzeugt wird, ist äquivalent zu $K_{1,2}(C, \omega_C) = 0$. Da die betrachteten Kurven $\mathrm{Cliff}(C) > 1$ erfüllen, entspricht Petris Satz der Richtung „$\Rightarrow$“ in Greens Vermutung für $l = 1$.

Das Hauptresultat der vorliegenden Arbeit ist der Satz von Voisin, welcher Greens Vermutung im Falle von Kurven von geradem Geschlecht, die in K3 Flächen liegen und deren Picard Gruppe erzeugen, impliziert. Der Beweis, der hier gegeben wird, ist von Kemeny [Kem20]. Der Beweis setzt grundlegende Kentnisse über K3 Flächen voraus, welche wiederholt werden und es werden Lazarsfeld–Mukai Bündel genutzt, welche eingeführt werden. Desweiteren wird Koszul Kohomologie eingeführt und in Garben Kohomologie umformuliert. Die Verbindung zwischen Greens Vermutung, Noethers Satz und Petris Satz wird danach erklärt. Zusätzlich wird der Beweis der Richtung „$\Leftarrow$“ gegeben. Anschließend werden die Sekanten Bündel Γ und $\mathcal{S}$ konstruiert. Diese passen in die kurz exakte Sequenz

$$0 \to \mathcal{S} \to \pi^*\mathcal{M} \to \Gamma \to 0.$$

Diese und weitere exakte Sequenzen werden genutzt um (Garben) Kohomologien zu berechnen, was schließlich zum Beweis von Voisins Satz führen wird. Als Anwendung werden die entwickelten Techniken zu einem Beweis eines Satzes führen, welcher einen Spezialfall der geometrischen Syzygy Vermutung, einer weiteren Vermutung über Koszul Kohomologie, impliziert.

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Contents

1 Notation and preliminaries

We write X, Y for schemes, $\mathcal{O}_X$ for the structure sheaf of X and ω_X for the canonical bundle on X. All schemes are schemes over the complex numbers and all vector spaces are vector spaces on the complex numbers. Sheaves of $\mathcal{O}_X$-modules are denoted by $\mathcal{F}$, $\mathcal{G}, \ldots$ and line bundles by $\mathcal{L}$. The setting of Kemeny's proof of Voisin's theorem consists of a K3 surface X, a smooth curve C contained in X and a line bundle on C. In these cases the line bundle on C is denoted by $\mathcal{A}$ so we can distinguish it from a line bundle on X, denoted by $\mathcal{L}$. The dual of sheaves of $\mathcal{O}_X$-modules is marked by $(-)^*$, e.g. $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

The next theorems will be used multiple times throughout this thesis.

Theorem 1.1 (Projection formula)**.** *[Har77, III Ex. 8.3] Let $f\colon X \to Y$ be a morphism between varieties over $\mathbb{C}$ and let $\mathcal{F} \in \mathrm{Coh}(X)$, $\mathcal{E} \in \mathrm{Coh}(Y)$ be coherent sheaves where $\mathcal{E}$ is locally free. Then there is an isomorphism between the higher direct images:*

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong R^i f_* \mathcal{F} \otimes \mathcal{E}.$$

Theorem 1.2. *[Har77, III Ex. 8.1] Let $f\colon X \to Y$ be a morphism and let $\mathcal{F}$ be a sheaf on X such that $R^i f_* \mathcal{F} = 0$ for all $i > 0$. Then*

$$H^n(X, \mathcal{F}) \cong H^n(Y, f_*\mathcal{F})$$

for all $n \geq 0$.

Theorem 1.3 (Künneth formula)**.** *[21, Tag 0BED] Let X, Y be varieties and let $\mathcal{F} \in \mathrm{Coh}(X)$, $\mathcal{G} \in \mathrm{Coh}(Y)$. Define $\mathcal{F} \boxtimes \mathcal{G} := p_X^* \mathcal{F} \otimes p_Y^* \mathcal{G} \in \mathrm{Coh}(X \times Y)$ where $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ are the projections. Then there is a canonical isomorphism for all $n \geq 0$:*

$$H^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \cong \bigoplus_{a+b=n} H^a(X, \mathcal{F}) \otimes H^b(Y, \mathcal{G}).$$

Theorem 1.4 (Serre duality)**.** *[Har77, III Cor. 7.7] Let X be a smooth projective scheme which is equidimensional of dimension n. Let $\mathcal{F}$ be a locally free sheaf on X. Then there is a natural isomorphism*

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^* \otimes \omega_X)^*.$$

Theorem 1.5 (Grauert's theorem)**.** *[Har77, III Cor. 12.9] Let $f\colon X \to Y$ be a projective morphism between varieties and suppose that Y is integral. Let $\mathcal{F} \in \mathrm{Coh}(X)$ be flat over Y such that the function $y \mapsto h^i(X_y, \mathcal{F}|_{X_y})$ is constant for some $i \geq 0$. Then $R^i f_* \mathcal{F}$ is a locally free sheaf on Y and for every $y \in Y$ there is a natural isomorphism*

$$R^i f_* \mathcal{F} \otimes \kappa(y) \xrightarrow{\sim} H^i(X_y, \mathcal{F}|_{X_y}).$$

Theorem 1.6 (Brill–Noether)**.** *[Ara+85, V Thm. 1.1, 1.5] Let C be a smooth projective curve of genus g and let $d \geq 1$ and $r \geq 0$ be integers. Suppose that C is general in the sense of Brill–Noether theory. Then there exists a line bundle $\mathcal{L}$ on C of degree d that satisfies $h^0(C, \mathcal{L}) - 1 = r$ if and only if $g - (r+1)(g - d + r) \geq 0$.*

1.1 K3 surfaces

K3 surfaces are special kinds of surfaces and very interesting in their own right. However, very little theory is needed, so this section is just a reminder of the most basic facts.

Definition 1.1. [Huy15, Ch. 1 Def. 1.1] Let k be a field. A *K3 surface* is a complete non-singular variety X of dimension two such that

(i) $\omega_{X/k} \cong \mathcal{O}_X$

(ii) $H^1(X, \mathcal{O}_X) = 0$

Remark. Let X be a K3 surface over $\mathbb{C}$. Then Serre duality implies $\chi(X, \mathcal{O}_X) = 2$ where $\chi(X, -)$ denotes the Euler–Poincaré characteristic.

Definition 1.2. [Huy15, Ch. 1] Let X be a non-singular complete surface over a field k and let $\mathcal{L}_1, \mathcal{L}_2 \in \operatorname{Pic}(X)$. We define the *intersection pairing* $(\mathcal{L}_1.\mathcal{L}_2)$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ as

$$(\mathcal{L}_1.\mathcal{L}_2) := \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}_1^*) - \chi(X, \mathcal{L}_2^*) + \chi(X, \mathcal{L}_1^* \otimes \mathcal{L}_2^*).$$

Lemma 1.7. *[Huy15, Ch. 1] Let X be a K3 surface and let $\mathcal{L}_1, \mathcal{L}_2 \in \operatorname{Pic}(X)$. Then the following assertions hold:*

(i) *The intersection pairing is a symmetric bilinear form $(-.-)\colon \operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \mathbb{Z}$.*

(ii) *If $\mathcal{L}_1 = \mathcal{O}(C)$ for a curve (i.e. a divisor) $C \subset X$ then $(\mathcal{L}_1.\mathcal{L}_2) = \deg(\mathcal{L}_2|_C)$.*

(iii) *If $\mathcal{L}_1$ is ample and $C \subset X$ is a curve then $(\mathcal{L}_1.C) := (\mathcal{L}_1.\mathcal{O}(C)) = \deg(\mathcal{L}_1|_C) > 0$.*

Theorem 1.8 (Riemann–Roch for line bundles on surfaces)**.** *Let X be a non-singular complete surface and let $\mathcal{L} \in \operatorname{Pic}(X)$. Then we can relate the Euler–Poincaré characteristic and the intersection form in the following way:*

$$\chi(X, \mathcal{L}) = \frac{1}{2}(\mathcal{L}.\mathcal{L} \otimes \omega_X^*) + \chi(X, \mathcal{O}_X)$$

Corollary 1.8.1 (Riemann–Roch for line bundles on K3 surfaces)**.** *Let X be a K3 surface and let $\mathcal{L} \in \operatorname{Pic}(X)$. Then*

$$\chi(X, \mathcal{L}) = \frac{1}{2}(\mathcal{L}.\mathcal{L}) + 2.$$

1.2 Lazarsfeld–Mukai bundles

Lazarsfeld–Mukai bundles are locally free sheaves on a K3 surface and arise from line bundles on curves contained in the considered K3 surface. They were introduced by Lazarsfeld in his paper [Laz86] where they were applied to Brill–Noether theory. Mukai used those vector bundles for the classification of prime Fano manifolds of coindex three in his paper [Muk89], see introduction of [Apr12]. We will give the construction and prove some basic results.

Definition 1.3. [Laz86, Ch. 1] Let X be a K3 surface, let $C \subset X$ be a smooth projective irreducible curve of genus g and let $\mathcal{A}$ be a line bundle on C such that $\mathcal{A}$ and $\mathcal{A}^* \otimes \omega_C$ are globally generated. Then the sheaf $\mathcal{F}_{C,\mathcal{A}}$ is defined as the kernel of the evaluation map $H^0(C, \mathcal{A}) \otimes \mathcal{O}_X \to i_*\mathcal{A}$ where $i\colon C \to X$ is the given closed immersion.

Remark. [Laz86, Ch. 1]

(i) Since $\mathcal{A}$ is globally generated, there is a short exact sequence

$$0 \to \mathcal{F}_{C,\mathcal{A}} \to H^0(C, \mathcal{A}) \otimes \mathcal{O}_X \to i_*\mathcal{A} \to 0. \tag{1}$$

(ii) Dualizing the above sequence results in

$$\begin{aligned} 0 \to \mathcal{H}om_X(i_*\mathcal{A}, \mathcal{O}_X) &\to \mathcal{H}om_X(H^0(C,\mathcal{A}) \otimes \mathcal{O}_X, \mathcal{O}_X) \to \mathcal{F}^*_{C,\mathcal{A}} \\ &\to \mathcal{E}xt^1_X(i_*\mathcal{A}, \mathcal{O}_X) \to \mathcal{E}xt^1_X(H^0(C,\mathcal{A}) \otimes \mathcal{O}_X, \mathcal{O}_X). \end{aligned}$$

As $H^0(\mathcal{A}) \otimes \mathcal{O}_X$ is locally free, the last $\mathcal{E}xt$-sheaf is trivial. On the other hand $i_*\mathcal{A}$ is a torsion sheaf and hence $\mathcal{H}om_X(i_*\mathcal{A}, \mathcal{O}_X) = 0$. Finally, we use [Huy15, Ch. 9 Lem. 2.1] to conclude that $\mathcal{E}xt^1_X(i_*\mathcal{A}, \mathcal{O}_X) \cong i_*(\omega_C \otimes \mathcal{A}^*)$. Therefore, the exact sequence becomes the short exact sequence

$$0 \to H^0(C,\mathcal{A})^* \otimes \mathcal{O}_X \to \mathcal{F}^*_{C,\mathcal{A}} \to i_*(\omega_C \otimes \mathcal{A}^*) \to 0. \tag{2}$$

(iii) The sheaf $\mathcal{A}$ has codimension one over $\mathcal{O}_X$. Ergo, the short exact sequence (1) implies that $\mathcal{F}_{C,\mathcal{A}}$ is a vector bundle and so is its dual.

Definition 1.4. Consider a K3 surface X, a smooth curve $C \subset X$, a line bundle $\mathcal{A}$ on C and the vector bundle $\mathcal{F}_{C,\mathcal{A}}$ as in the previous definition. The *Lazarsfeld–Mukai bundle* $\mathcal{E}_{C,\mathcal{A}}$ is defined as $\mathcal{F}^*_{C,\mathcal{A}}$.

Lemma 1.9. *[Laz86, Ch. 1] Let $\mathcal{E}$ be the associated Lazarsfeld–Mukai bundle to a smooth projective curve $C \subset X$ and a line bundle $\mathcal{A}$ on C such that $\mathcal{A}$ and $\omega_C \otimes \mathcal{A}^*$ are globally generated. Then the following assertions hold true:*

(i) *$\mathcal{E}$ is globally generated*

(ii) $H^0(X, \mathcal{F}_{C,\mathcal{A}}) = 0 = H^2(X, \mathcal{E})$

(iii) $H^1(X, \mathcal{F}_{C,\mathcal{A}}) = 0 = H^1(X, \mathcal{E})$

(iv) $h^0(X, \mathcal{E}) = h^0(C, \mathcal{A}) + h^1(C, \mathcal{A})$

(v) $\det(\mathcal{E}) \cong \mathcal{O}_X(C)$

Proof. (i) By construction, there is the short exact sequence

$$0 \to H^0(C,\mathcal{A})^* \otimes \mathcal{O}_X \to \mathcal{E} \to i_*(\omega_C \otimes \mathcal{A}^*) \to 0.$$

Since $\omega_C \otimes \mathcal{A}^*$ is globally generated, the evaluation map $H^0(C, \omega_C \otimes \mathcal{A}^*) \otimes \mathcal{O}_X \twoheadrightarrow i_*(\omega_C \otimes \mathcal{A}^*)$ is surjective. By the vanishing of $H^1(X, \mathcal{O}_X)$, we get the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^0(C,\mathcal{A})^* \otimes \mathcal{O}_X \arrow[r] & \mathcal{E} \arrow[r] & i_*(\omega_C \otimes \mathcal{A}^*) \arrow[r] & 0 \\ & & & H^0(C, \omega_C \otimes \mathcal{A}^*) \otimes \mathcal{O}_X. \arrow[ul, dashed] \arrow[u, two heads] & \end{tikzcd}$$

This yields a surjection $(H^0(C,\mathcal{A})^* \oplus H^0(C, \omega_C \otimes \mathcal{A}^*)) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{E}$, and hence $\mathcal{E}$ is globally generated.

(ii) By Serre duality, it suffices to prove $H^0(X, \mathcal{F}_{C,\mathcal{A}}) = 0$. This follows directly from sequence (1):

$$0 \to H^0(X, \mathcal{F}_{C,\mathcal{A}}) \to H^0(C,\mathcal{A}) \xrightarrow{\sim} H^0(X, i_*\mathcal{A}).$$

(iii) We continue the long exact sequence from (ii):

$$H^0(C,\mathcal{A}) \xrightarrow{\sim} H^0(X, i_*\mathcal{A}) \to H^1(X, \mathcal{F}_{C,\mathcal{A}}) \to H^1(X, H^0(C,\mathcal{A}) \otimes \mathcal{O}_X).$$

Hence, the map $H^0(X, i_*\mathcal{A}) \to H^1(X, \mathcal{F}_{C,\mathcal{A}})$ is the zero map. However, $H^1(X, H^0(X,\mathcal{A}) \otimes \mathcal{O}_X) = 0$ since X is a K3 surface. Ergo, $H^1(X, \mathcal{F}_{C,\mathcal{A}}) = 0$. The second equation follows from Serre duality.

(iv) Since $H^1(X, \mathcal{O}_X) = 0$, the proof of (i) implies $H^0(X, \mathcal{E}) \cong H^0(C, \mathcal{A})^* \oplus H^0(C, \omega_C \otimes \mathcal{A}^*)$. On the other hand Serre duality implies $H^0(C, \omega_C \otimes \mathcal{A}^*) \cong H^1(C, \mathcal{A})^*$.

(v) This proof is due to [Huy15, Ch. 9 Lem. 2.1]. Consider the short exact sequence

$$0 \to H^0(C, \mathcal{A})^* \otimes \mathcal{O}_X \to \mathcal{E} \to i_*(\omega_C \otimes \mathcal{A}^*) \to 0.$$

The sheaf $\omega_C \otimes \mathcal{A}^*$ is locally trivial on C. In other words, it is isomorphic to $\mathcal{O}_C$ on $C \setminus \{x_1, \cdots, x_n\}$. Thus, the pushforward $i_*(\omega_C \otimes \mathcal{A}^*)$ is isomorphic to $i_*\mathcal{O}_C$ on $X \setminus \{x_1, \cdots, x_n\}$. Locally free sheaves on X are determined by their restriction to $X \setminus \{x_1, \cdots, x_x\}$. Therefore, to compute $\det(\mathcal{E})$ we can assume that $i_*(\omega_C \otimes \mathcal{A}^*) \cong \mathcal{O}_C$. Now, use $\det(\mathcal{O}_C) \cong \mathcal{O}_X(C)$ to conclude the desired isomorphism. □

1.3 Introduction to the setting of Kemeny's proof of Voisin's theorem

The setting for section 3, 4 and parts of section 2 will be introduced along with basic observations.

Let X be a K3 surface with Picard group $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathcal{L}]$ where $\mathcal{L}$ is a globally generated ample line bundle on X. Assume that $(\mathcal{L}.\mathcal{L}) = \deg(\mathcal{L}|_C) = 2g - 2$ where $g = 2k$ and $C \in |\mathcal{L}|$. The curve C can be chosen irreducible because otherwise the irreducible components would generate the Picard group of X. Now, two important observations can be made. First, observe that the ideal sheaf $\mathcal{I}_C$ of $C \subset X$ is by definition isomorphic to $\mathcal{L}^*$. Therefore, the adjunction formula yields the isomorphism $\omega_C \cong \mathcal{L}|_C \otimes \omega_X|_C \cong \mathcal{L}|_C$. This immediatly implies that the genus of the curve C is $g = 2k$. Second, note that $H^1(X, \mathcal{L}^q) = 0$ for all q. Indeed, there is the short exact sequence

$$0 \to \mathcal{L}^* \to \mathcal{O}_X \to \mathcal{O}_C \to 0.$$

Taking cohomology yields

$$H^0(X, \mathcal{O}_X) \xrightarrow{\sim} H^0(C, \mathcal{O}_C) \to H^1(X, \mathcal{L}^*) \to H^1(X, \mathcal{O}_X) = 0.$$

Hence, $H^1(X, \mathcal{L}^*) = 0$. Now, we proceed via induction. Tensoring the above sequence by $\mathcal{L}^{-q+1}$ and taking cohomology results in

$$0 = H^0(C, \omega_C^{-q+1}) \to H^1(X, \mathcal{L}^{-q}) \to H^1(X, \mathcal{L}^{-q+1}) = 0.$$

Therefore, $H^1(X, \mathcal{L}^{-q}) = 0$ for all $q \geq 0$. Using Serre duality, this implies $H^1(X, \mathcal{L}^q) = 0$ for all q.

Define $\mathcal{M}_\mathcal{L}$ as the kernel of the evaluation map $H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{L}$, which is surjective since $\mathcal{L}$ is globally generated. Consequently, $\mathcal{M}_\mathcal{L}$ is locally free itself since it is the kernel of a surjection between locally free sheaves. This bundle is the object of interest: in the section on Koszul cohomology it will be explained how the vanishing of $H^1(X, \bigwedge^{k+1} \mathcal{M}_\mathcal{L})$ implies Voisin's theorem.

Throughout section 3 and section 4 a fixed line bundle $\mathcal{A}$ on the curve C will be considered. This bundle gives rise to a Lazarsfeld–Mukai bundle which is needed for the construction of the secant sheaves and later for the secant bundles. The choice of $\mathcal{A}$ is, up to degree and dimension of global sections, arbitrary and we are not interested in examining line bundles with these fixed invariants. The standard notation g^r_d summarizes these two numerical invariants of line bundles on curves:

Definition 1.5. [Har77, IV 5] Let C be a smooth projective curve and let $\mathcal{A}$ be a line bundle on C. We call $\mathcal{A}$ a g^r_d if $\deg(\mathcal{A}) = d$ and $\dim |\mathcal{A}| := h^0(C, \mathcal{A}) - 1 = r$.

Note that C is general by [Laz86]. Therefore, Brill–Noether theory implies that a g^1_d exists on C if and only if $d \geq k + 1$. Let $\mathcal{A}$ be a g^1_{k+1}. Observe that a base point $x \in C$ of $H^0(C, \mathcal{A})$ would yield the g^1_k $\mathcal{A}(-x)$, which cannot exist. Hence, $\mathcal{A}$ is globally generated. Similarly, $\omega_C \otimes \mathcal{A}^*$ is a g^{k-1}_{3k-3} and a g^{k-1}_d exists if and only if $d \geq 3k-3$. Ergo, it has to be globally generated as well. Thus, a Lazarsfeld–Mukai bundle $\mathcal{E}$ can be associated to $\mathcal{A}$. By definition $h^0(C,\mathcal{A}) = 2$ and by the Riemann–Roch formula $h^1(C,\mathcal{A}) = k$. Consequently, lemma 1.9 implies that $h^0(X,\mathcal{E}) = k+2$. Consider the short exact sequence $0 \to \mathcal{F}_{C,\mathcal{A}} \to H^0(C,\mathcal{A}) \otimes \mathcal{O}_X \to i_*\mathcal{A} \to 0$. Since $h^0(C,\mathcal{A}) = 2$, $\mathcal{F}_{C,\mathcal{A}}$, has to be locally free of rank two and so does $\mathcal{E}$.

2 Koszul cohomology, Green's conjecture and syzygies

The goal of this chapter is to indroduce Koszul cohomology, to explain Green's conjecture and to prove the known direction in Green's conjecture. Furthermore, some context for the geometric syzygy conjecture will be given.

2.1 Koszul cohomology

Koszul cohomology seems to be more algebraic than geometric so we will start with the definition right away and explain the connection to geometry afterwards.

Let V be a finite dimensional vector space and let $\operatorname{Sym}(V)$ be its symmetric algebra. Let $M = \bigoplus_{q\in\mathbb{Z}} M_q$ be a graded $\operatorname{Sym}(V)$-module. Consider the product $\pi\colon \bigwedge^{p-1} V^* \otimes V^* \to \bigwedge^{p} V^*$. Dualizing π yields $\bigwedge^{p} V \xrightarrow{f} \bigwedge^{p-1} V \otimes V$, which is the linear extension of

$$f\colon \bigwedge^{p} V \to \bigwedge^{p-1} V \otimes V, v_1 \wedge \cdots \wedge v_p \mapsto \frac{(-1)^p}{p}\sum_{i=1}^{p}(-1)^i v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p \otimes v_i.$$

Viewing V as the degree one part of $\operatorname{Sym}(V)$, we get the multiplication map $V \otimes M_q \xrightarrow{m_q} M_{q+1}$. Define the map $d_{p,q}$ as the composite

$$\bigwedge^{p} V \otimes M_q \xrightarrow{f\otimes id} \bigwedge^{p-1} V \otimes V \otimes M_q \xrightarrow{id\otimes m_q} \bigwedge^{p-1} V \otimes M_{q+1}.$$

Definition 2.1. [Gre84, Ch. 0] Consider the complex

$$\bigwedge^{p+1} V \otimes M_{q-1} \xrightarrow{d_{p+1,q-1}} \bigwedge^{p} V \otimes M_q \xrightarrow{d_{p,q}} \bigwedge^{p-1} V \otimes M_{q+1}.$$

(i) The *Koszul cohomology* $K_{p,q}(M,V)$ is defined as $K_{p,q}(M,V) := \ker(d_{p,q})/\operatorname{im}(d_{p+1,q-1})$.

(ii) Let X be a smooth projective variety, let $\mathcal{L}$ be a line bundle on X, let $\mathcal{F}$ be a coherent sheaf on X and let $V \subset H^0(X,\mathcal{L})$ be a linear subspace. Then the Koszul cohomology $K_{p,q}(X,\mathcal{F},\mathcal{L},V)$ is defined as

$$K_{p,q}(X,\mathcal{F},\mathcal{L},V) := K_{p,q}(\bigoplus_{n\in\mathbb{Z}} H^0(X,\mathcal{F}\otimes\mathcal{L}^n), V).$$

If $V = H^0(X,\mathcal{L})$, it will be dropped from the notation. Similarly, if $\mathcal{F} \cong \mathcal{O}_X$, it will be dropped from the notation as well.

Next, we will link Koszul cohomology to syzygies.

Definition 2.2. [Gre84, 1.b] Assume that M is a graded $\operatorname{Sym}(V)$-module and has a minimal free resolution of the form

$$\cdots \to \bigoplus_{q\geq q_1} \operatorname{Sym}(V)(-q) \otimes W_{1,q} \to \bigoplus_{q\geq q_0} \operatorname{Sym}(V)(-q) \otimes W_{0,q} \to M \to 0$$

where the $W_{p,q}$ are finite dimensional vector spaces. Then the *syzygies of order p and weight q* are defined as $W_{p,q}$.

By [Gre84] such a resolution exists if the dimensions of the vector spaces M_q are finite for all q and if the number of negative q's where M_q not trivial is finite. Syzygies and the Koszul cohomology are related by the following theorem.

Theorem 2.1. *[Gre84, Thm. 1.b.4] There is an isomorphism of vector spaces*

$$K_{p,q}(M,V) \cong W_{p,p+q}.$$

This is one of the main reasons why Koszul cohomology is interesting. Suppose that X is a projective variety and consider a very ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$. Then $\mathcal{L}$ determines a map to the projective space $\mathbb{P}(H^0(X,\mathcal{L})^*)$ by evaluating global sections. In nice cases, such as in Noether's theorem, the image is the projective spectrum of $\bigoplus_{n\in\mathbb{Z}} H^0(X,\mathcal{L}^n)$. Hence, the syzygies of the image as a closed subscheme of some projective space are described by Koszul cohomology.

The next step is to apply Koszul cohomology and theorem 2.1 to the construction of three important exact sequences. Let X be a smooth projective variety and let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of locally free sheaves of finite rank. Consider an open affine subscheme $\mathrm{Spec}(A) \subset X$ on which the sheaves are free and define $M_i := \mathcal{F}_i|_{\mathrm{Spec}(A)}$. The induced map $M_1 \to M_2$ turns $\mathrm{Sym}M_2$ into a $\mathrm{Sym}M_1$-module. The definition of the Koszul differential can be adapted to the map

$$\bigwedge^{j+1} M_1 \otimes \mathrm{Sym}^{i-j-1} M_2 \to \bigwedge^{j} M_1 \otimes \mathrm{Sym}^{i-j} M_2 \to \bigwedge^{j-1} M_1 \otimes \mathrm{Sym}^{i-j+1} M_2.$$

This glues into a map

$$\bigwedge^{j+1} \mathcal{F}_1 \otimes \mathrm{Sym}^{i-j-1} \mathcal{F}_2 \to \bigwedge^{j} \mathcal{F}_1 \otimes \mathrm{Sym}^{i-j} \mathcal{F}_2 \to \bigwedge^{j-1} \mathcal{F}_1 \otimes \mathrm{Sym}^{i-j+1} \mathcal{F}_2$$

for all $0 < j < i$. Composing these maps results in

$$\cdots \to \mathrm{Sym}^{i-2}\mathcal{F}_2 \otimes \bigwedge^{2} \mathcal{F}_1 \to \mathrm{Sym}^{i-1}\mathcal{F}_2 \otimes \mathcal{F}_1 \to \mathrm{Sym}^{i}\mathcal{F}_2 \to \mathrm{Sym}^{i}\mathcal{F}_3 \to 0. \tag{3}$$

Similarly, the map $M_2^* \to M_1^*$ turns $\mathrm{Sym}M_1^*$ into a $\mathrm{Sym}M_2^*$ algebra, which yields

$$\bigwedge^{j+1} M_2^* \otimes \mathrm{Sym}^{i-j-1} M_1^* \to \bigwedge^{j} M_2^* \otimes \mathrm{Sym}^{i-j} M_1^* \to \bigwedge^{j-1} M_2^* \otimes \mathrm{Sym}^{i-j+1} M_1^*.$$

This glues into

$$\bigwedge^{j+1} \mathcal{F}_2^* \otimes \mathrm{Sym}^{i-j-1} \mathcal{F}_1^* \to \bigwedge^{j} \mathcal{F}_2^* \otimes \mathrm{Sym}^{i-j} \mathcal{F}_1^* \to \bigwedge^{j-1} \mathcal{F}_2^* \otimes \mathrm{Sym}^{i-j+1} \mathcal{F}_1^*,$$

which is dual to

$$\bigwedge^{j-1} \mathcal{F}_2 \otimes \mathrm{Sym}^{i-j+1} \mathcal{F}_1 \to \bigwedge^{j} \mathcal{F}_2 \otimes \mathrm{Sym}^{i-j} \mathcal{F}_1 \to \bigwedge^{j+1} \mathcal{F}_2 \otimes \mathrm{Sym}^{i-j-1} \mathcal{F}_1.$$

Therefore, composition yields

$$\cdots \to \bigwedge^{i-2} \mathcal{F}_2 \otimes \mathrm{Sym}^2\mathcal{F}_1 \to \bigwedge^{i-1} \mathcal{F}_2 \otimes \mathcal{F}_1 \to \bigwedge^{i} \mathcal{F}_2 \to \bigwedge^{i} \mathcal{F}_3 \to 0. \tag{4}$$

Corollary 2.1.1. *The sequences*

(3)
$$\cdots \to \mathrm{Sym}^{i-2}\mathcal{F}_2 \otimes \bigwedge^{2} \mathcal{F}_1 \to \mathrm{Sym}^{i-1}\mathcal{F}_2 \otimes \mathcal{F}_1 \to \mathrm{Sym}^{i}\mathcal{F}_2 \to \mathrm{Sym}^{i}\mathcal{F}_3 \to 0$$

(4)
$$\cdots \to \bigwedge^{i-2} \mathcal{F}_2 \otimes \mathrm{Sym}^2\mathcal{F}_1 \to \bigwedge^{i-1} \mathcal{F}_2 \otimes \mathcal{F}_1 \to \bigwedge^{i} \mathcal{F}_2 \to \bigwedge^{i} \mathcal{F}_3 \to 0$$

are exact.

Proof. The ends of both sequences, i.e.

$$\mathrm{Sym}^{i-1}\mathcal{F}_2 \otimes \mathcal{F}_1 \to \mathrm{Sym}^i\mathcal{F}_2 \to \mathrm{Sym}^i\mathcal{F}_3 \to 0$$

and

$$\bigwedge^{i-1}\mathcal{F}_2 \otimes \mathcal{F}_1 \to \bigwedge^{i}\mathcal{F}_2 \to \bigwedge^{i}\mathcal{F}_3 \to 0$$

are exact by [Eis04, Prop. A2.2] so for the first sequence it suffices to show that

$$\bigwedge^{j+1} M_1 \otimes \mathrm{Sym}^{i-j-1}M_2 \xrightarrow{f} \bigwedge^{j} M_1 \otimes \mathrm{Sym}^{i-j}M_2 \xrightarrow{g} \bigwedge^{j-1} M_1 \otimes \mathrm{Sym}^{i-j+1}M_2$$

is exact for all $0 < j < i$. This is equivalent to the exactness of

$$0 \to \mathrm{im}(f) \to \bigwedge^{j} M_1 \otimes \mathrm{Sym}^{i-j}M_2 \to \mathrm{im}(g) \to 0.$$

Note that $\mathrm{im}(f)$ and $\mathrm{im}(g)$ are free since the Koszul differential sends free generators to free generators. Hence, the above sequence is exact if and only if

$$0 \to \mathrm{im}(f) \otimes \kappa(\mathfrak{p}) \to \bigwedge^{j} M_1 \otimes \mathrm{Sym}^{i-j}M_2 \otimes \kappa(\mathfrak{p}) \to \mathrm{im}(g) \otimes \kappa(\mathfrak{p}) \to 0$$

is exact for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Define $V_i := M_i \otimes \kappa(\mathfrak{p})$. Then exactness of the above sequence is equivalent to the exactness of

$$\bigwedge^{j+1} V_1 \otimes \mathrm{Sym}^{i-j-1}V_2 \to \bigwedge^{j} V_1 \otimes \mathrm{Sym}^{i-j}V_2 \to \bigwedge^{j-1} V_1 \otimes \mathrm{Sym}^{i-j+1}V_2,$$

which is equivalent to $K_{i-j,j}(\mathrm{Sym}V_2, V_1) = 0$. After choosing a suitable basis for V_1 and V_2 the injection $\mathrm{Sym}V_1 \hookrightarrow \mathrm{Sym}V_2$ can be identified with the inclusion $\mathbb{C}[x_1, \dots, x_m] \hookrightarrow \mathbb{C}[x_1, \dots, x_n]$ for some integers $m \le n$. The minimal free resolution of $\mathbb{C}[x_1, \dots, x_n]$ as a $\mathbb{C}[x_1, \dots, x_m]$-module has no syzygies of order p and weight q $W_{p,q}$ for $p \neq q$ and therefore $K_{j,i-j}(\mathrm{Sym}V_2, V_1) \cong W_{j,i} = 0$ for $0 < j < i$ by the theorem above. Considering a very similar argument for the second assertion, it suffices to show that the sequence

$$\bigwedge^{j+1} V_2^* \otimes \mathrm{Sym}^{i-j-1}V_1^* \to \bigwedge^{j} V_2^* \otimes \mathrm{Sym}^{i-j}V_1^* \to \bigwedge^{j-1} V_2^* \otimes \mathrm{Sym}^{i-j+1}V_1^*$$

is exact. This is equivalent to $K_{j,i-j}(\mathrm{Sym}V_1^*, V_2^*) = 0$. Choose a suitable basis for V_1^* and V_2^* to identify $\mathrm{Sym}V_2^* \to \mathrm{Sym}V_1^*$ with the projection $\mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow \mathbb{C}[x_1, \dots, x_n]/(x_1, \dots, x_m)$ for some integers $m \le n$. Expand this to a minimal free resolution of $\mathbb{C}[x_1, \dots, x_n]/(x_1, \dots, x_m)$ and note that the syzygies of order p and weight q $W'_{p,q}$ are zero for $p \neq q$. Hence, $K_{j-i,i}(\mathrm{Sym}V_1^*, V_2^*) \cong W'_{j-i,j} = 0$ for all $0 < j < i$ by the theorem above. $\square$

Now we dualize $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, apply Sym^i and dualize again to obtain

$$0 \to \mathrm{Sym}^i\mathcal{F}_1 \to \mathrm{Sym}^i\mathcal{F}_2 \to \mathrm{Sym}^{i-1}\mathcal{F}_2 \otimes \mathcal{F}_3 \to \mathrm{Sym}^{i-2}\mathcal{F}_2 \otimes \bigwedge^2 \mathcal{F}_3 \to \cdots \tag{5}$$

This is still exact since all involved sheaves are locally free.

In his paper [Gre84] Green established multiple methods for calculating Koszul cohomology. Two of them will be used in the next sections and are therefore stated here.

Theorem 2.2 (Koszul duality)**.** *[Gre84, Thm. 2.c.6] Let X be a smooth projective variety of dimension n, let $\mathcal{L} \in \operatorname{Pic}(X)$ be a line bundle and let $\mathcal{F}$ be a vector bundle on X. Take a base-point free linear subspace $W \subset H^0(X, \mathcal{L})$ of dimension $\dim W = r + 1$ and suppose that*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{q-i}) = 0 = H^i(X, \mathcal{F} \otimes \mathcal{L}^{q-i-1})$$

for all $1 \leq i \leq n-1$. Then there is an isomorphism

$$K_{p,q}(X, \mathcal{F}, \mathcal{L}, W)^* \cong K_{r-n-p,n+1-q}(X, \mathcal{F}^* \otimes \omega_X, \mathcal{L}, W).$$

Corollary 2.2.1 (Special case of the duality theorem for curves)**.** *Let C be a smooth projective curve of genus g. Then there is an isomorphism*

$$K_{p,q}(C, \omega_C)^* \cong K_{g-2-p,3-q}(C, \omega_C).$$

Corollary 2.2.2 (Special case of the duality theorem for K3 surfaces)**.** *Let X be a K3 surface with Picard group $\operatorname{Pic}(X) \cong \mathbb{Z}[\mathcal{L}]$ where $\mathcal{L}$ is an ample globally generated line bundle with $(\mathcal{L}.\mathcal{L}) = 4k-2$. Then there is an isomorphism*

$$K_{p,q}(X, \mathcal{L})^* \cong K_{2k-2-p,3-q}(X, \mathcal{L}).$$

The next theorem is sometimes called the Lefschetz theorem or the hyperplane restriction theorem.

Theorem 2.3. *[Gre84, Thm. 3.b.7] Let X be a smooth projective variety, let $\mathcal{L}$ be a line bundle on X and let $Y \in |\mathcal{L}|$ be a connected hypersurface. Furthermore, assume that the first cohomology group $H^1(X, \mathcal{L}^q)$ vanishes for all $q \geq 0$. Then there is an isomorphism*

$$K_{p,q}(X, \mathcal{L}) \cong K_{p,q}(Y, \mathcal{L}|_Y)$$

for all p, q.

Before we finish the section on Koszul cohomology, we will connect Koszul cohomology to sheaf cohomology. This discussion follows [Laz89, Prop. 1.3.3]. Let X be a projective variety with a globally generated line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ and a vector bundle $\mathcal{F}$ on X. Define $\mathcal{M}_\mathcal{L}$ as the kernel of the evaluation map $H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$, which is surjective since $\mathcal{L}$ is globally generated. The short exact sequence

$$0 \to \mathcal{M}_\mathcal{L} \to H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L} \to 0$$

yields the short exact sequence

$$0 \to \bigwedge^i \mathcal{M}_\mathcal{L} \to \bigwedge^i H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \to \bigwedge^{i-1} \mathcal{M}_\mathcal{L} \otimes \mathcal{L} \to 0 \tag{6}$$

by taking the i-th exterior power. Indeed, by [Har77, II Ex. 5.16] there is a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{i+1} = \bigwedge^i H^0(X, \mathcal{L}) \otimes \mathcal{O}_X$$

such that for all $0 \leq p \leq i$:

$$\mathcal{F}_{p+1}/\mathcal{F}_p \cong \bigwedge^{i-p} \mathcal{M}_\mathcal{L} \otimes \bigwedge^p \mathcal{L}.$$

Note that $\bigwedge^p \mathcal{L} = 0$ for $p > 1$. Thus, for $p > 1$ the inclusion $\mathcal{F}_p \subset \mathcal{F}_{p+1}$ is an equality. Consequently, the filtration can be shortened to

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \bigwedge^i H^0(X, \mathcal{L}) \otimes \mathcal{O}_X.$$

Using the required isomorphisms, one obtains

$$\mathcal{F}_1 \cong \bigwedge^i \mathcal{M}_\mathcal{L}$$

and hence the cokernel of the inclusion $\bigwedge^i \mathcal{M}_\mathcal{L} \to \bigwedge^i H^0(X,\mathcal{L}) \otimes \mathcal{O}_X$ is naturally isomorphic to $\bigwedge^{i-1} \mathcal{M}_\mathcal{L} \otimes \mathcal{L}$. Therefore, the sequence

$$0 \to \bigwedge^i \mathcal{M}_\mathcal{L} \to \bigwedge^i H^0(X,\mathcal{L}) \otimes \mathcal{O}_X \to \bigwedge^{i-1} \mathcal{M}_\mathcal{L} \otimes \mathcal{L} \to 0$$

is exact. Tensor the sequence (6) with powers of $\mathcal{L}$ and with $\mathcal{F}$ to obtain the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \bigwedge^{p+1} \mathcal{M}_\mathcal{L} \otimes \mathcal{F} \otimes \mathcal{L}^{q-1} \arrow[d] & & & \\
 & \bigwedge^{p+1} H^0(X,\mathcal{L}) \otimes \mathcal{F} \otimes \mathcal{L}^{q-1} \arrow[d] \arrow[dr, dashed] & & 0 \arrow[d] & \\
0 \arrow[r] & \bigwedge^{p} \mathcal{M}_\mathcal{L} \otimes \mathcal{F} \otimes \mathcal{L}^{q} \arrow[r] \arrow[d] & \bigwedge^{p} H^0(X,\mathcal{L}) \otimes \mathcal{F} \otimes \mathcal{L}^{q} \arrow[r] \arrow[dr, dashed] & \bigwedge^{p-1} \mathcal{M}_\mathcal{L} \otimes \mathcal{F} \otimes \mathcal{L}^{q+1} \arrow[r] \arrow[d] & 0 \\
 & 0 & & \bigwedge^{p-1} H^0(X,\mathcal{L}) \otimes \mathcal{F} \otimes \mathcal{L}^{q+1} \arrow[d] & \\
 & & & \bigwedge^{p-2} \mathcal{M}_\mathcal{L} \otimes \mathcal{F} \otimes \mathcal{L}^{q+2} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

with exact rows and columns. Taking global sections of the dashed morphisms induces

$$\bigwedge^{p+1} H^0(\mathcal{L}) \otimes H^0(\mathcal{F} \otimes \mathcal{L}^{q-1}) \dashrightarrow \bigwedge^{p} H^0(\mathcal{L}) \otimes H^0(\mathcal{F} \otimes \mathcal{L}^{q}) \dashrightarrow \bigwedge^{p-1} H^0(\mathcal{L}) \otimes H^0(\mathcal{F} \otimes \mathcal{L}^{q+1}),$$

which is in fact the Koszul complex. Observe that $K_{p,q}(X,\mathcal{F},\mathcal{L})$ is naturally isomorphic to $H^1(X, \bigwedge^{p+1} \mathcal{M}_\mathcal{L} \otimes \mathcal{F} \otimes \mathcal{L}^{q-1})$ provided $H^1(X, \mathcal{F} \otimes \mathcal{L}^{q-1}) = 0$. Indeed, taking global sections results in

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & H^0(\bigwedge^{p+1} \mathcal{M}_{\mathcal{L}} \otimes \mathcal{F} \otimes \mathcal{L}^{q-1}) \arrow[d] & & \\
 & \bigwedge^{p+1} H^0(\mathcal{L}) \otimes H^0(\mathcal{F} \otimes \mathcal{L}^{q-1}) \arrow[d] \arrow[dr, dashed] & & 0 \arrow[d] \\
0 \arrow[r] & H^0(\bigwedge^{p} \mathcal{M}_{\mathcal{L}} \otimes \mathcal{F} \otimes \mathcal{L}^{q}) \arrow[r] \arrow[d] & \bigwedge^{p} H^0(\mathcal{L}) \otimes H^0(\mathcal{F} \otimes \mathcal{L}^{q}) \arrow[r] \arrow[dr, dashed] & H^0(\bigwedge^{p-1} \mathcal{M}_{\mathcal{L}} \otimes \mathcal{F} \otimes \mathcal{L}^{q+1}) \arrow[d] \\
 & H^1(\bigwedge^{p+1} \mathcal{M}_{\mathcal{L}} \otimes \mathcal{F} \otimes \mathcal{L}^{q-1}) \arrow[d] & & \bigwedge^{p-1} H^0(\mathcal{L}) \otimes H^0(\mathcal{F} \otimes \mathcal{L}^{q+1}) \arrow[d] \\
 & 0 & & H^0(\bigwedge^{p-2} \mathcal{M}_{\mathcal{L}} \otimes \mathcal{F} \otimes \mathcal{L}^{q+2}).
\end{tikzcd}$$

A diagram chase yields natural isomorphisms

$$K_{p,q}(X, \mathcal{F}, \mathcal{L}) \cong H^0(X, \bigwedge^{p} \mathcal{M}_{\mathcal{L}} \otimes \mathcal{F} \otimes \mathcal{L}^{q})/\mathrm{im}(f_2) \cong H^1(X, \bigwedge^{p+1} \mathcal{M}_{\mathcal{L}} \otimes \mathcal{F} \otimes \mathcal{L}^{q-1}).$$

This is called the kernel bundle description of Koszul cohomology. Observe that if X is a K3 surface with Picard group $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathcal{L}]$ where $\mathcal{L}$ is ample, globally generated and $(\mathcal{L}.\mathcal{L}) = 4k - 2$, the cohomology $H^1(X, \mathcal{O}_X \otimes \mathcal{L}^{q-1})$ vanishes for all q. Hence, the kernel bundle description gives an isomorphism

$$K_{p,q}(X, \mathcal{L}) \cong H^1(X, \bigwedge^{p+1} \mathcal{M}_{\mathcal{L}} \otimes \mathcal{L}^{q-1}).$$

2.2 Green's conjecture

In this section it will be argued how Voisin's theorem implies a partial result of Green's conjecture. Furthermore, it will be explained why Noether's theorem and Petri's theorem are special cases of Green's conjecture. This is the main result of this thesis:

Theorem 2.4 (Voisin)**.** *[Voi02, Thm. 1] Let X be a K3 surface with Picard group $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathcal{L}]$ where $\mathcal{L}$ is an ample globally generated line bundle with $(\mathcal{L}.\mathcal{L}) = 2g - 2$ and $g = 2k$. Then $K_{k,1}(X, \mathcal{L}) = 0$.*

Suppose that Voisin's theorem has already been proven. One can show that $K_{p+1,1}(X, \mathcal{L})$ injects into $H^0(X, \mathcal{L}) \otimes K_{p,1}(X, \mathcal{L})$ for all $p \geq 1$, see [AN09, Cor. 2.10]. Therefore, Voisin's theorem implies $K_{p,1}(X, \mathcal{L}) = 0$ for all $p \geq k$. Now, consider $C \in |\mathcal{L}|$ and use theorem 2.3. This yields the vanishing of $K_{p,1}(C, \omega_C)$ for all $p \geq k$, which is Koszul dual to $K_{2k-p-2,2}(C, \omega_C)^*$. Thus, one obtains $K_{p',2}(C, \omega_C) = 0$ for all $p' \leq k - 2$. Finally, suppose that $\mathrm{Cliff}(C) = k - 1$. Then Voisin's theorem implies the direction "$\Rightarrow$" in Green's conjecture. Recall that the direction "$\Leftarrow$" was proven by Green and Lazarsfeld.

Lemma 2.5. *Let C be as in the discussion above. Then $\mathrm{Cliff}(C) = k - 1$.*

Proof. Note that C is Brill–Noether general by [Laz86]. Thus, [KK89, Thm. 1] implies that $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 2$ where gon denotes the gonality. Hence, it is sufficient to show $\mathrm{gon}(C) = k+1$. Let $f\colon C \to \mathbb{P}^1$ be a finite morphism of minimal degree. Then $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ is a $g^1_{\mathrm{gon}(C)}$. The curve C admits a g^1_d if and only if $2d \geq k+1$ due to Brill–Noether theory. Hence, $\mathrm{gon}(C) \geq k+1$. On the other hand, there is a g^1_{k+1} by the same theorems. Let $\mathcal{L}$ be this line bundle. In section 1.3 it was argued that in this case $H^0(C, \mathcal{L})$ is base point free. Therefore, $\mathcal{L}$ determines a morphism to $\mathbb{P}^1$ of degree $k+1$ and hence $\mathrm{gon}(C) \leq k+1$. $\square$

The next goal is to use Koszul cohomology to rephrase the classical results of Max Noether and Petri. In addition, we will see that these results can be viewed as special cases of Green's conjecture, which is another motivation for the search of a proof.

Theorem 2.6 (Max Noether)**.** *[Ara+85, III §2] Let C be a smooth projective curve which is not hyperelliptic. Then the canonical ring of C is generated in degree one.*

Recall that the canonical ring of a curve C is defined as:

$$R(C) := \bigoplus_{n\in\mathbb{N}} H^0(C, \omega_C^n).$$

Hence, Noether's theorem states that for every $n \geq 1$ the natural map

$$\mathrm{Sym}^n H^0(C, \omega_C) \twoheadrightarrow H^0(C, \omega_C^n)$$

is surjective. Writing down the definition of the Koszul cohomology of bidegree $(0, q)$, one sees that this map is surjective if $K_{0,q}(C, \omega_C) = 0$ for all $1 \leq q \leq n$. On the other hand $K_{0,n}(C, \omega_C) = 0$ if the map is surjective. Thus, Noether's theorem is equivalent to the vanishing of $K_{0,q}(C, \omega_C)$ for all $q \geq 1$. Observe that $K_{0,1}(C, \omega_C)$ is always trivial. For $q \geq 3$ use the duality theorem to obtain an isomorphism $K_{0,q}(C, \omega_C) \cong K_{g-2,3-q}(C, \omega_C)^*$. Hence, $K_{0,q}(C, \omega_C)$ vanishes for all $q \geq 3$ and consequently Noether's theorem can be reformulated in the following way:

Theorem 2.7. *Let C be a smooth projective curve which is not hyperelliptic. Then $K_{0,2}(C, \omega_C) = 0$.*

Remark. Observe that this is just the direction "$\Rightarrow$" in Green's conjecture in the case of $l = 0$.

Another example is provided by Petri's theorem. If C is not hyperelliptic, the canonical bundle ω_C is very ample [Har77, IV Prop. 5.2]. Hence, it determines a closed embedding $\varphi_{\omega_C}\colon C \to \mathbb{P}^{g-1}$. This embedding factors over the projective spectrum of the canonical ring $R(C)$ due to Noether's theorem:

$$\begin{tikzcd} C \arrow[rr, "\varphi_{\omega_C}"] \arrow[dr] & & \mathbb{P}^{g-1} \\ & \mathrm{Proj}(R(C)) \arrow[ur, "i"] & \end{tikzcd}$$

The morphism $C \to \mathrm{Proj}(R(C))$ is an isomorphism by [21, Tag 01Q1]. It only seems natural to ask the question of how the ideal sheaf $\mathcal{I}$ of the closed immersion i looks like.

Theorem 2.8 (Petri)**.** *[Ara+85, III §3] Let C be a canonical curve of genus $g \geq 4$. Then the ideal sheaf $\mathcal{I}$ is generated by elements of degree two unless C is trigonal or a plane quintic.*

Let $\mathfrak{a}$ be the kernel of the surjection $\mathrm{Sym}(H^0(C, \omega_C)) \twoheadrightarrow R(C)$ and consider the minimal free resolution of $R(C)$

$$\cdots \to \bigoplus_{q\geq q_1} \mathrm{Sym}(H^0(C,\omega_C))(-q) \otimes U_{1,q} \xrightarrow{f} \bigoplus_{q\geq q_0} \mathrm{Sym}(H^0(C,\omega_C))(-q) \otimes U_{0,q} \to R(C) \to 0.$$

From the minimality of this resolution it follows that $U_{0,0} \cong \mathbb{C}$ and $U_{0,q} = 0$ for all $q \neq 0$. Hence, the image of f is nothing but the ideal $\mathfrak{a}$. Now, consider the minimal free resolution of $\mathfrak{a}$:

$$\cdots \to \bigoplus_{q \geq q_1} \mathrm{Sym}(H^0(C, \omega_C))(-q) \otimes W_{1,q} \to \bigoplus_{q \geq q_0} \mathrm{Sym}(H^0(C, \omega_C))(-q) \otimes W_{0,q} \to \mathfrak{a} \to 0$$

Since both resolutions are minimal, they should coincide after a shift: $U_{p,q} \cong W_{p-1,q}$ for all p and q. Thus, the ideal $\mathfrak{a}$ is generated by elements of degree two if and only if $W_{0,q} \cong U_{1,q} = 0$ for $q = 1$ and for all $q \geq 3$. The second vector space is isomorphic to $K_{1,q-1}(C, \omega_C)$ by theorem 2.1. Hence, Petri's theorem translates to $K_{1,q}(C, \omega_C) = 0$ for $q = 0$ and $q \geq 2$. The vanishing for $q = 0$ is obvious. For $q \geq 3$ one could argue similarly as in the reformulation of Noether's theorem. We conclude that Petri's theorem can be rephrased as:

Theorem 2.9. *Let C be a canonical curve of genus $g \geq 4$. Then $K_{1,2}(C, \omega_C) = 0$ unless C is trigonal or a plane quintic.*

Remark. One can show that curves of genus $g \leq 2$ are hyperelliptic and curves of genus $g = 3$ are hyperelliptic or trigonal. Furthermore, it is known that curves with Clifford index equal to one are exactly those curves that are trigonal or plane quintics, see [Eis05, 9A]. Thus, the condition $g \geq 4$ can be replaced with $\mathrm{Cliff}(C) > 1$ and hence Petri's theorem corresponds to the direction "$\Rightarrow$" in Green's conjecture in the case of $l = 1$.

2.3 Proof of the known direction in Green's conjecture

As already mentioned the direction "$\Leftarrow$" in Green's conjecture was proven in a joint appendix with Lazarsfeld in Greens's paper [Gre84]. Their proof used the following theorem:

Theorem 2.10. *[Gre84, Thm. appendix] Let X be a smooth projective variety and let $\mathcal{L}$, $\mathcal{M}_1$, $\mathcal{M}_2$ be line bundles on X such that $\mathcal{L} \cong \mathcal{M}_1 \otimes \mathcal{M}_2$. Assume that $h^0(X, \mathcal{M}_1)$ and $h^0(X, \mathcal{M}_2)$ are both larger than one and define $r_i := h^0(X, \mathcal{M}_i) - 1$. Then $K_{r_1+r_2-1,1}(X, \mathcal{L}) \neq 0$.*

Proof. First, choose two global sections $d_i \in H^0(X, \mathcal{M}_i)$. The injections $\mathcal{M}_i^* \cong \mathcal{O}_X(-Z(d_i)) \hookrightarrow \mathcal{O}_X$ give rise to surjections $H^0(X, \mathcal{L})^* \twoheadrightarrow H^0(X, \mathcal{L} \otimes \mathcal{M}_i)^*$. Define

$$\overline{D_1} := \ker(H^0(X, \mathcal{L})^* \twoheadrightarrow H^0(X, \mathcal{L} \otimes \mathcal{M}_2^* \cong \mathcal{M}_1)^*) \subset \mathbb{C}^{r+1},$$

$$\overline{D_2} := \ker(H^0(X, \mathcal{L})^* \twoheadrightarrow H^0(X, \mathcal{L} \otimes \mathcal{M}_1^* \cong \mathcal{M}_2)^*) \subset \mathbb{C}^{r+1}$$

and choose a basis $s_0, \dots, s_r$ for $H^0(X, \mathcal{L})$ with dual basis $e_0, \dots, e_r$ such that

(i) $e_1, \dots, e_{r-r_1}$ is a basis for $\overline{D_1}$,

(ii) $e_{r_2+1}, \dots, e_r$ is a basis for $\overline{D_2}$ and

(iii) $e_{r_2+1}, \dots, e_{r-r_1}$ is a basis for $\overline{D_1} \cap \overline{D_2}$.

Consequently, one obtains

(i) $H^0(X, \mathcal{M}_1 \cong \mathcal{L} \otimes \mathcal{M}_2^*) = \langle s_0, s_{r-r_1+1}, \dots, s_r \rangle$

(ii) $H^0(X, \mathcal{M}_2 \cong \mathcal{L} \otimes \mathcal{M}_1^*) = \langle s_0, \dots, s_{r_2} \rangle$.

Now, define

$$\iota := \sum_{i=1}^{r-r_1} e_i \otimes s_i,$$
$$s := \sum_{i=0}^{r} e_i \otimes s_i,$$
$$\alpha := \iota \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} = \sum_{i=1}^{r_2} e_i \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} \otimes s_i$$

and consider

$$\begin{aligned} s \wedge \alpha &= \sum_{j=0}^{r} \sum_{i=1}^{r_2} e_j \wedge e_i \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} \otimes (s_j \otimes s_i) \\ &= \sum_{i=1}^{r_2} e_0 \wedge e_i \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} \otimes (s_0 \otimes s_i) \\ &+ \sum_{j=1}^{r_2} \sum_{i=1}^{r_2} e_j \wedge e_i \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} \otimes (s_j \otimes s_i) \\ &+ \sum_{j=r-r_1+1}^{r} \sum_{i=1}^{r_2} e_j \wedge e_i \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} \otimes (s_j \otimes s_i). \end{aligned}$$

Note that in the second last sum all summands occur twice but with different signs. Ergo,

$$\begin{aligned} s \wedge \alpha &= \sum_{i=1}^{r_2} e_0 \wedge e_i \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} \otimes (s_0 \otimes s_i) \\ &+ \sum_{j=r-r_1+1}^{r} \sum_{i=1}^{r_2} e_j \wedge e_i \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} \otimes (s_j \otimes s_i). \end{aligned}$$

The tensor products $s_j \otimes s_i$ that occur in this sum are contained in

$$H^0(\mathcal{L} \otimes \mathcal{M}_2^* \otimes \mathcal{L} \otimes \mathcal{M}_1^*) \cong H^0(\mathcal{L})$$

due to the range of the indices. Thus,

$$s \wedge \alpha \in \bigwedge^{r-r_1-r_2+2} H^0(X, \mathcal{L})^* \otimes H^0(X, \mathcal{L}).$$

Recall that there are isomorphisms

$$\bigwedge^{m} H^0(X, \mathcal{L})^* \cong \bigwedge^{r+1-m} H^0(X, \mathcal{L})$$

for all $0 \leq m \leq r+1$, using the given basis. Explicit calculations will show that the composite

$$\begin{aligned} &\bigwedge^{m} H^0(X, \mathcal{L})^* \otimes H^0(X, \mathcal{L}) \xrightarrow{\sim} \bigwedge^{r+1-m} H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L}) \\ &\xrightarrow{d} \bigwedge^{r-m} H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L}^2) \xrightarrow{\sim} \bigwedge^{m+1} H^0(X, \mathcal{L})^* \otimes H^0(X, \mathcal{L}^2) \end{aligned}$$

is, up to scaling, the map $s \wedge -$ for all $0 \leq m \leq r+1$. Here, d is the Koszul differential. Hence, the fact that $s \wedge (s \wedge \alpha) = 0$ implies that $s \wedge \alpha$ yields an element in $K_{r_1+r_2-1,1}(X, \mathcal{L})$. If $s \wedge \alpha \in K_{r_1+r_2-1,1}(X, \mathcal{L})$ is not trivial, the assertion follows. Suppose in contrary that $s \wedge \alpha = s \wedge \beta$ for some $\beta \in \bigwedge^{r-r_1-r_2+1} H^0(X, \mathcal{L})^*$. Consequently,

$$s \wedge e_j \wedge \beta = \pm s \wedge \alpha \wedge e_j = 0$$

for all $r_2 + 1 \leq j \leq r - r_1$. On the other hand

$$s \wedge e_j \wedge \beta = \sum_{i=0}^{r} e_i \wedge e_j \wedge \beta \otimes s_i$$

where the s_i are all linearly independent. Hence, $e_i \wedge e_j \wedge \beta = 0$ for all i and for all $r_2 + 1 \leq j \leq r - r_1$. The element $e_j \wedge \beta$ lies in the $(r - r_1 - r_2 + 2)$-nd exterior power of $H^0(X, \mathcal{L})^*$ where $r - r_1 - r_2 + 2 < r + 1$. Therefore, $e_j \wedge \beta = 0$ for all $r_2 + 1 \leq j \leq r$. The element β can be written as a linear combination of elementary wedge products where each elementary wedge product consists of a, up to permutation, unique wedge of $(r - r_1 - r_2 + 1)$-many basis vectors. Since $e_j \wedge \beta = 0$ for all $r_2 + 1 \leq j \leq r - r_1$, e_j has to participate in each product for all $r_2 + 1 \leq j \leq r - r_1$. Thus, β can be written as $\beta = c \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1}$. Now, write c as a linear combination of the basis $e_0, \ldots, e_r$ and plug in all definitions in the equation $s \wedge \alpha = s \wedge c \wedge e_{r_2} \wedge \cdots \wedge e_{r-r_1}$. After a very lengthy calculation this yields $s \wedge \alpha = 0$, which is absurd. □

Corollary 2.10.1. *[Gre84] Let C be a smooth projective curve over a field of characteristic zero. Suppose that there is an $l \geq 0$ such that $K_{p,2}(C, \omega_C)$ vanishes for all $p \leq l$. Then* $\mathrm{Cliff}(C) > l$.

Proof. We will prove the contraposition. Suppose that $\mathrm{Cliff}(C) \leq l$ and let $\mathcal{L}$ be one of the line bundles with the smallest Clifford index that satisfy $h^0(\mathcal{L}) = r + 1 \geq 2$ and $h^1(\mathcal{L}) \geq 2$. Let d be the degree of $\mathcal{L}$. Thus, by assumption $d - 2r \leq l$. Define $\mathcal{M} := \omega_C \otimes \mathcal{L}^*$. Then Serre duality implies $h^0(\mathcal{M}) = h^1(\mathcal{L}) \geq 2$ and $h^1(\mathcal{M}) = h^0(\mathcal{L}) \geq 2$. Note that by the Riemann–Roch formula $h^1(\mathcal{L}) = r - d + g$. Hence, the above theorem states $K_{g+2r-d-2,1}(C, \omega_C) \neq 0$. Koszul duality yields $K_{g+2r-d-2,1}(C, \omega_C) \cong K_{d-2r,2}(C, \omega_C)^*$ so $K_{d-2r,2}(C, \omega_C) \neq 0$. □

2.4 Geometric syzygy conjecture

Recall that Voisin's theorem predicts the vanishing of $K_{l,1}(C, \omega_C)$ for all $l \geq k$. The geometric syzygy conjecture on the other hand is about the Koszul cohomology group $K_{k-1,1}(C, \omega_C)$. Since we do not expect this group to vanish, we are interested in finding a basis with certain properties. The considered property is the rank of a linear syzygy.

Definition 2.3. [Kem19] Let X be a projective variety and let $\mathcal{L}$ be a very ample line bundle on X. Consider a linear syzygy $\alpha \in K_{p,1}(X, \mathcal{L})$. The *rank* of α is defined as the dimension of the minimal linear subspace $V \subset H^0(X, \mathcal{L})$ such that α is an element in $K_{p,1}(X, \mathcal{L}, V)$.

Assume $\alpha \in K_{p,1}(X, \mathcal{L})$ has rank k. By definition, there is a syzygy $\alpha' \in K_{p,1}(X, \mathcal{L}, V)$ that it mapped to α under $K_{p,1}(X, \mathcal{L}, V) \to K_{p,1}(X, \mathcal{L})$ where $V \subset H^0(X, \mathcal{L})$ is a k-dimensional subspace. The map $K_{p,1}(X, \mathcal{L}, V) \to K_{p,1}(X, \mathcal{L})$ is induced by

$$\begin{tikzcd}
\bigwedge^{p+1} V \arrow[r] \arrow[d, hook] & \bigwedge^{p} V \otimes H^0(X, \mathcal{L}) \arrow[r, "d_{p,1}"] \arrow[d, hook] & \bigwedge^{p-1} V \otimes H^0(X, \mathcal{L}^2) \arrow[d, hook] \\
\bigwedge^{p+1} H^0(X, \mathcal{L}) \arrow[r] & \bigwedge^{p} H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L}) \arrow[r] & \bigwedge^{p-1} H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L}^2).
\end{tikzcd}$$

If the rank k of α and therefore the dimension of V is smaller than p, then $K_{p,1}(X,\mathcal{L},V)=0$ and thus α' and α have to be zero. If one assumes $k=p$, the Koszul cohomology group $K_{p,1}(X,\mathcal{L},V)$ equals the kernel of the Koszul differential $d_{p,1}$. Choose a basis $v_1,\dots,v_p$ of V which induces the only basis element $v_1\wedge\cdots\wedge v_p$ of $\bigwedge^p V$. Because $\bigwedge^p V$ is one-dimensional, every element in $\bigwedge^p V\otimes H^0(X,\mathcal{L})$ can be expressed as $v_1\wedge\cdots\wedge v_p\otimes v$ with $v\in H^0(X,\mathcal{L})$ being an arbitrary element. The Koszul differential sends this element to

$$\frac{(-1)^p}{p}\sum_{i=1}^{p}(-1)^i v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge v_p\otimes(v_i\otimes v)$$

where $v_i\otimes v$ is viewed as an element in $H^0(X,\mathcal{L}^2)$. Since the elements $v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge v_p$ are linearly independent, the only way the expression above can become zero is that $v_i\otimes v$ is zero for all $1\le i\le p$. Take an open affine subscheme $\mathrm{Spec}A\subset X$ on which the line bundle $\mathcal{L}$ becomes trivial. Let M be the corresponding A module and denote by $\varphi\colon M\overset{\cong}{\to}A$ the trivialization. Then $v_i\otimes v$ beeing zero implies

$$\varphi(v_i|_{\mathrm{Spec}(A)})\cdot\varphi(v|_{\mathrm{Spec}(A)})=0.$$

If X is integral, this implies that $v_i|_{\mathrm{Spec}(A)}=0$ or $v|_{\mathrm{Spec}(A)}=0$. The zero locus is closed so $v_i=0$ or $v=0$ and hence $d_{p,1}$ is injective. Ergo, the linear syzygy α has rank larger or equal to $p+1$. On the other hand syzygies of rank $p+1$ or $p+2$ have a geometric meaning: If there is a syzygy of rank $p+1$, then X lies on a rational normal scroll and syzygies of rank $p+2$ come from linear sections of Grassmannians, see [Bot07] and [AN07]. Consequently, syzygies of rank $p+1$ or $p+2$ are called geometric. This explains the word "geometric" in the following conjecture:

Conjecture 2.11 (Geometric syzygy conjecture in even genus)**.** *[Kem20, Cor. 0.4] Let C be a general curve of genus $g=2k$. Then $K_{k-1,1}(C,\omega_C)$ is generated by geometric syzygies of rank k which come from $K_{k-1,1}(C,\omega_C,H^0(C,\omega_C\otimes\mathcal{O}_C(-A)))$ with $A\in W^1_{k+1}(C)$.*

Recall that $W^r_d(C)$ is defined as the set of divisors D on C with $\deg D=d$ and $\dim|D|\ge r$, see [Ara+85, IV]. The result we will actually prove is:

Theorem 2.12. *[Kem20, Thm. 0.2] Let X be a K3 surface with Picard group $\mathrm{Pic}(X)\cong\mathbb{Z}[\mathcal{L}]$ where $\mathcal{L}$ is an ample globally generated line bundle with $(\mathcal{L}.\mathcal{L})=4k-2$. Let $\mathcal{E}$ be the Lazarsfeld–Mukai bundle associated to a g^1_{k+1} on $C\in|\mathcal{L}|$. Take $0\ne s\in H^0(X,\mathcal{E})$, then $K_{k-1,1}(X,\mathcal{L},H^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)}))$ is a one-dimensional subspace of $K_{k-1,1}(X,\mathcal{L})$ and the morphism*

$$\begin{aligned}\psi\colon\mathbb{P}(H^0(X,\mathcal{E}))&\to\mathbb{P}(K_{k-1,1}(X,\mathcal{L}))\\ [s]&\mapsto K_{k-1,1}(X,\mathcal{L},H^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)}))\end{aligned}$$

is the Veronese embedding of degree $k-2$, i.e. ψ induces an isomorphism

$$\mathrm{Sym}^{k-2}H^0(X,\mathcal{E})\xrightarrow{\sim}K_{k-1,1}(X,\mathcal{L}).$$

For the argument of how this theorem implies the geometric syzygy conjecture in even genus we refer to [Kem20].

3 Universal secant bundles and proof of Voisin's theorem

The goal of this section will be to present the universal construction of the secant bundles Γ and $\mathcal{S}$ and to prove Voisin's theorem. Those bundles will fit into an exact sequence

$$0 \to \mathcal{S} \to \pi^*\mathcal{M}_{\mathcal{L}} \to \Gamma \to 0.$$

This sequence will be used to prove the vanishing of $H^1(X, \bigwedge^{k+1} \mathcal{M}_{\mathcal{L}}) \cong K_{k,1}(X, \mathcal{L})$.

Before we start the actual construction of Γ and $\mathcal{S}$ we will construct the secant sheaves Γ'_s and $\mathcal{S}'_s$. The parametrization of these sheaves along $s \in H^0(X, \mathcal{E})$ will give Γ and $\mathcal{S}$ after passing to a blow-up. In addition, the secant sheaves and their relation to the secant bundles will come up in the application to the geometric syzygy conjecture. The whole section follows [Kem20]. From now on we work in the setting introduced and examined in section 1.3.

3.1 Secant sheaves

Take a global section $0 \neq s \in H^0(X, \mathcal{E})$. The closed subscheme $Z(s) \subset X$ consists of finitely many points in X with multiplicities. Multiplication with the section s defines an injection $\mathcal{O}_X \overset{s}{\hookrightarrow} \mathcal{E}$. Composing with the map $\mathcal{E} \overset{\wedge s}{\to} \mathcal{E} \wedge \mathcal{E}$ yields an exact sequence

$$0 \to \mathcal{O}_X \overset{s}{\to} \mathcal{E} \overset{\wedge s}{\to} \mathcal{E} \wedge \mathcal{E}.$$

Using lemma 1.9, we have an isomorphism $\mathcal{E} \wedge \mathcal{E} = \det(\mathcal{E}) \cong \mathcal{L}$. Note that under this isomorphism the image of $\wedge s$ is isomorphic to $\mathcal{L} \otimes \mathcal{I}_{Z(s)} \hookrightarrow \mathcal{L}$. Thus, the sequence

$$0 \to \mathcal{O}_X \overset{s}{\to} \mathcal{E} \overset{\wedge s}{\to} \mathcal{L} \otimes \mathcal{I}_{Z(s)} \to 0 \tag{7}$$

is exact. By the commutative diagram

$$\begin{tikzcd}
H^0(X,\mathcal{E}) \otimes \mathcal{O}_X \arrow[r] \arrow[d, two heads] & H^0(X, \mathcal{I}_{Z(s)} \otimes \mathcal{L}) \otimes \mathcal{O}_X \arrow[d, "ev"'] \\
\mathcal{E} \arrow[r, two heads] & \mathcal{L} \otimes \mathcal{I}_{Z(s)}
\end{tikzcd}$$

the evaluation map ev has to be surjective. Hence, $\mathcal{I}_{Z(s)} \otimes \mathcal{L}$ is globally generated. Consider the exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{I}_{Z(s)} \to \mathcal{L} \to \mathcal{L}|_{Z(s)} \to 0. \tag{8}$$

Applying the global section functor, we obtain a map

$$W_s := H^0(X, \mathcal{L})/H^0(X, \mathcal{L} \otimes \mathcal{I}_{Z(s)}) \to H^0(X, \mathcal{L}|_{Z(s)}).$$

Definition 3.1. [Kem20] The *secant sheaves* associated to $Z(s)$ are defined as

$$\Gamma'_s := \ker(W_s \otimes \mathcal{O}_X \to H^0(X, \mathcal{L}|_{Z(s)}) \otimes \mathcal{O}_X \to \mathcal{L}|_{Z(s)}),$$
$$\mathcal{S}'_s := \ker(H^0(X, \mathcal{L} \otimes \mathcal{I}_{Z(s)}) \otimes \mathcal{O}_X \to \mathcal{L} \otimes \mathcal{I}_{Z(s)}).$$

Remark. Take a global section $0 \neq t \in H^0(X, \mathcal{L} \otimes \mathcal{I}_{Z(s)})$ and consider its zero locus $Z(t)$. Since $H^0(X, \mathcal{L} \otimes \mathcal{I}_{Z(s)})$ is a subspace of $H^0(X, \mathcal{L})$, the zero locus of t is an effective divisor on X linearly equivalent to C. Furthermore, the zero locus of t contains $Z(s)$ as a closed subscheme so elements of $H^0(X, \mathcal{L} \otimes \mathcal{I}_{Z(s)})$ are equations for curves in X which are linearly equivalent to C and contain $Z(s)$. Since $\mathcal{L}$ is globally generated, it is base point free and determines a morphism $X \to \mathbb{P}(\mathrm{H}^0(X, \mathcal{L})^*)$.

Now, elements $t \in H^0(X, \mathcal{L}\otimes\mathcal{I}_{Z(s)})$ yield hyperplanes in $\mathbb{P}(H^0(X,\mathcal{L})^*)$ that contain $Z(s)$. Similarly, a hyperplane that contains $Z(s)$ comes from some non-zero global section $t \in H^0(X, \mathcal{L}\otimes\mathcal{I}_{Z(s)})$, up to multiplication with a scalar. Hence, we get a correspondence between the hyperplanes that contain $Z(s)$ and elements in $\mathbb{P}(H^0(X, \mathcal{L}\otimes\mathcal{I}_{Z(s)}))$. Note that every hyperplane intersects X non-trivially since there is no global section of $\mathcal{L}$ with empty zero locus. Now, take a point $x \in X$ and consider the fibre of $\mathcal{S}'_s$ over x. In this case the fibre is the kernel of the map $H^0(X, \mathcal{L}\otimes\mathcal{I}_{Z(s)}) \to \mathcal{L}\otimes\mathcal{I}_{Z(s)}\otimes\kappa(x)$. If we suppose that x does not lie in $Z(s)$ this simplifies to $H^0(X, \mathcal{L}\otimes\mathcal{I}_{Z(s)}) \to \kappa(x)$. Consequently, the kernel of this map parametrizes those hyperplanes that contain $Z(s)$ and x. On the other hand, suppose that x is contained in $Z(s)$. In this case the kernel consists of global sections that have a zero of order at least two at x and hence parametrizes the hyperplanes that intersect X in x tangentially. This justifies the name secant bundle for $\mathcal{S}'_s$.

Consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{S}'_s \arrow[r] \arrow[d, dashed] & H^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)})\otimes\mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{L}\otimes\mathcal{I}_{Z(s)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M}_{\mathcal{L}} \arrow[r] \arrow[d, dashed] & H^0(X,\mathcal{L})\otimes\mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma'_s \arrow[r] \arrow[d] & W_s\otimes\mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{L}|_{Z(s)} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

All rows and the middle and the right column are short exact sequences. The dashed maps are obtained by the universal property of the kernels and therefore this diagram is commutative. The 3×3 lemma implies the exactness of the left column.

The main issue with this construction is that Γ'_s is not locally free since $Z(s)$ is not a divisor. This could be resolved by blowing up X along $Z(s)$ turning $Z(s)$ into the exceptional divisor. In addition there is no natural choice for $s \in H^0(X,\mathcal{E})$ so we would like to work with all sections s simultaneously to obtain a parametrization of Γ'_s and $\mathcal{S}'_s$ over all non-zero $s \in H^0(X,\mathcal{E})$. This leads to the universal construction of the secant bundles.

3.2 Secant bundles

This section follows [Kem20]. Recall that $h^0(X,\mathcal{E}) = k+2$ and hence $\mathbb{P} := \mathbb{P}(H^0(X,\mathcal{E}))$ is isomorphic to $\mathbb{P}^{k+1}$. Consider the product $X\times\mathbb{P}$ with the two projections $p\colon X\times\mathbb{P}\to X$ and $q\colon X\times\mathbb{P}\to\mathbb{P}$ and consider the closed subscheme $Z := \{(x,s) | s(x) = 0\} \subset X\times\mathbb{P}$. Note that the fibre of $q|_Z$ over $[s]\in\mathbb{P}$ is $Z(s)$ and thus Z is a parameterization of the subschemes $Z(s)$. Hence, the projection $q|_Z$ is quasi finite, projective and, by [Har77, III Ex. 11.2], also finite. The Lazarsfeld–Mukai bundle $\mathcal{E}$ is globally generated thus Z is isomorphic to the projective bundle of the kernel of the evaluation map $H^0(X,\mathcal{E})\otimes\mathcal{O}_X\to\mathcal{E}$. Ergo, Z is smooth. Using miracle flatness [21, Tag 00R4], we see that $Z\to\mathbb{P}$ is flat since both schemes are smooth and the morphism is finite. Define $\mathcal{M} := p^*\mathcal{M}_{\mathcal{L}}$. Then Künneth yields $H^1(X\times\mathbb{P}, \bigwedge^{k+1}\mathcal{M}) \cong H^1(X, \bigwedge^{k+1}\mathcal{M}_{\mathcal{L}})$ and thus proving Voisin's theorem only requires showing that the first vector space vanishes.

Apply the Künneth formula to get $H^0(X\times\mathbb{P}, \mathcal{E}\boxtimes\mathcal{O}_{\mathbb{P}}(1)) \cong \operatorname{Hom}(H^0(X,\mathcal{E}), H^0(X,\mathcal{E}))$. Multiplying with the element corresponding to $id \in \operatorname{Hom}(H^0(X,\mathcal{E}), H^0(X,\mathcal{E}))$ gives rise to an injection

$$0 \to \mathcal{O}_X\boxtimes\mathcal{O}_{\mathbb{P}}(-2) \xrightarrow{\cdot id} \mathcal{E}\boxtimes\mathcal{O}_{\mathbb{P}}(-1).$$

Compare this to (7). By a similar argument, the injection has cokernel $p^*\mathcal{L} \otimes \mathcal{I}_Z$. Therefore, we obtain an exact sequence

$$0 \to \mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(-2) \xrightarrow{\cdot id} \mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(-1) \to p^*\mathcal{L} \otimes \mathcal{I}_Z \to 0. \tag{9}$$

Now, let $\pi \colon B \to X \times \mathbb{P}$ be the blow-up of the closed subscheme $Z \subset X \times \mathbb{P}$ and let D be the exceptional divisor. Then some basic facts state that [Kem20]

(i) $\pi_* \mathcal{O}_B \cong \mathcal{O}_{X \times \mathbb{P}}$

(ii) $\pi_* \mathcal{I}_D \cong \mathcal{I}_Z$

(iii) $R^i \pi_* \mathcal{O}_B = 0 = R^i \pi_* \mathcal{I}_D$ for all $i > 0$.

Moreover, by setting p' and q' as the composites in the diagram

$$\begin{tikzcd}
B \arrow[dr, "\pi"] \arrow[drr, bend left, "q'"] \arrow[ddr, bend right, "p'"'] & & \\
& X \times \mathbb{P} \arrow[r, "q"] \arrow[d, "p"] & \mathbb{P} \\
& X, &
\end{tikzcd}$$

one gets the following isomorphisms due to the projection formula:

(iv) $q'_*(p'^*\mathcal{L} \otimes \mathcal{I}_D) \cong q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z)$

(v) $q'_* p'^* \mathcal{L} \cong q_* p^* \mathcal{L}$.

Note that the fibre dimensions of q' and p' are constant. This implies that these morphisms are flat due to miracle flatness [21, Tag 00R4].

Take the ideal-sequence of the exceptional divisor D and twist it with the line bundle $p'^*\mathcal{L}$ to get

$$0 \to p'^*\mathcal{L} \otimes \mathcal{I}_D \to p'^*\mathcal{L} \to p'^*\mathcal{L}|_D \to 0.$$

Now, apply the pushforward q'_* and define $\mathcal{W} := \operatorname{coker}(q'_*(p'^*\mathcal{L} \otimes \mathcal{I}_D) \to q'_* p'^* \mathcal{L})$.

Lemma 3.1. *[Kem20, Lem. 1.1] The above defined sheaf $\mathcal{W} \in \mathrm{Coh}(\mathbb{P})$ is locally free of rank k.*

Proof. Consider the sequence

$$0 \to p^*\mathcal{L} \otimes \mathcal{I}_Z \to p^*\mathcal{L} \to p^*\mathcal{L}|_Z \to 0$$

and apply q_* to get

$$0 \to q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z) \to q_* p^* \mathcal{L} \to q_*(p^*\mathcal{L}|_Z) \to R^1 q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z) \to R^1 q_* p^* \mathcal{L}.$$

Note that

$$R^1 q_* p^* \mathcal{L} \cong H^1(X, \mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}} = 0.$$

By construction, $\mathcal{W}$ participates in the short exact sequence

$$0 \to \mathcal{W} \to q_*(p^*\mathcal{L}|_Z) \to R^1 q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z) \to R^1 q_* p^* \mathcal{L} = 0$$

and therefore $\mathcal{W}$ is locally free of rank k if $q_*(p^*\mathcal{L}|_Z)$ and $R^1 q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z)$ are locally free of rank $k+1$ and 1. To use Grauert's theorem observe that the morphism $q \colon X \times \mathbb{P} \to \mathbb{P}$ is projective.

(i) The line bundle $p^*\mathcal{L}|_Z \in \mathrm{Pic}(Z)$ is flat over $\mathbb{P}$ since $q|_Z\colon Z \to \mathbb{P}$ is a flat morphism. The fibre $(X\times\mathbb{P})_{[s]}$ is isomorphic to X and $p^*\mathcal{L}|_Z|_{(X\times\mathbb{P})_{[s]}} \cong \mathcal{L}|_{Z(s)} \cong \mathcal{O}_{Z(s)}$. Therefore,

$$h^0((X\times\mathbb{P})_{[s]}, p^*\mathcal{L}|_Z|_{(X\times\mathbb{P})_{[s]}}) = h^0(X,\mathcal{L}|_{Z(s)}) = h^0(X,\mathcal{O}_{Z(s)}) = k+1$$

for all $0 \neq s \in H^0(X,\mathcal{E})$. By [Har77, III Thm. 12.8], the function $\mathbb{P} \to \mathbb{N}$, $y \mapsto h^0(y, p^*\mathcal{L}|_Z|_{(X\times\mathbb{P})_y})$ is upper semicontinuous ergo it is the constant function with value $k+1$.

(ii) The line bundle $p^*\mathcal{L}$ is flat over $\mathbb{P}$ since q is a flat morphism. The ideal sheaf $\mathcal{I}_Z$ sits in the short exact sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_{X\times\mathbb{P}} \to \mathcal{O}_Z \to 0$ where the two (non-trivial) sheaves on the right hand side are flat over $\mathbb{P}$. Hence, $\mathcal{I}_Z$ is flat over $\mathbb{P}$ as well. Tensor products of flat sheaves are flat, so $p^*\mathcal{L}\otimes\mathcal{I}_Z$ is flat over $\mathbb{P}$. Using the short exact sequence (7)

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_{Z(s)}\otimes\mathcal{L} \to 0$$

together with lemma 1.9, one sees that $H^1(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)}) \cong H^2(X,\mathcal{O}_X) \overset{SD}{\cong} H^0(X,\mathcal{O}_X)$. Ergo, $h^1((X\times\mathbb{P})_{[s]}, (\mathcal{L}\otimes\mathcal{I}_Z)|_{(X\times\mathbb{P})_{[s]}}) = 1$ for all $0 \neq s \in H^0(X,\mathcal{E})$. Conclude similarly to the above argument.

□

Remark. (i) The short exact sequence

$$0 \to \mathcal{L}\otimes\mathcal{I}_{Z(s)} \to \mathcal{L} \to \mathcal{L}|_{Z(s)} \to 0$$

yields

$$0 \to H^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)}) \to H^0(X,\mathcal{L}) \to H^0(X,\mathcal{L}|_{Z(s)}) \to H^1(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)}) \to 0.$$

In the proof above it was shown that $h^0(X,\mathcal{L}|_{Z(s)})$ and $h^1(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)})$ are independent of $s \in H^0(X,\mathcal{E})$. This implies that $h^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)})$ does not depend on $s \in H^0(X,\mathcal{E})$ either and therefore Grauert's theorem implies that $q_*(p^*\mathcal{L}\otimes\mathcal{I}_Z)$ is locally free and that $q_*(p^*\mathcal{L}\otimes\mathcal{I}_Z)\otimes\kappa([s]) \cong H^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)})$ for all $0 \neq s \in H^0(X,\mathcal{E})$.

(ii) Consider the short exact sequence

$$0 \to q_*(p^*\mathcal{L}\otimes\mathcal{I}_Z) \to q_*p^*\mathcal{L} \to \mathcal{W} \to 0$$

and take the fibre over a closed point $[s] \in \mathbb{P}$ to get

$$0 \to H^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)}) \to H^0(X,\mathcal{L}) \to \mathcal{W}\otimes\kappa([s]) \to 0.$$

It follows that $\mathcal{W}\otimes\kappa([s]) \cong W_s$ and therefore $\mathcal{W}$ parametrizes the vector spaces W_s.

Lemma 3.2. *[Kem20] The canonical maps*

(i) $q'^*q'_*(p'^*\mathcal{L}\otimes\mathcal{I}_D) \twoheadrightarrow p'^*\mathcal{L}\otimes\mathcal{I}_D$ *and*

(ii) $q'^*q'_*p'^*\mathcal{L} \twoheadrightarrow p'^*\mathcal{L}$ *are surjective.*

(iii) *Moreover, the kernel of* $q'^*q'_*p'^*\mathcal{L} \twoheadrightarrow p'^*\mathcal{L}$ *is isomorphic to* $\pi^*\mathcal{M}$.

Proof. (i) Consider the canonical map $q^*q_*(p^*\mathcal{L}\otimes\mathcal{I}_Z)\to p^*\mathcal{L}\otimes\mathcal{I}_Z$ and restrict it to a closed fibre $(X\times\mathbb{P})_{[s]}\cong X$. The obtained map can be identified with the evaluation homomorphism

$$H^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)})\otimes\mathcal{O}_X\twoheadrightarrow\mathcal{L}\otimes\mathcal{I}_{Z(s)}$$

by the previous remark. The surjectivity comes from the fact that $\mathcal{L}\otimes\mathcal{I}_{Z(s)}$ is globally generated. Hence, $q^*q_*(p^*\mathcal{L}\otimes\mathcal{I}_Z)\to p^*\mathcal{L}\otimes\mathcal{I}_Z$ is surjective on all closed fibres and thus surjective itself. Applying π^* preserves surjectivity and yields $q'^*q'_*(p'^*\mathcal{L}\otimes\pi^*\mathcal{I}_Z)\twoheadrightarrow p'^*\mathcal{L}\otimes\pi^*\mathcal{I}_Z$. Pulling back the ideal sheaf sequence of Z results in

$$\pi^*\mathcal{I}_Z\to\mathcal{O}_B\to\mathcal{O}_D\to 0,$$

which shows that the natural map $\pi^*\mathcal{I}_Z\twoheadrightarrow\mathcal{I}_D$ is surjective. Composing these two surjections yields the desired one.

(ii) Since $\mathcal{L}$ is globally generated, the pull-back $p'^*\mathcal{L}$ is globally generated as well and [Har77, III Thm. 8.8] implies that $q'^*q'_*p'^*\mathcal{L}\twoheadrightarrow p'^*\mathcal{L}$ is surjective.

(iii) Note that there are canonical isomorphism $q'^*q'_*p'^*\mathcal{L}\cong q'^*(H^0(X,\mathcal{L})\otimes\mathcal{O}_{\mathbb{P}})\cong H^0(X,\mathcal{L})\otimes\mathcal{O}_B$. Apply p'^*, which is flat, to

$$0\to\mathcal{M}_{\mathcal{L}}\to H^0(X,\mathcal{L})\otimes\mathcal{O}_X\stackrel{ev}{\to}\mathcal{L}\to 0$$

to get

$$0\to\pi^*\mathcal{M}\to H^0(X,\mathcal{L})\otimes\mathcal{O}_B\stackrel{ev}{\to}p'^*\mathcal{L}\to 0.$$

Under the canonical isomorphism $q'^*q'_*p'^*\mathcal{L}\cong H^0(X,\mathcal{L})\otimes\mathcal{O}_B$ the map ev becomes the natural map $q'^*q'_*p'^*\mathcal{L}\twoheadrightarrow p'^*\mathcal{L}$.

□

By construction of $\mathcal{W}$, there is a short exact sequence

$$0\to q'_*(p'^*\mathcal{L}\otimes\mathcal{I}_D)\to q'_*p'^*\mathcal{L}\to\mathcal{W}\to 0.$$

Apply q'^* to obtain

$$0\to q'^*q'_*(p'^*\mathcal{L}\otimes\mathcal{I}_D)\to q'^*q'_*p'^*\mathcal{L}\to q'^*\mathcal{W}\to 0.$$

Note that the left-exactness is preserved since q' is flat. This sequence fits into the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & q'^*q'_*(p'^*\mathcal{L}\otimes\mathcal{I}_D) \arrow[r] \arrow[d, two heads] & q'^*q'_*p'^*\mathcal{L} \arrow[r] \arrow[d, two heads] & q'^*\mathcal{W} \arrow[r] \arrow[d, two heads, dashed] & 0 \\
0 \arrow[r] & p'^*\mathcal{L}\otimes\mathcal{I}_D \arrow[r] & p'^*\mathcal{L} \arrow[r] & p'^*\mathcal{L}|_D \arrow[r] & 0
\end{tikzcd}$$

The left vertical map and the middle vertical map are the canonical ones and considering the lemma above they are surjective. The dashed map exists by the universal property of the cokernel and is surjective because the middle vertical map is surjective.

Definition 3.2. [Kem20] The *secant bundles* Γ and $\mathcal{S}$ are defined as the kernel of

(i) the dashed map in the above diagram: $\Gamma:=\ker(q'^*\mathcal{W}\twoheadrightarrow q'^*\mathcal{L}|_D)$

(ii) the left vertical map in the above diagram: $\mathcal{S}:=\ker(q'^*q'_*(p'^*\mathcal{L}\otimes\mathcal{I}_D)\twoheadrightarrow p'^*\mathcal{L}\otimes\mathcal{I}_D)$.

Note that Γ and $\mathcal{S}$ are locally free since D is a divisor. Furthermore, observe that Γ has rank k because $\mathcal{W}$ has rank $k+1$. The secant bundles fit into the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d, dashed] & q'^*q'_*(p'^*\mathcal{L}\otimes\mathcal{I}_D) \arrow[r] \arrow[d] & p'^*\mathcal{L}\otimes\mathcal{I}_D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^*\mathcal{M} \arrow[r] \arrow[d, dashed] & q'^*q'_*p'^*\mathcal{L} \arrow[r, "ev"] \arrow[d] & p'^*\mathcal{L} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma \arrow[r] \arrow[d] & q'^*\mathcal{W} \arrow[r] \arrow[d] & p'^*\mathcal{L}|_D \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

The dashed maps exist by the universal property of the kernel and by the 3×3 lemma the sequence

$$0 \to \mathcal{S} \to \pi^*\mathcal{M} \to \Gamma \to 0$$

is exact. Applying the $(k+1)$-th exterior power yields the long exact sequence

$$\cdots \to \bigwedge^{k-1}\pi^*\mathcal{M}\otimes \mathrm{Sym}^2\mathcal{S} \to \bigwedge^{k}\pi^*\mathcal{M}\otimes\mathcal{S} \to \bigwedge^{k+1}\pi^*\mathcal{M} \to \bigwedge^{k+1}\Gamma = 0. \tag{10}$$

3.3 Proof of Voisin's theorem

Recall that the goal is to prove the vanishing of $H^1(X\times\mathbb{P}, \bigwedge^{k+1}\mathcal{M})$. Theorem 1.2 combined with the projection formula yields an isomorphism $H^1(B, \bigwedge^{k+1}\pi^*\mathcal{M}) \cong H^1(X\times\mathbb{P}, \bigwedge^{k+1}\mathcal{M})$. The long exact sequence (10)

$$\cdots \xrightarrow{f_{k-2}} \bigwedge^{k-1}\pi^*\mathcal{M}\otimes\mathrm{Sym}^2\mathcal{S} \xrightarrow{f_{k-1}} \bigwedge^{k}\pi^*\mathcal{M}\otimes\mathcal{S} \xrightarrow{f_k} \bigwedge^{k+1}\pi^*\mathcal{M} \to 0$$

gives rise to the short exact sequence

$$0 \to \ker(f_k) \to \bigwedge^{k}\pi^*\mathcal{M}\otimes\mathcal{S} \xrightarrow{f_k} \bigwedge^{k+1}\pi^*\mathcal{M} \to 0.$$

Taking cohomology results in

$$H^1(B, \bigwedge^{k}\pi^*\mathcal{M}\otimes\mathcal{S}) \to H^1(B, \bigwedge^{k+1}\pi^*\mathcal{M}) \to H^2(B, \ker(f_k)).$$

Thus, it suffices to show that $H^1(B, \bigwedge^{k}\pi^*\mathcal{M}\otimes\mathcal{S}) = 0 = H^2(B, \ker(f_k))$. Similarly, consider

$$0 \to \ker(f_{k-1}) \to \bigwedge^{k-1}\pi^*\mathcal{M}\otimes\mathrm{Sym}^2\mathcal{S} \xrightarrow{f_{k-1}} \ker(f_k) \to 0$$

and apply H^i:

$$H^2(B, \bigwedge^{k-1}\pi^*\mathcal{M}\otimes\mathrm{Sym}^2\mathcal{S}) \to H^2(B, \ker(f_k)) \to H^3(B, \ker(f_{k-1}))$$

Hence, $H^2(B, \ker(f_k)) = 0$ if $H^2(B, \bigwedge^{k-1} \pi^*\mathcal{M} \otimes \mathrm{Sym}^2\mathcal{S}) = 0 = H^3(B, \ker(f_{k-1}))$.
By continuing this inductively, one concludes that Voisin's theorem holds true if

$$H^i(B, \bigwedge^{k+1-i} \pi^*\mathcal{M} \otimes \mathrm{Sym}^i\mathcal{S}) = 0$$

for $1 \leq i \leq k+1$.

To simplify the notation define $\mathcal{G} := q^*q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z)$.

Lemma 3.3. *[Kem20, Lem. 1.2] With $\mathcal{G}$ defined as above the following holds true:*

(i) $H^{k+1}(X \times \mathbb{P}, \mathrm{Sym}^{k+1}\mathcal{G}) = 0$

(ii) *there is a natural isomorphism* $H^k(X \times \mathbb{P}, \mathrm{Sym}^{k+1}\mathcal{G}) \cong \mathrm{Sym}^k H^0(X, \mathcal{E})$

Proof. Apply q_* to the exact sequence (9) to get

$$0 \to q_*(\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(-2)) \to q_*(\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(-1)) \to q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z) \to R^1q_*(\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(-2)).$$

The projection formula yields the isomorphisms

(i) $R^1q_*(\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(-2)) \cong H^1(X, \mathcal{O}_X) \otimes \mathcal{O}_{\mathbb{P}}(-2) = 0$

(ii) $q_*(\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(-2)) \cong \mathcal{O}_{\mathbb{P}}(-2)$

(iii) $q_*(\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(-1)) \cong H^0(X, \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}}(-1)$.

Thus, the above sequence can be rewritten as

$$0 \to \mathcal{O}_{\mathbb{P}}(-2) \to H^0(X, \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}}(-1) \to q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z) \to 0.$$

Applying q^* is exact since q is a flat morphism. Hence, we get the exact sequence

$$0 \to q^*\mathcal{O}_{\mathbb{P}}(-2) \to H^0(X, \mathcal{E}) \otimes q^*\mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{G} \to 0. \tag{11}$$

By applying the $(k+1)$-th symmetric product, one obtains the short exact sequence

$$0 \to \mathrm{Sym}^k H^0(X, \mathcal{E}) \otimes q^*\mathcal{O}_{\mathbb{P}}(-k-2) \to \mathrm{Sym}^{k+1} H^0(X, \mathcal{E}) \otimes q^*\mathcal{O}_{\mathbb{P}}(-k-1) \to \mathrm{Sym}^{k+1}\mathcal{G} \to 0.$$

For simplicity define $V_i := \mathrm{Sym}^i H^0(X, \mathcal{E})$. Therefore, taking cohomology results in

$$\begin{aligned} V_{k+1} \otimes H^k(q^*\mathcal{O}_{\mathbb{P}}(-k-1)) &\to H^k(\mathrm{Sym}^{k+1}\mathcal{G}) \to V_k \otimes H^{k+1}(q^*\mathcal{O}_{\mathbb{P}}(-k-2)) \\ \to V_{k+1} \otimes H^{k+1}(q^*\mathcal{O}_{\mathbb{P}}(-k-1)) &\to H^{k+1}(\mathrm{Sym}^{k+1}\mathcal{G}) \to V_k \otimes H^{k+2}(q^*\mathcal{O}_{\mathbb{P}}(-k-2)). \end{aligned}$$

By exactness of this sequence, it suffices to prove

(1) $H^k(X \times \mathbb{P}, q^*\mathcal{O}_{\mathbb{P}}(-k-1)) = 0$

(2) $H^{k+1}(X \times \mathbb{P}, q^*\mathcal{O}_{\mathbb{P}}(-k-2)) \cong \mathbb{C}$

(3) $H^{k+1}(X \times \mathbb{P}, q^*\mathcal{O}_{\mathbb{P}}(-k-1)) = 0$

(4) $H^{k+2}(X \times \mathbb{P}, q^*\mathcal{O}_{\mathbb{P}}(-k-2)) = 0$.

These assertions are direct consequences of the Künneth formula. □

Lemma 3.4. *[Kem20, Lem. 1.3] There is a natural isomorphism*

$$H^k(X \times \mathbb{P}, \mathrm{Sym}^k\mathcal{G} \otimes p^*\mathcal{L} \otimes \mathcal{I}_Z) \cong \mathrm{Sym}^k H^0(X, \mathcal{E}).$$

Proof. Taking the k-th symmetric power of sequence (11) and tensoring it with $p^*\mathcal{L}\otimes\mathcal{I}_Z$ results in

$$0\to(V_{k-1}\otimes\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-1))\otimes\mathcal{I}_Z\to(V_k\otimes\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k))\otimes\mathcal{I}_Z\to\mathrm{Sym}^k\mathcal{G}\otimes p^*\mathcal{L}\otimes\mathcal{I}_Z\to 0$$

where $V_i := \mathrm{Sym}^i H^0(X,\mathcal{E})$. Note that left-exactness is preserved since tensoring an injection between two locally free sheaves with an ideal sheaf corresponds locally to restricting the injection $A^n\hookrightarrow A^m$ to $\mathfrak{a}^n\hookrightarrow\mathfrak{a}^m$, which is still injective (here, $\mathrm{Spec}(A)\subset X\times\mathbb{P}$ is an open affine subscheme and $\mathfrak{a}$ is the restriction of $\mathcal{I}_Z$ to $\mathrm{Spec}(A)$). Applying cohomology yields

$$\begin{aligned}&V_{k-1}\otimes H^k((\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-1))\otimes\mathcal{I}_Z)\to V_k\otimes H^k((\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k))\otimes\mathcal{I}_Z)\\ \to& H^k(\mathrm{Sym}^k\mathcal{G}\otimes p^*\mathcal{L}\otimes\mathcal{I}_Z)\to V_{k-1}\otimes H^{k+1}((\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-1))\otimes\mathcal{I}_Z).\end{aligned}$$

Ergo, it is enough to prove

(i) $H^k(X\times\mathbb{P},(\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-1))\otimes\mathcal{I}_Z)=0$

(ii) $H^k(X\times\mathbb{P},(\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k))\otimes\mathcal{I}_Z)\cong\mathbb{C}$

(iii) $H^{k+1}(X\times\mathbb{P},(\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-1))\otimes\mathcal{I}_Z)=0$.

For the vanishings in (i) and (iii) take sequence (9) and twist it by $q^*\mathcal{O}_{\mathbb{P}}(-k-1)$ to get

$$0\to\mathcal{O}_X\boxtimes\mathcal{O}_{\mathbb{P}}(-k-3)\overset{\cdot id}{\to}\mathcal{E}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-2)\to\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-1)\otimes\mathcal{I}_Z\to 0.$$

Using the Künneth formula yields

(1) $H^k(X\times\mathbb{P},\mathcal{E}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-2))=0$

(2) $H^{k+1}(X\times\mathbb{P},\mathcal{O}_X\boxtimes\mathcal{O}_{\mathbb{P}}(-k-3))\cong H^0(X,\mathcal{E})$

(3) $H^{k+1}(X\times\mathbb{P},\mathcal{E}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-2))\cong H^0(X,\mathcal{E})$

(4) $H^{k+2}(X\times\mathbb{P},\mathcal{O}_X\boxtimes\mathcal{O}_{\mathbb{P}}(-k-3))=0$.

Observe that by construction the map

$$H^{k+1}(X\times\mathbb{P},\mathcal{O}_X\boxtimes\mathcal{O}_{\mathbb{P}}(-k-3))\to H^{k+1}(X\times\mathbb{P},\mathcal{E}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-2))$$

can be identified with $id\colon H^0(X,\mathcal{E})\to H^0(X,\mathcal{E})$. Thus, taking cohomology of the short exact sequence above yields

$$0\to H^k((\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-1))\otimes\mathcal{I}_Z)\to H^0(X,\mathcal{E})\overset{id}{\to}H^0(X,\mathcal{E})\to H^{k+1}((\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-1))\otimes\mathcal{I}_Z)\to 0.$$

For the isomorphism in (ii) consider the sequence (9) twisted by $q^*\mathcal{O}_{\mathbb{P}}(-k)$:

$$0\to\mathcal{O}_X\boxtimes\mathcal{O}_{\mathbb{P}}(-k-2)\overset{\cdot id}{\to}\mathcal{E}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-1)\to(\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k))\otimes\mathcal{I}_Z\to 0$$

Applying Künneth, one sees that $H^i(X\times\mathbb{P},\mathcal{E}\boxtimes\mathcal{O}_{\mathbb{P}}(-k-1))=0$ for all i. Consequently, the boundary map $H^k(X\times\mathbb{P},(\mathcal{L}\boxtimes\mathcal{O}_{\mathbb{P}}(-k))\otimes\mathcal{I}_Z)\overset{\sim}{\to}H^{k+1}(X\times\mathbb{P},\mathcal{O}_X\boxtimes\mathcal{O}_{\mathbb{P}}(-k-2))$ is an isomorphism and the latter vector space is naturally isomorphic to $\mathbb{C}$ due to Künneth's formula and Serre duality. $\square$

Lemma 3.5. *[Kem20, Lem. 1.4] The canonical map $\mathcal{G}:=q^*q_*(p^*\mathcal{L}\otimes\mathcal{I}_Z)\to p^*\mathcal{L}\otimes\mathcal{I}_Z$ induces an isomorphism*

$$H^k(\mathrm{Sym}^k\mathcal{G}\otimes\mathcal{G})\overset{\sim}{\to}H^k(\mathrm{Sym}^k\mathcal{G}\otimes p^*\mathcal{L}\otimes\mathcal{I}_Z).$$

Proof. The proof will show that the vector spaces $H^k(\mathrm{Sym}^k\mathcal{G}\otimes\mathcal{G})$ and $H^k(\mathrm{Sym}^k\mathcal{G}\otimes p^*\mathcal{L}\otimes\mathcal{I}_Z)$ are both naturally isomorphic to $\mathrm{Sym}^k H^0(X,\mathcal{E})$. Composition yields a natural endomorphism of $\mathrm{Sym}^k H^0(X,\mathcal{E})$ which has to be the identity. Due to lemma 3.4, it is already known that there is a natural isomorphism $H^k(\mathrm{Sym}^k\mathcal{G}\otimes p^*\mathcal{L}\otimes\mathcal{I}_Z)\cong\mathrm{Sym}^k H^0(X,\mathcal{E})$. Hence, it suffices to show the other isomorphism. Twist (11) by $q^*\mathcal{O}_{\mathbb{P}}(-k-1)$ to get

$$0\to q^*\mathcal{O}_{\mathbb{P}}(-k-3)\to H^0(X,\mathcal{E})\otimes q^*\mathcal{O}_{\mathbb{P}}(-k-2)\to\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k-1)\to 0.$$

Künneth's formula yields the isomorphisms

(i) $H^k(X\times\mathbb{P},q^*\mathcal{O}_{\mathbb{P}}(-k-2))=0$

(ii) $H^{k+1}(X\times\mathbb{P},q^*\mathcal{O}_{\mathbb{P}}(-k-3))\cong H^0(X,\mathcal{E})$

(iii) $H^{k+1}(X\times\mathbb{P},q^*\mathcal{O}_{\mathbb{P}}(-k-2))\cong\mathbb{C}$

(iv) $H^{k+2}(X\times\mathbb{P},q^*\mathcal{O}_{\mathbb{P}}(-k-3))=0.$

Thus, taking cohomology of the above short exact sequence results in

$$0\to H^k(\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k-1))\to H^0(X,\mathcal{E})\xrightarrow{id}H^0(X,\mathcal{E})\to H^{k+1}(\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k-1))\to 0$$

and we obtain the two vanishings

$$H^k(X\times\mathbb{P},\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k-1))=0=H^{k+1}(X\times\mathbb{P},\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k-1)).$$

Now, take the k-th symmetric power of (11) and tensor it with $\mathcal{G}$ to get

$$0\to V_{k-1}\otimes\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k-1)\to V_k\otimes\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k)\to\mathrm{Sym}^k\mathcal{G}\otimes\mathcal{G}\to 0$$

where $V_i:=\mathrm{Sym}^i H^0(X,\mathcal{E})$. Thus, applying cohomology to this sequence yields

$$0\to V_k\otimes H^k(X\times\mathbb{P},\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k))\xrightarrow{\sim}H^k(X\times\mathbb{P},\mathrm{Sym}^k\mathcal{G}\otimes\mathcal{G})\to 0.$$

Finally, it suffices to show that $H^k(X\times\mathbb{P},\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k))\cong\mathbb{C}$. For this twist (11) by $q^*\mathcal{O}_{\mathbb{P}}(-k)$ to get

$$0\to q^*\mathcal{O}_{\mathbb{P}}(-k-2)\to H^0(X,\mathcal{E})\otimes q^*\mathcal{O}_{\mathbb{P}}(-k-1)\to\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k)\to 0.$$

Observe that $H^i(X\times\mathbb{P},q^*\mathcal{O}_{\mathbb{P}}(-k-1))=0$ for $i=k,\,k+1$ and that $H^{k+1}(X\times\mathbb{P},q^*\mathcal{O}_{\mathbb{P}}(-k-2))\cong\mathbb{C}$ via Künneth. This produces the exact sequence

$$0\to H^k(X\times\mathbb{P},\mathcal{G}\otimes q^*\mathcal{O}_{\mathbb{P}}(-k))\xrightarrow{\sim}\mathbb{C}\to 0.$$

□

Proposition 3.6. *[Kem20, Prop. 1.5] There is a natural isomorphism*

$$H^k(X\times\mathbb{P},\mathrm{Sym}^{k+1}\mathcal{G})\xrightarrow{\sim}H^k(X\times\mathbb{P},\mathrm{Sym}^k\mathcal{G}\otimes p^*\mathcal{L}\otimes\mathcal{I}_Z).$$

Proof. Consider the composite

$$H^k(X\times\mathbb{P},\mathrm{Sym}^{k+1}\mathcal{G})\to H^k(X\times\mathbb{P},\mathrm{Sym}^k\mathcal{G}\otimes\mathcal{G})\to H^k(X\times\mathbb{P},\mathrm{Sym}^k\mathcal{G}\otimes p^*\mathcal{L}\otimes\mathcal{I}_Z)$$

where the first map is obtained from the natural map $\mathrm{Sym}^{k+1}\mathcal{G}\to\mathrm{Sym}^k\mathcal{G}\otimes\mathcal{G}$ and the second map is the natural isomorphism of lemma 3.5. The composition $\mathrm{Sym}^{k+1}\mathcal{G}\to\mathrm{Sym}^k\mathcal{G}\otimes\mathcal{G}\to\mathrm{Sym}^{k+1}\mathcal{G}$ is just multiplication by $k+1$ so $H^k(X\times\mathbb{P},\mathrm{Sym}^{k+1}\mathcal{G})\hookrightarrow H^k(X\times\mathbb{P},\mathrm{Sym}^k\mathcal{G}\otimes\mathcal{G})$ and consequently the composite is injective. Lemma 3.3 and 3.4 imply that the left hand side and the right hand side of the composite have the same dimensions. □

Lemma 3.7. *Here is a collection of isomorphisms for later application:*

(i) $R^1\pi_*\mathcal{S} = 0$

(ii) $R^1\pi_*\Gamma = 0$

Let $\mathcal{F}$ be a locally free coherent sheaf on $X \times \mathbb{P}$, then

(iii) $\pi_*(\pi^*\mathcal{F} \otimes \mathcal{I}_D) \cong \mathcal{F} \otimes \mathcal{I}_Z$

(iv) $R^j\pi_*(\pi^*\mathcal{F} \otimes \mathcal{I}_D) = 0$ *for all* $j > 0$

(v) $R^j\pi_*\pi^*\mathcal{F} = 0$ *for all* $j > 0$.

Proof. The last three assertions are direct consequences of the projection formula using $\pi_*\mathcal{I}_D \cong \mathcal{I}_Z$ and $R^j\pi_*\mathcal{I}_D = 0 = R^j\pi_*\mathcal{O}_D = 0$ for all $j > 0$. Consider the short exact sequence

$$0 \to \mathcal{S} \to \pi^*\mathcal{G} \to p'^*\mathcal{L} \otimes \mathcal{I}_D \to 0.$$

Pushing down on $X \times \mathbb{P}$ and using the last three assertions, one obtains the exact sequences

$$0 \to \pi_*\mathcal{S} \to \mathcal{G} \to p^*\mathcal{L} \otimes \mathcal{I}_Z \to R^1\pi_*\mathcal{S} \to 0$$

and

$$0 \to R^2\pi_*\mathcal{S} \to 0.$$

Hence, one sees immediately that $R^2\pi_*\mathcal{S} = 0$. Since the map $\mathcal{G} \twoheadrightarrow p^*\mathcal{L} \otimes \mathcal{I}_Z$ is the surjection from the beginning of the construction of the secant bundles, $R^1\pi_*\mathcal{S} = 0$ as well. Now, consider

$$0 \to \mathcal{S} \to \pi^*\mathcal{M} \to \Gamma \to 0$$

and push it on $X \times \mathbb{P}$ to get

$$0 \to R^1\pi_*\Gamma \to R^2\pi_*\mathcal{S}.$$

Since $R^2\pi_*\mathcal{S}$ vanishes, $R^1\pi_*\Gamma$ does as well. □

Now, we can proof Voisin's theorem.

Theorem 3.8 (Voisin)**.** *[Kem20, Thm. 1.6]* $H^i(B, \bigwedge^{k+1-i} \pi^*\mathcal{M} \otimes \mathrm{Sym}^i\mathcal{S}) = 0$ *for* $1 \le i \le k+1$.

Proof. Note that $\pi^*\mathcal{G} \cong q'^*q'_*(p'^*\mathcal{L} \otimes \mathcal{I}_D)$. By construction, the vector bundle $\mathcal{S}$ sits in the short exact sequence

$$0 \to \mathcal{S} \to \pi^*\mathcal{G} \to p'^*\mathcal{L} \otimes \mathcal{I}_D \to 0.$$

Applying the i-th symmetric power results in

$$0 \to \mathrm{Sym}^i\mathcal{S} \to \mathrm{Sym}^i\pi^*\mathcal{G} \to \mathrm{Sym}^{i-1}\pi^*\mathcal{G} \otimes p'^*\mathcal{L} \otimes \mathcal{I}_D \to 0. \tag{12}$$

Pushing the exact sequence down on $X \times \mathbb{P}$ yields

$$0 \to \pi_*\mathrm{Sym}^i\mathcal{S} \to \mathrm{Sym}^i\mathcal{G} \to \mathrm{Sym}^{i-1}\mathcal{G} \otimes p^*\mathcal{L} \otimes \mathcal{I}_Z \to 0$$

by the last lemma. As a consequence of lemma 3.7 and theorem 1.2, we can identifiy

$$H^l(B, \mathrm{Sym}^i\pi^*\mathcal{G}) \to H^l(B, \mathrm{Sym}^{i-1}\pi^*\mathcal{G} \otimes p'^*\mathcal{L} \otimes \mathcal{I}_D)$$

with

$$H^l(X \times \mathbb{P}, \mathrm{Sym}^i\mathcal{G}) \to H^l(X \times \mathbb{P}, \mathrm{Sym}^{i-1}\mathcal{G} \otimes p^*\mathcal{L} \otimes \mathcal{I}_Z)$$

for every l. Set $i = k+1$ and take cohomology of (12) to obtain

$$H^k(\mathrm{Sym}^{k+1}\pi^*\mathcal{G}) \to H^k(\mathrm{Sym}^k\pi^*\mathcal{G} \otimes p'^*\mathcal{L} \otimes \mathcal{I}_D) \to H^{k+1}(\mathrm{Sym}^{k+1}\mathcal{S}) \to H^{k+1}(\mathrm{Sym}^{k+1}\pi^*\mathcal{G}).$$

By proposition 3.6, the first map is an isomorphism and by lemma 3.3 the last vector space is zero. Thus, $H^{k+1}(B, \mathrm{Sym}^{k+1}\mathcal{S}) = 0$. It remains to show that $H^i(B, \bigwedge^{k+1-i}\pi^*\mathcal{M} \otimes \mathrm{Sym}^i\mathcal{S}) = 0$ for $1 \leq i \leq k$. To simplify the notation define $\mathcal{F}_j := \bigwedge^{k+1-j}\mathcal{M}$. Tensoring the sequence (12) with the bundle $\pi^*\mathcal{F}_i \cong \bigwedge^{k+1-i}\pi^*\mathcal{M}$ results in

$$0 \to \pi^*\mathcal{F}_i \otimes \mathrm{Sym}^i\mathcal{S} \to \pi^*\mathcal{F}_i \otimes \mathrm{Sym}^i\pi^*\mathcal{G} \to \pi^*\mathcal{F}_i \otimes \mathrm{Sym}^{i-1}\pi^*\mathcal{G} \otimes p'^*\mathcal{L} \otimes \mathcal{I}_D \to 0.$$

Hence, it suffices to show

$$H^i(B, \pi^*\mathcal{F}_i \otimes \mathrm{Sym}^i\pi^*\mathcal{G}) = 0 = H^{i-1}(B, \pi^*\mathcal{F}_i \otimes \mathrm{Sym}^{i-1}\pi^*\mathcal{G} \otimes p'^*\mathcal{L} \otimes \mathcal{I}_D)$$

for all $1 \leq i \leq k$.

(i) For the first vanishing, note that by theorem 1.2 and lemma 3.7 it is enough to show that

$$H^i(X \times \mathbb{P}, \mathcal{F}_i \otimes \mathrm{Sym}^i\mathcal{G}) = 0$$

for all $1 \leq i \leq k$. Take the i-th symmetric power of (11) and tensor it with $\mathcal{F}_i$ to obtain

$$0 \to V_{i-1} \otimes \mathcal{F}_i \otimes q^*\mathcal{O}_{\mathbb{P}}(-i-1) \to V_i \otimes \mathcal{F}_i \otimes q^*\mathcal{O}_{\mathbb{P}}(-i) \to \mathcal{F}_i \otimes \mathrm{Sym}^i\mathcal{G} \to 0.$$

Using the Künneth formula, one sees that

$$H^i(X \times \mathbb{P}, \mathcal{F}_i \otimes q^*\mathcal{O}_{\mathbb{P}}(-i)) = 0 = H^{i+1}(X \times \mathbb{P}, \mathcal{F}_i \otimes q^*\mathcal{O}_{\mathbb{P}}(-i-1))$$

for all $1 \leq i \leq k$.

(ii) Again, by theorem 1.2 lemma 3.7, it suffices to prove the vanishing of

$$H^{i-1}(X \times \mathbb{P}, \mathcal{F}_i \otimes \mathrm{Sym}^{i-1}\mathcal{G} \otimes p^*\mathcal{L} \otimes \mathcal{I}_Z)$$

for all $1 \leq i \leq k$. Taking the $(i-1)$-th symmetric power of sequence (11) and tensoring it with $\mathcal{F}_i \otimes p^*\mathcal{L} \otimes \mathcal{I}_Z$ results in

$$\begin{aligned} 0 &\to (V_{i-2} \otimes \mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i)) \otimes \mathcal{I}_Z \to (V_{i-1} \otimes \mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i+1)) \otimes \mathcal{I}_Z \\ &\to \mathcal{F}_i \otimes \mathrm{Sym}^{i-1}\mathcal{G} \otimes p^*\mathcal{L} \otimes \mathcal{I}_Z \to 0. \end{aligned}$$

Thus, it is enough to show that

$$H^{i-1}(X \times \mathbb{P}, (\mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i+1)) \otimes \mathcal{I}_Z) = 0 = H^i(X \times \mathbb{P}, (\mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i)) \otimes \mathcal{I}_Z)$$

for all $1 \leq i \leq k$. Twisting the sequence (9) with $\mathcal{F}_i \otimes q^*\mathcal{O}_{\mathbb{P}}(-i+1)$, one obtains

$$0 \to \mathcal{F}_i \otimes q^*\mathcal{O}_{\mathbb{P}}(-i-1) \to \mathcal{F}_i \otimes \mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(-i) \to (\mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i+1)) \otimes \mathcal{I}_Z \to 0.$$

The Künneth formula provides

$$H^{i-1}(X \times \mathbb{P}, \mathcal{F}_i \otimes \mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(-i)) = 0 = H^i(X \times \mathbb{P}, \mathcal{F}_i \otimes \mathcal{O}_{\mathbb{P}}(-i-1))$$

for all $1 \leq i \leq k$. Ergo, $H^{i-1}(X \times \mathbb{P}, (\mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i+1)) \otimes \mathcal{I}_Z) = 0$ for all $1 \leq i \leq k$. The other vanishing is proven similarly.

□

4 Application to the geometric syzygy conjecture

In this section the techniques in the proof of Voisin's theorem will be used to prove a result that implies the geometric syzygy conjecture.

Theorem 4.1. *[Kem20, Thm. 2.1] The natural homomorphism*

$$\psi\colon H^1(B, \bigwedge^k \pi^*\mathcal{M} \otimes p'^*\mathcal{L}) \twoheadrightarrow H^1(B, \bigwedge^k \Gamma \otimes p'^*\mathcal{L})$$

is surjective.

Remark. Using the kernel bundle description, this theorem can be translated to: There is a natural surjection

$$K_{k-1,2}(X, \mathcal{L}) \twoheadrightarrow H^1(B, \bigwedge^k \Gamma \otimes p'^*\mathcal{L}).$$

Proof. The map ψ is obtained by twisting (10) with $p'^*\mathcal{L}$

$$\cdots \to \bigwedge^{k-2} \pi^*\mathcal{M} \otimes \mathrm{Sym}^2\mathcal{S} \otimes p'^*\mathcal{L} \to \bigwedge^{k-1} \pi^*\mathcal{M} \otimes \mathcal{S} \otimes p'^*\mathcal{L} \to \bigwedge^{k} \pi^*\mathcal{M} \otimes p'^*\mathcal{L} \to \bigwedge^{k} \Gamma \otimes p'^*\mathcal{L} \to 0$$

and taking cohomology. Using the same trick as in the beginning of section 3.3, one concludes that it suffices to show

$$H^{i+1}(B, \bigwedge^{k-i} \pi^*\mathcal{M} \otimes \mathrm{Sym}^i\mathcal{S} \otimes p'^*\mathcal{L}) = 0$$

for all $1 \leq i \leq k$. Tensoring the sequence (12) with $\bigwedge^{k-i} \pi^*\mathcal{M} \otimes p'^*\mathcal{L}$, results in

$$\begin{aligned} 0 \to \bigwedge^{k-i} \pi^*\mathcal{M} \otimes \mathrm{Sym}^i\mathcal{S} \otimes p'^*\mathcal{L} &\to \bigwedge^{k-i} \pi^*\mathcal{M} \otimes \mathrm{Sym}^i\pi^*\mathcal{G} \otimes p'^*\mathcal{L} \\ &\to \bigwedge^{k-i} \pi^*\mathcal{M} \otimes \mathrm{Sym}^{i-1}\pi^*\mathcal{G} \otimes p'^*\mathcal{L}^2 \otimes \mathcal{I}_D \to 0. \end{aligned}$$

Therefore, using theorem 1.2 and lemma 3.7, it is enough to prove

(i) $H^{i+1}(X \times \mathbb{P}, \bigwedge^{k-i} \mathcal{M} \otimes \mathrm{Sym}^i\mathcal{G} \otimes p^*\mathcal{L}) = 0$

(ii) $H^{i}(X \times \mathbb{P}, \bigwedge^{k-i} \mathcal{M} \otimes \mathrm{Sym}^{i-1}\mathcal{G} \otimes p^*\mathcal{L}^2 \otimes \mathcal{I}_Z) = 0.$

To shorten the notation define $\mathcal{F}_i := \bigwedge^{k-i} \mathcal{M}$ and $V_i := \mathrm{Sym}^i H^0(X, \mathcal{E})$.

(i) Take Sym^i of the short exact sequence (11) and tensor it with $\mathcal{F}_i \otimes p^*\mathcal{L}$ to get

$$0 \to V_{i-1} \otimes \mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i-1) \to V_i \otimes \mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i) \to \mathcal{F}_i \otimes \mathrm{Sym}^i\mathcal{G} \otimes p^*\mathcal{L} \to 0.$$

Thus, it suffices to show the vanishings

$$H^{i+1}(X \times \mathbb{P}, \mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i)) = 0 = H^{i+2}(X \times \mathbb{P}, \mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i-1))$$

for all $1 \leq i \leq k$, which follow from the Künneth formula.

(ii) Taking the $(i-1)$-th symmetric power of (11) and tensoring it with $\mathcal{F}_i \otimes p^*\mathcal{L}^2 \otimes \mathcal{I}_Z$ yields

$$0 \to (V_{i-2} \otimes \mathcal{F}_i \otimes \mathcal{L}^2 \boxtimes \mathcal{O}_{\mathbb{P}}(-i)) \otimes \mathcal{I}_Z \to (V_{i-1} \otimes \mathcal{F}_i \otimes \mathcal{L}^2 \boxtimes \mathcal{O}_{\mathbb{P}}(-i+1)) \otimes \mathcal{I}_Z$$
$$\to \mathcal{F}_i \otimes \mathrm{Sym}^{i-1}\mathcal{G} \otimes p^*\mathcal{L}^2 \otimes \mathcal{I}_Z \to 0.$$

Ergo, it suffices to show

$$H^{i+1}(X \times \mathbb{P}, (\mathcal{F}_i \otimes \mathcal{L}^2 \boxtimes \mathcal{O}_{\mathbb{P}}(-i)) \otimes \mathcal{I}_Z) = 0 = H^i(X \times \mathbb{P}, (\mathcal{F}_i \otimes \mathcal{L}^2 \boxtimes \mathcal{O}_{\mathbb{P}}(-i+1)) \otimes \mathcal{I}_Z).$$

For the first vanishing, the sequence (9) is tensored with $\mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i)$ to get

$$0 \to \mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i-2) \to \mathcal{F}_i \otimes (\mathcal{E} \otimes \mathcal{L}) \boxtimes \mathcal{O}_{\mathbb{P}}(-i-1) \to (\mathcal{F}_i \otimes \mathcal{L}^2 \boxtimes \mathcal{O}_{\mathbb{P}}(-i)) \otimes \mathcal{I}_Z \to 0.$$

Therefore, it is enough to show

$$H^{i+1}(X \times \mathbb{P}, \mathcal{F}_i \otimes (\mathcal{E} \otimes \mathcal{L}) \boxtimes \mathcal{O}_{\mathbb{P}}(-i-1)) = 0 = H^{i+2}(X \times \mathbb{P}, \mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i-2))$$

for $1 \leq i \leq k$, which follows from the Künneth formula. For the second vanishing, one proceeds as before but tensors (9) with $\mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i+1)$ instead of $\mathcal{F}_i \otimes \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(-i)$. □

Lemma 4.2. *[Kem20, Lem. 2.2]* $\bigwedge^k \Gamma \cong q'^*\mathcal{O}_{\mathbb{P}}(k) \otimes \mathcal{I}_D$.

Proof. The bundle Γ has rank k so $\bigwedge^k \Gamma = \det(\Gamma)$. Taking determinants of

$$0 \to \Gamma \to q'^*\mathcal{W} \to p'^*\mathcal{L}|_D \to 0$$

yields $\det(\Gamma) \cong q'^* \det(\mathcal{W}) \otimes \det(p'^*\mathcal{L}|_D)^*$. Taking determinants of

$$0 \to p'^*\mathcal{L} \otimes \mathcal{I}_D \to p'^*\mathcal{L} \to p'^*\mathcal{L}|_D \to 0$$

implies $\det(p'^*\mathcal{L}|_D) \cong \mathcal{I}_D^*$ so $\det(\Gamma) \cong q'^* \det(\mathcal{W}) \otimes \mathcal{I}_D$. Consider

$$0 \to q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z) \to H^0(X, \mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{W} \to 0$$

to obtain $\det(\mathcal{W}) \cong \det(q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z))^*$. Sequence (7)

$$0 \to \mathcal{O}_{\mathbb{P}}(-2) \to H^0(X, \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}}(-1) \to q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z) \to 0$$

combined with lemma 1.9, yields $\det(q_*(p^*\mathcal{L} \otimes \mathcal{I}_Z)) \cong \mathcal{O}_{\mathbb{P}}(-k-2) \otimes \mathcal{O}_{\mathbb{P}}(2) \cong \mathcal{O}_{\mathbb{P}}(-k)$. Hence, $\det(\mathcal{W}) \cong \mathcal{O}_{\mathbb{P}}(k)$ and $\det(\Gamma) \cong q'^*\mathcal{O}_{\mathbb{P}}(k) \otimes \mathcal{I}_D$. □

Corollary 4.2.1. *[Kem20, Cor. 2.3] The map*

$$\psi\colon H^1(B, \bigwedge^k \pi^*\mathcal{M} \otimes p'^*\mathcal{L}) \xrightarrow{\sim} H^1(B, \bigwedge^k \Gamma \otimes p'^*\mathcal{L})$$

is an isomorphism and induces a natural isomorphism $K_{k-1,1}(X, \mathcal{L}) \cong \mathrm{Sym}^{k-2} H^0(X, \mathcal{E})$.

Proof. The map ψ is surjective hence it is an isomorphism if both vector spaces have the same dimension. Combine the previous lemma with theorem 1.2 and lemma 3.7 to obtain

$$H^1(B, \bigwedge^k \Gamma \otimes p'^*\mathcal{L}) \cong H^1(B, q'^*\mathcal{O}_{\mathbb{P}}(k) \otimes \mathcal{I}_D \otimes p'^*\mathcal{L}) \cong H^1(X \times \mathbb{P}, (\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z).$$

Twist sequence (9) by $q^*\mathcal{O}_{\mathbb{P}}(k)$ to get

$$0 \to \mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(k-2) \to \mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(k-1) \to (\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z \to 0.$$

The Künneth formula implies

(i) $H^1(X \times \mathbb{P}, \mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(k-1)) = 0$

(ii) $H^2(X \times \mathbb{P}, \mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(k-1)) = 0$

(iii) $H^2(X \times \mathbb{P}, \mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(k-2)) \cong \operatorname{Sym}^{k-2} H^0(X, \mathcal{E})^*$.

Consequently, the boundary map yields an isomorphism

$$H^1(B, \bigwedge^k \Gamma \otimes p'^*\mathcal{L}) \xrightarrow{\sim} \operatorname{Sym}^{k-2} H^0(X, \mathcal{E})^*.$$

Therefore, pre-composing this isomorphism with ψ gives a surjection

$$K_{k-1,2}(X, \mathcal{L}) \twoheadrightarrow \operatorname{Sym}^{k-2} H^0(X, \mathcal{E})^*$$

due to theorem 4.1 and the following remark. Now, dualize and use Koszul duality to obtain an injective map

$$\operatorname{Sym}^{k-2} H^0(X, \mathcal{E}) \hookrightarrow K_{k-1,1}(X, \mathcal{L}).$$

Voisin's theorem implies that both vector spaces have the same dimension [Kem19], which implies the result. ☐

Theorem 4.3. *[Kem20, Thm. 0.2] There is a map of sheaves $f \colon K_{k-1,1}(X, \mathcal{L})^* \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(k-2)$ such that the diagram*

$$\begin{tikzcd} K_{k-1,1}(X, \mathcal{L})^* \arrow[r, "H^0(f)"] \arrow[d, "\cong"'] & \operatorname{Sym}^{k-2} H^0(X, \mathcal{E})^* \arrow[d, "\cong"'] \\ K_{k-2,1}(X, \mathcal{L}) \arrow[r, "\psi"] & H^1(B, \bigwedge^k \Gamma \otimes p'^*\mathcal{L}) \end{tikzcd}$$

commutes (the isomorphism of the right hand side was constructed in the proof of corollary 4.2.1).

Proof. Pushing the short exact sequence $0 \to \Gamma \to q'^*\mathcal{W} \to p'^*\mathcal{L}|_D \to 0$ on $X \times \mathbb{P}$ yields

$$0 \to \Gamma' \to q^*\mathcal{W} \to p^*\mathcal{L}|_Z \to 0$$

since $R^1\pi_*\Gamma = 0$ due to lemma 3.7. Taking the k-th exterior power yields a natural map $\bigwedge^k \Gamma' \to \bigwedge^k q^*\mathcal{W} \cong q^*\mathcal{O}_{\mathbb{P}}(k)$ with image equal to $\mathcal{I}_Z \otimes q^*\mathcal{O}_{\mathbb{P}}(k)$, see [EL12, Cor. 3.7]. This map can be identified with

$$\bigwedge^k \pi_*\Gamma \to \pi_* \bigwedge^k \Gamma \cong \pi_*(\mathcal{I}_D \otimes q'^*\mathcal{O}_{\mathbb{P}}(k)) \hookrightarrow \pi_* \bigwedge^k q'^*\mathcal{W} \cong \bigwedge^k q^*\mathcal{W}$$

and hence the map $\psi \colon H^1(B, \bigwedge^k \pi^*\mathcal{M} \otimes p'^*\mathcal{L}) \to H^1(B, \bigwedge^k \Gamma \otimes p'^*\mathcal{L})$ can be identified with the composition

$$\psi \colon H^1(X \times \mathbb{P}, \bigwedge^k \mathcal{M} \otimes p^*\mathcal{L}) \to H^1(X \times \mathbb{P}, \bigwedge^k \Gamma' \otimes p^*\mathcal{L}) \to H^1(X \times \mathbb{P}, (\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z).$$

Note that $q_*(\bigwedge^k \mathcal{M} \otimes p^*\mathcal{L}) \cong H^0(X, \bigwedge^k \mathcal{M}_{\mathcal{L}} \otimes \mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}}$ and therefore $H^i(\mathbb{P}, q_*(\bigwedge^k \mathcal{M} \otimes p^*\mathcal{L})) = 0$ for all $i > 0$. The Leray spectral sequence then yields an isomorphism

$$H^1(X \times \mathbb{P}, \bigwedge^k \mathcal{M} \otimes p^*\mathcal{L}) \cong H^0(\mathbb{P}, R^1 q_*(\bigwedge^k \mathcal{M} \otimes p^*\mathcal{L})).$$

Twisting the sequence (11) with $q^*\mathcal{O}_{\mathbb{P}}(k)$ and taking direct image under q results in

$$0 \to \mathcal{O}_{\mathbb{P}}(k-2) \to H^0(X,\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}}(k-1) \to q_*((\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z) \to R^1 q_*(\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(k-2)) = 0$$

since

$$R^1 q_*(\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(k-2)) \cong H^1(X,\mathcal{O}_X) \otimes \mathcal{O}_{\mathbb{P}}(k-2) = 0.$$

The sheaf on the left hand side and the sheaf in the middle have no higher cohomology, so $H^i(\mathbb{P}, q_*((\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z)) = 0$ for all $i > 0$. Using the Leray spectral sequence, one obtains an isomorphism

$$H^1(X \times \mathbb{P}, (\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z) \cong H^0(\mathbb{P}, R^1 q_*((\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z)).$$

Ergo, taking global sections of

$$K_{k-1,1}(X,\mathcal{L})^* \otimes \mathcal{O}_{\mathbb{P}} \cong R^1 q_*(\bigwedge^k \mathcal{M} \otimes p^*\mathcal{L}) \to R^1 q_*((\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z)$$

results in ψ. Again, twist the sequence (11) with $q^*\mathcal{O}_{\mathbb{P}}(k)$ and apply q_* to obtain

$$R^1 q_*(\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(k-1)) \to R^1 q_*((\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z) \to R^2 q_*(\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(k-2)) \to R^2 q_*(\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(k-1)).$$

Using the Künneth formula, one sees that

(i) $R^1 q_*(\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(k-1)) = 0$

(ii) $R^2 q_*(\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}}(k-1)) = 0$

(iii) $R^2 q_*(\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(k-2)) \cong R^2 q_*(\mathcal{O}_{X\times\mathbb{P}}) \otimes \mathcal{O}_{\mathbb{P}}(k-2)$.

The flat base change theorem [Har77, III Prop. 9.3] applied to $\mathbb{P} \to \mathrm{Spec}(\mathbb{C})$ implies $R^2 q_* \mathcal{O}_{X\times\mathbb{P}} \cong H^2(X,\mathcal{O}_X) \otimes \mathcal{O}_{\mathbb{P}}$. Hence this yields an isomorphism

$$R^1 q_*((\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z) \cong R^2 q_* \mathcal{O}_{X\times\mathbb{P}} \otimes \mathcal{O}_{X\times\mathbb{P}}(k-2) \cong \mathcal{O}_{\mathbb{P}}(k-2)$$

and therefore ψ can be identified with taking global sections of

$$f\colon K_{k-1,1}(X,\mathcal{L})^* \otimes \mathcal{O}_{\mathbb{P}} \to R^1 q_*((\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z) \cong \mathcal{O}_{\mathbb{P}}(k-2). \tag{13}$$

□

Theorem 4.4. *[Kem20, Thm. 0.2]* $K_{k-1,1}(X,\mathcal{L},H^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)}))$ *is a one-dimensional subspace of* $K_{k-1,1}(X,\mathcal{L})$ *and the morphism*

$$\begin{aligned} \tilde{\psi}\colon \mathbb{P}(H^0(X,\mathcal{E})) &\to \mathbb{P}(K_{k-1,1}(X,\mathcal{L})) \\ [s] &\mapsto K_{k-1,1}(X,\mathcal{L},H^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)})) \end{aligned}$$

is the Veronese embedding of degree $k-2$, *i.e.* ψ *induces an isomorphism*

$$\mathrm{Sym}^{k-2} H^0(X,\mathcal{E})) \xrightarrow{\sim} K_{k-1,1}(X,\mathcal{L}).$$

Proof. Consider the map (13) and note that the sheaves on both sides are globally generated. By construction, this map is surjective on global sections and hence surjective itself. Use the natural isomorphism $K_{k-1,1}(X,\mathcal{L}) \cong \mathrm{Sym}^{k-2} H^0(X,\mathcal{E})$ and pass to the level of graded rings to get

$$\bigoplus_{n\geq 0} \mathrm{Sym}^{k-2} H^0(X,\mathcal{E})^* \otimes \mathrm{Sym}^n H^0(X,\mathcal{E})^* \twoheadrightarrow \bigoplus_{n\geq 0} \mathrm{Sym}^{n+k-2} H^0(X,\mathcal{E})^*.$$

This yields a Veronese embedding

$$\tilde{\psi}\colon \mathbb{P}(H^0(X,\mathcal{E})) \hookrightarrow \mathbb{P}(\mathrm{Sym}^{k-2} H^0(X,\mathcal{E})) \xrightarrow{\sim} \mathbb{P}(K_{k-1,1}(X,\mathcal{L}))$$

of degree $k-2$. It remains to prove that for all $t \in H^0(X,\mathcal{E}) \setminus \{0\}$

$$\tilde{\psi}([t]) = K_{k-1,1}(X,\mathcal{L},H^0(X,\mathcal{L}\otimes\mathcal{I}_{Z(s)})).$$

Take a point $[t] \in \mathbb{P}(H^0(X,\mathcal{E})^*)$ and note that $\tilde{\psi}$ sends $[t]$ to the point in $\mathbb{P}(K_{k-1,1}(X,\mathcal{L}))$ that corresponds to the zero ideal in

$$\bigoplus_{n\geq 0} \mathrm{Sym}^{k-2} H^0(X,\mathcal{E})^* \otimes \mathrm{Sym}^n H^0(X,\mathcal{E})^* \otimes \kappa([t]).$$

Going through the identifications of (13) backwards, provides

$$\begin{aligned} &H^0(\mathbb{P}, K_{k-1,1}(X,\mathcal{L})^* \otimes \mathcal{O}_{\mathbb{P}} \otimes \kappa([t]) \to \mathcal{O}_{\mathbb{P}}(k-2) \otimes \kappa([t])) \\ \widehat{=}\ &H^0(\mathbb{P}, R^1 q_*(\bigwedge^k \mathcal{M} \otimes p^*\mathcal{L}) \otimes \kappa([t]) \to R^1 q_*((\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}(k)) \otimes \mathcal{I}_Z) \otimes \kappa([t])). \end{aligned}$$

Note that Grauert's theorem implies

(i) $R^1 q_*(\bigwedge^k \mathcal{M} \otimes p^*\mathcal{L}) \otimes \kappa([t]) \cong H^1(X, \bigwedge^k \mathcal{M}_{\mathcal{L}} \otimes \mathcal{L})$

(ii) $R^1 q_*((\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}}) \otimes \mathcal{I}_Z) \otimes \kappa([t]) \cong H^1(X, \mathcal{L} \otimes \mathcal{I}_{Z(t)})$.

Using these isomorphisms, the fibre over $[t]$ can be identified with

$$H^1(X, \bigwedge^k \mathcal{M}_{\mathcal{L}} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{I}_{Z(t)}).$$

Looking at the first identifications, one sees that this map factors into

$$H^1(X, \bigwedge^k \mathcal{M}_{\mathcal{L}} \otimes \mathcal{L}) \to H^1(X, \bigwedge^k \Gamma'_t \otimes \mathcal{L}) \to H^1(X, \mathcal{L} \otimes \mathcal{I}_{Z(t)})$$

and was obtained by restricting the morphism $\bigwedge^k \pi^*\mathcal{M} \otimes p'^*\mathcal{L} \to \bigwedge^k \Gamma \otimes p'^*\mathcal{L}$ to the fibre B_t of q' over $[t]$. Ergo, we can identify the fibre of (13) with the natural surjection

$$H^1(B_t, \bigwedge^k \pi_t^*\mathcal{M}_{\mathcal{L}} \otimes \mathcal{L}) \twoheadrightarrow H^1(B_t, \bigwedge^k \Gamma_t \otimes \pi_t^*\mathcal{L}).$$

There are isomorphisms

(i) $\bigwedge^k \mathcal{M}_{\mathcal{L}} \otimes \mathcal{L} \cong \bigwedge^k \mathcal{M}_{\mathcal{L}}^*$: The bundle $\mathcal{M}_{\mathcal{L}}$ is locally free of rank $2k$ and hence the wedge product yields a bilinear form $\bigwedge^k \mathcal{M}_{\mathcal{L}} \times \bigwedge^k \mathcal{M}_{\mathcal{L}} \to \det(\mathcal{M}_{\mathcal{L}}) \cong \mathcal{L}^*$. This bilinear form is non-degenerate, which implies the desired isomorphism.

(ii) $\bigwedge^k \Gamma_t \otimes \pi_t^*\mathcal{L} \cong \bigwedge^k \mathcal{S}_t^*$: This is obtained by taking the determinants of $0 \to \mathcal{S}_t \to \pi_t^*\mathcal{M}_{\mathcal{L}} \to \Gamma_t \to 0$ since $\mathcal{S}_t$ and Γ_t are locally free of rank k.

Using these isomorphisms, one can identify the fibre of (13) with the map

$$H^1(B_t, \bigwedge^k \pi_t^*\mathcal{M}_{\mathcal{L}}^*) \twoheadrightarrow H^1(B_t, \bigwedge^k \mathcal{S}_t^*).$$

Note that from dualizing the short exact sequence

$$0 \to \mathcal{S}_t \to \pi_t^*\mathcal{M}_{\mathcal{L}} \to \Gamma_t \to 0$$

and taking the k-th exterior power, the same map in H^1 arises. Now, dualize the above surjection and apply Serre duality to get

$$H^1(B_t, \bigwedge^k \mathcal{S}_t \otimes \mathcal{O}_{B_t}(D_t)) \hookrightarrow H^1(B_t, \bigwedge^k \pi_t^*\mathcal{M}_{\mathcal{L}} \otimes \mathcal{O}_{B_t}(D_t))$$

(the canonical sheaf of the blowup of X along $Z(t)$ is isomorphic to $\mathcal{O}_{B_t}(D_t)$). Note that there are short exact sequences

$$0 \to \mathcal{S}_t \to H^0(X, \mathcal{L} \otimes \mathcal{I}_{Z(t)}) \otimes \mathcal{O}_{B_t} \to \pi_t^*\mathcal{L} \otimes \mathcal{I}_{D_t} \to 0$$

and

$$0 \to \pi_t^*\mathcal{M}_{\mathcal{L}} \to H^0(X, \mathcal{L}) \otimes \mathcal{O}_{B_t} \to \pi_t^*\mathcal{L} \to 0$$

with $H^0(X, \mathcal{L} \otimes \mathcal{I}_{Z(t)}) \cong H^0(B_t, \pi_t^*\mathcal{L} \otimes \mathcal{I}_{D_t})$ and $H^0(X, \mathcal{L}) \cong H^0(B_t, \pi_t^*\mathcal{L})$. Consequently, the kernel bundle description identifies the injection above with

$$K_{k-1,1}(B_t, \mathcal{O}_{B_t}(D_t), \pi_t^*\mathcal{L} \otimes \mathcal{I}_{D_t}) \hookrightarrow K_{k-1,1}(B_t, \mathcal{O}_{B_t}(D_t), \pi_t^*\mathcal{L}).$$

Consider the fibre product diagram

$$\begin{tikzcd} D_t \arrow[r] \arrow[d] & B_t \arrow[d, "\pi_t"] \\ Z(t) \arrow[r] & X. \end{tikzcd}$$

The commutativity implies $\pi_t^*\mathcal{L}^n|_{D_t} \cong \mathcal{O}_{D_t}$ and hence

$$H^0(D_t, (\pi_t^*\mathcal{L}^n \otimes \mathcal{O}_{B_t}(D_t))|_{D_t}) \cong H^0(D_t, \mathcal{O}_{B_t}(D_t)|_{D_t}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$

for all n. Thus, the short exact sequence

$$0 \to \pi_t^*\mathcal{L}^n \to \pi_t^*\mathcal{L}^n \otimes \mathcal{O}_{B_t}(D_t) \to (\pi_t^*\mathcal{L}^n \otimes \mathcal{O}_{B_t}(D_t))|_{D_t} \to 0$$

induces an isomorphism

$$H^0(B_t, \pi_t^*\mathcal{L}^n \otimes \mathcal{O}_{B_t}(D_t)) \cong H^0(B_t, \pi_t^*\mathcal{L}^n) \cong H^0(X, \mathcal{L}^n)$$

for any n. Therefore, we can naturally identify the fibre of (13) with

$$K_{k-1,1}(X, \mathcal{L}, H^0(X, \mathcal{L} \otimes \mathcal{I}_{Z(t)})) \hookrightarrow K_{k-1,1}(X, \mathcal{L}).$$

$\square$

A List of exact sequences

Here is a list with the most frequently used exact sequences, including their numbering:

(1)
$$0 \to \mathcal{F}_{C,\mathcal{A}} \to H^0(C,\mathcal{A}) \otimes \mathcal{O}_X \to i_*\mathcal{A} \to 0$$

(2)
$$0 \to H^0(C,\mathcal{A})^* \otimes \mathcal{O}_X \to \mathcal{F}^*_{C,\mathcal{A}} \to i_*(\omega_C \otimes \mathcal{A}^*) \to 0$$

(3)
$$\cdots \to \mathrm{Sym}^{i-2}\mathcal{F}_2 \otimes \bigwedge^2 \mathcal{F}_1 \to \mathrm{Sym}^{i-1}\mathcal{F}_2 \otimes \mathcal{F}_1 \to \mathrm{Sym}^i\mathcal{F}_2 \to \mathrm{Sym}^i\mathcal{F}_3 \to 0$$

(4)
$$\cdots \to \bigwedge^{i-2} \mathcal{F}_2 \otimes \mathrm{Sym}^2\mathcal{F}_1 \to \bigwedge^{i-1} \mathcal{F}_2 \otimes \mathcal{F}_1 \to \bigwedge^i \mathcal{F}_2 \to \bigwedge^i \mathcal{F}_3 \to 0$$

(5)
$$0 \to \mathrm{Sym}^i\mathcal{F}_1 \to \mathrm{Sym}^i\mathcal{F}_2 \to \mathrm{Sym}^{i-1}\mathcal{F}_2 \otimes \mathcal{F}_3 \to \mathrm{Sym}^{i-2}\mathcal{F}_2 \otimes \bigwedge^2 \mathcal{F}_3 \to \cdots$$

(6)
$$0 \to \bigwedge^i \mathcal{M}_\mathcal{L} \to \bigwedge^i H^0(X,\mathcal{L}) \otimes \mathcal{O}_X \to \bigwedge^{i-1} \mathcal{M}_\mathcal{L} \otimes \mathcal{L} \to 0$$

(7)
$$0 \to \mathcal{O}_X \stackrel{s}{\to} \mathcal{E} \stackrel{\wedge s}{\to} \mathcal{I}_{Z(s)} \otimes \mathcal{L} \to 0$$

(8)
$$0 \to \mathcal{I}_{Z(s)} \otimes \mathcal{L} \to \mathcal{L} \to \mathcal{L}|_{Z(s)} \to 0$$

(9)
$$0 \to \mathcal{O}_X \boxtimes \mathcal{O}_\mathbb{P}(-2) \stackrel{\cdot id}{\to} \mathcal{E} \boxtimes \mathcal{O}_\mathbb{P}(-1) \to p^*\mathcal{L} \otimes \mathcal{I}_Z \to 0$$

(10)
$$\cdots \to \bigwedge^{k-1} \pi^*\mathcal{M} \otimes \mathrm{Sym}^2\mathcal{S} \to \bigwedge^k \pi^*\mathcal{M} \otimes \mathcal{S} \to \bigwedge^{k+1} \pi^*\mathcal{M} \to 0$$

(11)
$$0 \to q^*\mathcal{O}_\mathbb{P}(-2) \to H^0(X,\mathcal{E}) \otimes q^*\mathcal{O}_\mathbb{P}(-1) \to \mathcal{G} \to 0$$

(12)
$$0 \to \mathrm{Sym}^i\mathcal{S} \to \mathrm{Sym}^i\pi^*\mathcal{G} \to \mathrm{Sym}^{i-1}\pi^*\mathcal{G} \otimes p'^*\mathcal{L} \otimes \mathcal{I}_D \to 0$$

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