Period-index conjecture for (hyperkähler) varieties

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I report on recent work partially in collaboration and in progress with Alessio Bottini, Frank Gounelas, Moritz Hartlieb, and Dominique Mattei.

1. Introduction

We work with quasi-projective varieties X over the complex numbers. Recall that a Brauer–Severi variety over X is a smooth and proper morphism $P \to X$ such that all fibres are isomorphic to a projective space $\mathbb{P}^{n-1}$. The standard example is the projectivisation $P = \mathbb{P}(E) \to X$ of a locally free sheaf of rank n which comes endowed with a relative tautological line bundle $\mathcal{O}(1)$.

Attached to a Brauer–Severi variety $P \to X$ are two numerical invariants:

(i) The *period* is the minimal positive fibre degree of all line bundles on P.

(ii) The *index* can be either defined as the minimum of all m, such that P contains a closed subscheme $P' \subset P$ that over a dense Zariski open subset is a linear Brauer–Severi variety of dimension $m-1$, or as the minimum of all m such that P contains a closed subscheme $Z \subset P$ which is generically finite of degree m over X.

Note that there is a clear geometric recipe that allows one to pass from P' to Z and back, e.g. to Z one associates its fibrewise span which describes a natural P'.

Also recall that the *Brauer group* $\mathrm{Br}(X)$ is the abelian group of all equivalence classes of Brauer–Severi varieties $P \to X$. Here, P is declared to be equivalent to $P \otimes \mathbb{P}(E)$ for all locally free sheaves E.

Then for $\alpha = [P]$ the period of P is just the order of the element $\alpha \in \mathrm{Br}(X)$ and we write $\mathrm{per}(\alpha)$. Also the index is well defined and we write $\mathrm{ind}(\alpha)$. It is classically known that

$$\mathrm{per}(\alpha) \mid \mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{\mathrm{ind}(\alpha)-1}.$$

In particular, period and index share the same prime factors.

If X is smooth and projective, then

$$\mathrm{Br}(X) \cong (\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \oplus \mathrm{Tors}H^3(X,\mathbb{Z}),$$

but the Brauer group grows when passing to smaller open subsets $U \subset X$. The direct limit is the Brauer group $\mathrm{Br}(K(X))$ which is a huge divisible group (by a result of Merkujev–Suslin [17]).

2. Period-index conjecture

A uniform bound for the index in terms of the period is predicted:

Conjecture 1 (Colliot-Thélène)**.** *For all classes $\alpha \in \mathrm{Br}(K(X))$ one has:*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{\dim(X)-1}. \tag{1}$$

By work of Kresch [16], Hotchkiss [8], and de Jong and Perry [4], it was felt that the conjecture is probably not true in this strong form but possibly only for 'coprime' classes. This was shortly after the meeting confirmed by Perry [18] (AI assisted).

However, the conjecture is trivially true in dimension one (Tsen's theorem) and in dimension two [3]. In [15] we proved the conjecture for K3 surfaces (algebraic and analytic) by methods that turned out to be useful much later for hyperkähler manifolds [1, 10].

The following result was first proved by de Jong [3] for surfaces and then later with Starr [5] in general (the result was again only stated for surfaces, but experts knew how to interpret this as the more general assertion).

Theorem 2 (Discriminant-avoidance)**.** *If (1) holds for all X smooth projective of dimension N and all $\alpha \in \mathrm{Br}(X)$, then it also holds for all $\alpha \in \mathrm{Br}(K(X))$ over the function field of any variety of dimension N.*

Together with Gounelas [7] we redesigned the original proof and introduced the notion of universal Brauer–Sever varieties $Q(d,n) \to B(d,n)$ of fixed period and index. Moreover, we computed basic invariants: $\mathrm{Br}(B) = \langle [Q] \rangle$, $\pi_1(B) = \{1\}$, $\mathrm{Pic}(B) \cong \mathbb{Z}$, and $H^*(B,\mathbb{Q})$.

3. Some known cases

Beyond surfaces only very few results towards the period-index conjecture are known. Of course, sometimes the Brauer group of a projective variety X is just trivial but in dimension > 1 the Brauer group is always big and it is generally believed that the period-index conjecture for function fields is equally complicated for any two projective varieties (a wisdom that I learned from O. Benoist).

According to [6], we know that $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^g$ on all g-dimensional complex tori. The bound cannot be improved unless one restricts to abelian varieties (Hotchkiss). In dimension three this was verified by Hotchkiss and Perry [9], i.e. we now know that $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^2$ on abelian threefolds. In higher dimensions some special abelian varieties, e.g. products of elliptic curves, have been dealt with by Ruoxi Li [19].

4. Hodge theory

Initiated originally by Kresch [16], but fully developed by Hotchkiss [8] and de Jong and Perry [4], we now have Hodge theoretic techniques at our disposal to approach the period-index problem. There are essentially two ways: By using the Hodge theory of a Brauer–Severi variety or by using the twisted Hodge theory of a variety as introduced in [15].

(i) Start with a topologically trivial Brauer–Severi variety $P \to X$. Any $P' \subset P$ of relative dimension $m-1$ as above gives rise to an integral Hodge class $\Delta = [P'] \in H^{n-m,n-m}(P,\mathbb{Z})$. Thus, if there is no Hodge class which on the generic fibre describes the class of a linear subspace of codimension $n-m$, then the index of $\alpha = [P]$ does not divide m.

Similarly, if $Z \subset P$ is generically finite of degree m over X, then its fundamental class $\delta = [Z] \in H^{n-1,n-1}(P, \mathbb{Z})$ is a Hodge class which is fibrewise of degree m. If such a class does not exist, then the index cannot divide m.

It is curious to observe that, although geometrically we know exactly how to pass from P' to Z and back, cohomologically this is not possible. Even worse, if $m = \mathrm{per}(\alpha)^{\dim(X)-1}$, then a Hodge class δ as above always exists while this is not clear for Δ. (That there is a priori no reason for the existence of Δ goes back to [16, 8, 4]. It has now been confirmed in [18].)

Another way to view the discussion for δ (which always exists) is to say that if the integral Hodge conjecture holds for P then the period-index conjecture holds in its original form. In other words, using that it does not hold, allows us to conclude that over the varieties considered in [18] there exist Brauer–Severi varieties for which the integral Hodge conjecture does not hold.

(ii) The second way to apply Hodge theory is via the twisted Hodge structures $\widetilde{H}(X, \alpha, \mathbb{Z})$ introduced in [15]. It has become clear by know that it is better to work with topological K-theory instead, see [8], but for the purpose of this talk we stick to cohomology.

Roughly, it turns out that for a topologically trivial Brauer class $\alpha \in \mathrm{Br}(X)$ of period $d = \mathrm{per}(\alpha)$, there always exists an integral Hodge class in $\widetilde{H}(X, \alpha, \mathbb{Z})$ of rank $r = d^{\dim(X)-1} \cdot (\dim(X) - 1)!$. Any type of integral Hodge conjecture for the twisted variety (X, α) would then imply that there exists an α-twisted coherent sheaf on X of rank r and, therefore, $\mathrm{ind}(\alpha) \mid r$. Hence, if $\mathrm{per}(\alpha)$ and $(\dim(X)-1)!$ are coprime to each other, then (1) holds.

5. Uniform bounds

It seems we are currently still far away from proving the period-index conjecture for coprime classes. But we do know that uniform bounds exist at least for unramified classes. This is the following result joint with Mattei [12]. A conditional (on a standard Lefschetz conjecture) result of a similar kind was obtained by de Jong and Perry [4].

Theorem 3. *Assume X is a smooth projective variety. Then there exists an integer $g(X)$ such that for all $\alpha \in \mathrm{Br}(X)$ one has*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{g(X)}.$$

At this point, no uniform bounds are known for ramified classes.

The key idea is to consider a covering family of complete intersection curves $B = \mathbb{P}^{N-1} \leftarrow \mathcal{C} \to X$ and the associated relative Jacobian $\mathrm{Pic}^0(\mathcal{C}/B) \to B$. A Brauer class $\alpha \in \mathrm{Br}(X) = \mathrm{Br}(\mathcal{C})$ induces a twist Pic_α of this relative Jacobian and constructing sheaves on it eventually leads to twisted sheaves on X and then to a bound of the index.

6. Better bounds on hyperkähler manifolds

For special classes of varieties one can expect better bounds. In [11] I suggested the following.

Conjecture 4. *Assume X is a projective hyperkähler manifold of dimension $2N$. Then all coprime $\alpha \in \mathrm{Br}(X)$ satisfy*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^N. \tag{2}$$

This upper bound is sharper then the general one in every dimension except two. Bottini informed me that Hotchkiss and Belmans (and then independently confirmed by him) imitated [18] to produce (AI assisted) explicit classes on hyperkähler varieties of $\mathrm{K3}^{[2]}$-type which have to be excluded. The form of the expected coprimality factor is not clear at the moment.

In [11] I was able to provide the following evidence for the conjecture.

Theorem 5. *The conjectured upper bound (2) holds in the following cases:*
(i) *If X admits a Lagrangian fibration and α is coprime.*
(ii) *If X is the Hilbert scheme $S^{[n]}$ of a K3 surface.*

The conjecture was taken up shortly afterwards in [10] and proved in more case.

Theorem 6. *The upper bound (2) holds for all hyperkähler varieties of* $\mathrm{K3}^{[n]}$*-type of Picard number at least two and α is coprime and non-special.*

Note that the varieties described by these two theorems cover dense subsets of the corresponding moduli spaces.

Combining techniques of Markman as in [10] with techniques of O'Grady, together with Bottini we managed to prove (2) in the $\mathrm{K3}^{[n]}$-case. The results of that paper are summarised as follows.

Theorem 7. (i) *If X is a projective hyperkähler manifold of dimension $2N$ and $\alpha \in \mathrm{Br}(X)$ is coprime, then*

$$\mathrm{ind}_H \mid \mathrm{per}(\alpha)^N.$$

(ii) *Assume X is a general projective hyperkähler manifold (not necessarily of* $\mathrm{K3}^{[n]}$*-type) of dimension $2N$ and α is coprime and non-special. Then*

$$\mathrm{per}(\alpha)^N \mid \mathrm{ind}_H(\alpha) \mid \mathrm{ind}(\alpha).$$

(iii) *If X is of* $\mathrm{K3}^{[n]}$*-type and $\alpha \in \mathrm{Br}(X)$ is coprime and non-special, then (2) holds.*

The first statement uses a HK version (there are in fact several of those and the comparison is not totally trivial) of the Hodge-theoretic index introduced by Hotchkiss. It should be read as saying that if the integral Hodge conjecture holds for the twisted variety (X, α), then (2) holds. The second statement shows that the upper bound proposed by (2) is optimal. The third statement settles the $\mathrm{K3}^{[n]}$-case. The generalised Kummer case would be the next step in settling the conjecture for the known examples.

Note that the hyperkähler version of the two classes Δ and δ are potentially obstructed which should be interpreted as saying that the hyperkähler assumption has some yet unclear cohomological consequences.

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